Super-Additivity and Discontinuity Behavior of AVCs under List Decoding

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joint work with Holger Boche and H. Vincent Poor

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Fr2-4: Channel Capacity 4

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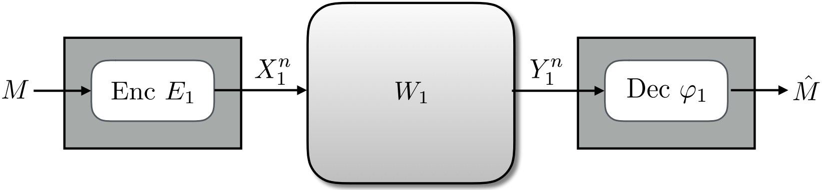
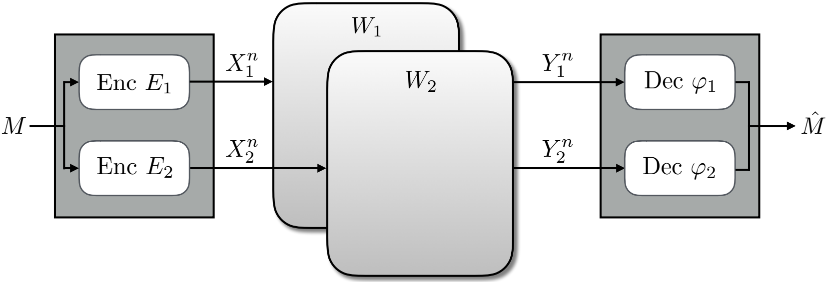
Outline

Capacity of DMC



- Capacity: $C(W_1)$

Capacity of Orthogonal DMCs



- Independent encoding/decoding: $C(W_1) + C(W_2)$

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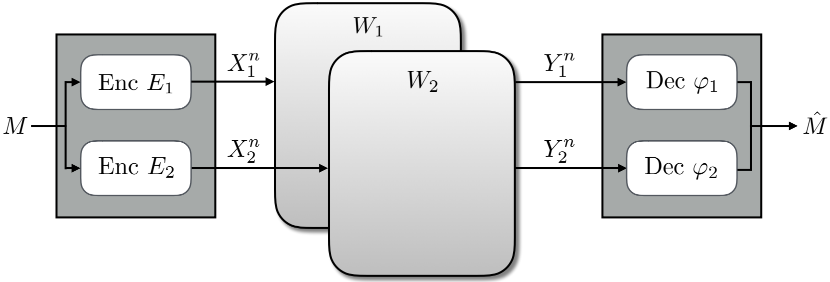

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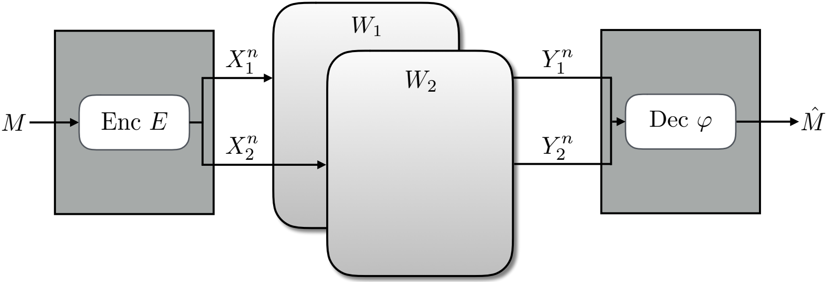

- Independent encoding/decoding: $C(W_1) + C(W_2)$
- Joint encoding/decoding: $C(W_1 \otimes W_2)$

$$C(W_1 \otimes W_2) = C(W_1) + C(W_2)$$

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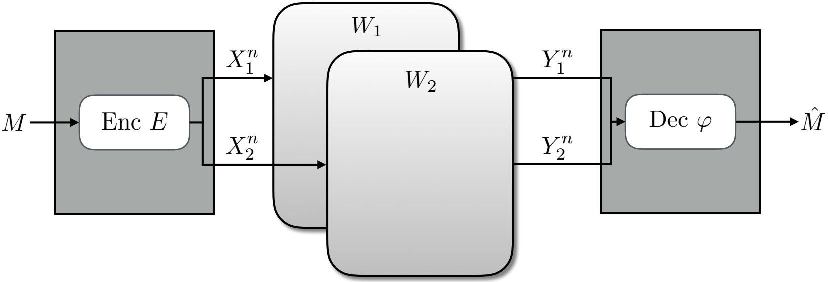

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Zero Error Capacity

- Shannon conjectured in 1956 the zero-error capacity to be additive:

$$C_0(W_1 \otimes W_2) \overset{?}{=} C_0(W_1) + C_0(W_2)$$

> Theorem 4, of course, is analogous to known results for ordinary capacity C, where the product channel has the sum of the ordinary capacities and the sum channel has an equivalent number of letters equal to the sum of the equivalent numbers of letters for the individual channels. We conjecture but have not been able to prove that the equalities in Theorem 4 hold in general, not just under the conditions given.

C. E. Shannon, "The zero error capacity of a noisy channel," *IRE Trans. Inf. Theory*, vol. 2, no. 3, pp. 8–19, Sep. 1956

- Subsequently restated in 1979 by Lovász

L. Lovász, "On the Shannon capacity of a graph," *IEEE Trans. Inf. Theory*, vol. 25, no. 1, pp. 1–7, Jan. 1979

Zero Error Capacity and AVCs

- Later disproved constructing explicit counter-examples with:

$$C_0(W_1 \otimes W_2) > C_0(W_1) + C_0(W_2)$$

- However, complete characterization is still an open problem

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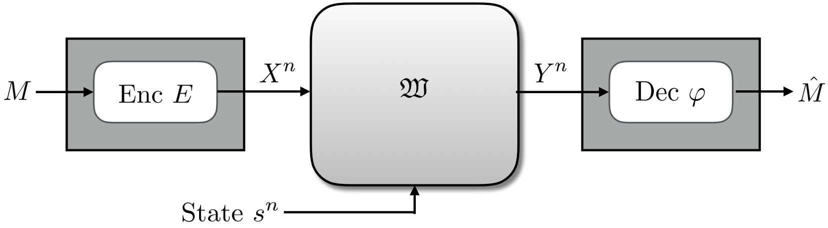
Outline

Arbitrarily Varying Channel



- Uncertainty set $\mathcal{S}$
  - actual state sequence $s^n \in \mathcal{S}^n$ unknown
  - channel may vary in an unknown and arbitrary manner

The *arbitrarily varying channel (AVC)* $\mathfrak{W}$ is given by the family

$$\mathfrak{W} := \{W_s\}_{s\in\mathcal{S}} = \{W(\cdot|\cdot, s)\}_{s\in\mathcal{S}}$$

List Code

Definition (List-L Code)

A (n, M_n)-*list-L code $\mathcal{C}_L$* consists of a deterministic encoder at the transmitter

$$f : \mathcal{M} \to \mathcal{X}^n$$

with a set of messages $\mathcal{M} = \{1, ..., M_n\}$ and a list decoder at the receiver

$$\varphi_L : \mathcal{Y}^n \to \mathfrak{P}_L(\mathcal{M})$$

with $\mathfrak{P}_L(\mathcal{M})$ the set of all subsets of $\mathcal{M}$ with cardinality at most L.

- For a state sequence $s^n \in \mathcal{S}^n$ the average probability of error of such a list code is
$$\bar{e}_L(s^n) = \frac{1}{|\mathcal{M}|} \sum_{m \in \mathcal{M}} \sum_{y^n : m \notin \varphi_L(y^n)} W^n(y^n | x_m^n, s^n).$$

Achievable Rate and Capacity

Definition (Achievable List-L-Rate and Capacity)

A rate $R > 0$ is an *achievable list-L rate* for an AVC $\mathfrak{W}$ if for every $\tau > 0$ there exists an $n(\tau) \in \mathbb{N}$ and a sequence $\{\mathcal{C}_{L,n}\}_{n\in\mathbb{N}}$ such that for all $n \geq n(\tau)$ we have

$$\frac{1}{n}\log\frac{M_n}{L} \geq R - \tau$$

$$\max_{s^n \in \mathcal{S}^n} \bar{e}_L(s^n) \leq \lambda_n$$

with $\lambda_n \to 0$ as $n \to \infty$.
The *list-L capacity $C_L(\mathfrak{W})$* of an AVC $\mathfrak{W}$ is given by the supremum of all achievable list-L rates R.

- For list size $L = 1$ the list-L code $\mathcal{C}_L$ reduces to a traditional deterministic code $\mathcal{C}$ whose decoder outputs only one specific message, i.e., $\varphi : \mathcal{Y}^n \to \mathcal{M}$

CR-Assisted Codes

- List codes use pre-specified encoders and decoders
- If coordination resources such as *common randomness (CR)* are available, transmitter and receiver can coordinate their choice of encoder and decoder

Definition (CR-Assisted Code)

A *CR-assisted* $(n, M_n, \mathcal{G}_n, P_\Gamma)$*-code* $\mathcal{C}_{CR}$ is given by a family of deterministic codes $\{\mathcal{C}(\gamma) : \gamma \in \mathcal{G}_n\}$ together with a random variable Γ taking values in $\mathcal{G}_n$ according to $P_\Gamma \in \mathcal{P}(\mathcal{G}_n)$.

- The average error extends to CR-assisted codes as

$$\bar{e}_{\mathsf{CR}}(s^n) = \frac{1}{|\mathcal{M}|} \sum_{m \in \mathcal{M}} \sum_{\gamma \in \mathcal{G}_n} \sum_{y^n : m \neq \varphi^\gamma(y^n)} W^n(y^n | x_m^{\gamma n}, s^n) P_\Gamma(\gamma)$$

  where φ^γ is decoder for CR realization $\gamma \in \mathcal{G}_n$ and $x_m^{\gamma n}$ is the codeword for message $m \in \mathcal{M}$ of the encoder f^γ for CR realization $\gamma \in \mathcal{G}_n$
- The definitions of a *CR-assisted achievable rate* and the *CR-assisted capacity* $C_{CR}(\mathfrak{W})$ of an AVC $\mathfrak{W}$ follow accordingly.

CR-Assisted Capacity

Theorem (CR-Assisted Capacity [BBT'60])

The CR-assisted capacity $C_{CR}(\mathfrak{W})$ of an AVC $\mathfrak{W}$ is

$$C_{CR}(\mathfrak{W}) = \max_{P_X \in \mathcal{P}(\mathcal{X})} \inf_{q \in \mathcal{P}(\mathcal{S})} I(X; \overline{Y}_q)$$

where $\overline{Y}_q$ represents the output of the averaged channel $\overline{W}_q(y|x) = \sum_{s \in \mathcal{S}} W(y|x,s)q(s)$ for some $q \in \mathcal{P}(\mathcal{S})$.

D. Blackwell, L. Breiman, and A. J. Thomasian, "The capacities of certain channel classes under random coding," *Ann. Math. Statist.*, vol. 31, no. 3, pp. 558–567, Sep. 1960

Symmetrizability

- To characterize the list-L capacity, we need the concept of symmetrizability

Definition

An AVC is called *L-symmetrizable* if there exists a stochastic matrix $\sigma \in \mathrm{CH}(\mathcal{X}^L; \mathcal{S})$ such that for all permutations $\pi \in \mathrm{Sym}[L+1]$ the following holds:

$$\sum_{s\in\mathcal{S}} W(y|x_1,s)\sigma(s|x_2,...,x_{L+1}) = \sum_{s\in\mathcal{S}} W(y|x_{\pi(1)},s)\sigma(s|x_{\pi(2)},...,x_{\pi(L+1)})$$

for all $x_1, x_2, ..., x_{L+1} \in \mathcal{X}$ and $y \in \mathcal{Y}$.

➠ Roughly speaking, AVC can *"simulate" additional valid inputs making it impossible for the decoder to decide on the correct codeword*

List Capacity

Theorem (List-L Capacity [BNP'95], [Hug'97])

The list-L capacity $C_L(\mathfrak{W})$ of an AVC $\mathfrak{W}$ is

$$C_L(\mathfrak{W}) = \begin{cases} C_{\mathsf{CR}}(\mathfrak{W}) & \text{if } \mathfrak{W} \text{ is not } L\text{-symmetrizable} \\ 0 & \text{otherwise.} \end{cases}$$

V. M. Blinvosky, P. Narayan, and M. S. Pinsker, "Capacity of the arbitrarily varying channel under list decoding," *Problems Inform. Transmission*, vol. 31, no. 2, pp. 99–113, 1995

B. L. Hughes, "The smallest list for the arbitrarily varying channel," *IEEE Trans. Inf. Theory*, vol. 43, no. 3, pp. 803–815, May 1997

- For list size $L = 1$, this reduces to the characterization of the deterministic capacity $C(\mathfrak{W})$ of the an AVC $\mathfrak{W}$ as

$$C(\mathfrak{W}) = \begin{cases} C_{\mathsf{CR}}(\mathfrak{W}) & \text{if } \mathfrak{W} \text{ is non-symmetrizable} \\ 0 & \text{otherwise.} \end{cases}$$

I. Csiszár and P. Narayan, "The capacity of the arbitrarily varying channel revisited: Positivity, constraints," *IEEE Trans. Inf. Theory*, vol. 34, no. 2, pp. 181–193, Mar. 1988

Outline

A Fundamental Function

- We introduce the function

$$F_L(\mathfrak{W}) = \inf_{\sigma \in \mathrm{CH}(\mathcal{X}^L;\mathcal{S})} \max_{x^{L+1} \in \mathcal{X}^{L+1}} \max_{\pi \in \mathrm{Sym}[L+1]} \sum_{y \in \mathcal{Y}} \Big| \sum_{s \in \mathcal{S}} W(y|x_1, s)\sigma(s|x_2, \ldots, x_{L+1}) - \sum_{s \in \mathcal{S}} W(y|x_{\pi(1)}, s)\sigma(s|x_{\pi(2)}, \ldots, x_{\pi(L+1)}) \Big|$$

➠ We have $F_L(\mathfrak{W}) \geq 0$ with equality if and only if $\mathfrak{W}$ is L-symmetrizable

Distance

- For two DMCs $W_1, W_2 \in \mathrm{CH}(\mathcal{X};\mathcal{Y})$ we define the distance between W_1 and W_2 based on the total variation distance as

$$d(W_1, W_2) := \max_{x\in\mathcal{X}} \sum_{y\in\mathcal{Y}} \big|W_1(y|x) - W_2(y|x)\big|.$$

- To extend this concept to AVCs, we consider $\mathfrak{W}_i = \{W_i(\cdot|\cdot, s_i)\}_{s_i\in\mathcal{S}_i}$, $i = 1, 2$, and define

$$G(\mathfrak{W}_1, \mathfrak{W}_2) := \max_{s_1\in\mathcal{S}_1} \min_{s_2\in\mathcal{S}_2} d\big(W_1(\cdot|\cdot, s_1), W_2(\cdot|\cdot, s_2)\big)$$

➠ Describes how well one AVC can be approximated by the other one

- Accordingly, we define the distance between $\mathfrak{W}_1$ and $\mathfrak{W}_2$ as

$$D(\mathfrak{W}_1, \mathfrak{W}_2) := \max\big\{G(\mathfrak{W}_1, \mathfrak{W}_2), G(\mathfrak{W}_2, \mathfrak{W}_1)\big\}$$

Definition of Continuity

Definition (Continuity of C_L)

Let $\mathfrak{W}$ be a finite AVC. The list-L capacity $C_L(\mathfrak{W})$ is said to be *continuous* in all finite AVCs $\mathfrak{W}$, if for all sequences of finite AVCs $\{\mathfrak{W}_n\}_{n=1}^{\infty}$ with

$$\lim_{n\to\infty} D(\mathfrak{W}_n, \mathfrak{W}) = 0 \tag{1}$$

we have

$$\lim_{n\to\infty} C_L(\mathfrak{W}_n) = C_L(\mathfrak{W}).$$

- Based on this definition, the list-L capacity $C_L(\mathfrak{W})$ is discontinuous in $\mathfrak{W}$ if and only if there is a sequence $\{\mathfrak{W}_n\}_{n=1}^{\infty}$ of finite AVCs satisfying (1) but

$$\limsup_{n\to\infty} C_L(\mathfrak{W}_n) > \liminf_{n\to\infty} C_L(\mathfrak{W}_n)$$

Discontinuity Points of List-L Capacity

- With the previous concept, we are now in the position to give a complete characterization of the discontinuity points of the list-L capacity

Theorem (Discontinuity Points of C_L)

The list-L capacity $C_L(\mathfrak{W})$ is discontinuous in the finite AVC $\mathfrak{W}$ if and only if the following conditions hold:

1. $C_{CR}(\mathfrak{W}) > 0$
2. $F_L(\mathfrak{W}) = 0$ *and for every* $\epsilon > 0$ *there exists a finite AVC* $\hat{\mathfrak{W}}$ *with* $D(\mathfrak{W}, \hat{\mathfrak{W}}) < \epsilon$ *and* $F_L(\hat{\mathfrak{W}}) > 0$.

- Note that for every list size $L \in \mathbb{N}$ the set of discontinuity points of C_L is non-emtpy!

Robustness Properties of List-L Capacity

Theorem (Robustness of C_L)

Let $\mathfrak{W}$ be a finite AVC with $F_L(\mathfrak{W}) > 0$. Then there exists an $\hat{\epsilon} > 0$ such that all finite AVCs $\hat{\mathfrak{W}}$ with

$$D(\hat{\mathfrak{W}}, \mathfrak{W}) < \hat{\epsilon}$$

are continuity points of $C_L(\mathfrak{W})$.

Super-Additivity and Super-Activation

Definition (Super-Additivity)

Let $\mathfrak{W}_1$ and $\mathfrak{W}_2$ be two finite AVCs and $\mathfrak{W}_1 \otimes \mathfrak{W}_2$ an orthogonal combination. Then, the list-L capacity is said to be *super-additive* if

$$C_L(\mathfrak{W}_1 \otimes \mathfrak{W}_2) > C_L(\mathfrak{W}_1) + C_L(\mathfrak{W}_2),$$

i.e., a joint use of both channels yields a higher list-L capacity than the sum of their individual uses.

- If $C_L(\mathfrak{W}_1) = C_L(\mathfrak{W}_2) = 0$ but $C_L(\mathfrak{W}_1 \otimes \mathfrak{W}_2) > 0$, we have the extreme case of non-additivity which we call *super-activation* of $\mathfrak{W}_1 \otimes \mathfrak{W}_2$

Super-Activation Not Possible

Theorem (No Super-Activation for C_L)

Let $\mathfrak{W}_1$ and $\mathfrak{W}_2$ be two orthogonal AVCs. Then

$$C_L(\mathfrak{W}_1 \otimes \mathfrak{W}_2) = 0$$

if and only if

$$C_L(\mathfrak{W}_1) = C_L(\mathfrak{W}_2) = 0.$$

➠ Super-activation is not possible for orthogonal AVCs

- This extends previous studies where it has been shown for traditional decoding, i.e., $L = 1$, that super-activation is a unique feature of secure communication and that it is not possible for public communication

R. F. Schaefer, H. Boche, and H. V. Poor, "Super-activation as a unique feature of secure communication in malicious environments," *Information*, vol. 7, no. 2, May 2016

Super-Additivity

Theorem (Super-Additivity of C_L)

Let $\mathfrak{W}_1$ and $\mathfrak{W}_2$ be two orthogonal AVCs. Then

$$C_L(\mathfrak{W}_1 \otimes \mathfrak{W}_2) > C_L(\mathfrak{W}_1) + C_L(\mathfrak{W}_2)$$

if and only if

$$\min\left\{F_L(\mathfrak{W}_1), F_L(\mathfrak{W}_2)\right\} = 0,$$
$$\max\left\{F_L(\mathfrak{W}_1), F_L(\mathfrak{W}_2)\right\} > 0,$$

and

$$\min\left\{C_{CR}(\mathfrak{W}_1), C_{CR}(\mathfrak{W}_2)\right\} > 0.$$

➡ List-L capacity is super-additive

- Set of channels which are super-additive is non-empty!

Outline

Randomized Encoding

- Study the Shannon capacity of AVCs under the maximum error criterion and randomized encoding
- This concept differs from the previously considered CR-assisted codes in the sense that the randomization is only performed locally at the encoder and not available at the decoder

Definition (Code with Randomized Encoding)

A (n, M_n)-*code* $\mathcal{C}_{ran}$ *with randomized encoding* consists of a stochastic encoder at the transmitter

$$E : \mathcal{M} \to \mathcal{P}(\mathcal{X}^n)$$

with a set of messages $\mathcal{M} = \{1, ..., M_n\}$ and a deterministic decoder at the receiver

$$\varphi : \mathcal{Y}^n \to \mathcal{M}.$$

Here, the conditional probability $E(x^n|m)$ describes the probability that the message $m \in \mathcal{M}$ is encoded into the codeword $x^n \in \mathcal{X}^n$.

Randomized Encoding (2)

- For a state sequence $s^n \in \mathcal{S}^n$ the maximum probability of error of such a code is

$$e_{\mathsf{max}}(s^n) = \max_{m \in \mathcal{M}} \sum_{y^n : \varphi(y^n) \neq m} \sum_{x^n \in \mathcal{X}^n} W^n(y^n | x^n, s^n) E(x^n | m).$$

Definition (Achievable Rate and Capacity)

A rate $R > 0$ is an *achievable rate* for an AVC $\mathfrak{W}$ under the maximum error criterion and randomized encoding if for every $\tau > 0$ there exists an $n(\tau) \in \mathbb{N}$ and a sequence $\{\mathcal{C}_{\mathsf{ran},n}\}_{n \in \mathbb{N}}$ such that for all $n \geq n(\tau)$ we have

$$\frac{1}{n} \log M_n \geq R - \tau$$

$$\max_{s^n \in \mathcal{S}^n} e_{\mathsf{max}}(s^n) \leq \lambda_n$$

with $\lambda_n \to 0$ as $n \to \infty$. The *Shannon capacity* $C_{\mathsf{max}}^{\mathsf{ran}}(\mathfrak{W})$ of an AVC $\mathfrak{W}$ under the maximum error criterion and randomized encoding is given by the supremum of all achievable rates R.

Capacity

- The following is known for the capacity $C_{\mathsf{max}}^{\mathsf{ran}}(\cdot)$ under the maximum error criterion and randomized encoding.

Theorem (Capacity [Ahl'78])

The capacity $C_{max}^{ran}(\mathfrak{W})$ of a finite AVC $\mathfrak{W}$ under the maximum error criterion and randomized encoding is

$$C_{max}^{ran}(\mathfrak{W}) = C_1(\mathfrak{W})$$

where $C_1(\mathfrak{W})$ is the capacity for deterministic encoding under the average error criterion (i.e., the list-L capacity for $L = 1$).

R. Ahlswede, "Elimination of correlation in random codes for arbitrarily varying channels," *Z. Wahrscheinlichkeitstheorie verw. Gebiete*, vol. 44, no. 2, pp. 159–175, Jun. 1978

Capacity (2)

- With this we immediately obtain the following corollary

Corollary

The continuity and super-additivity behavior of $C_{\mathsf{max}}^{\mathsf{ran}}(\cdot)$ is completely characterized by the list-L capacity $C_L(\cdot)$ for $L = 1$.

Remark

Note that a characterization of $C_{\mathsf{max}}(\cdot)$, i.e., the capacity of an AVC under the maximum error criterion for deterministic encoding, is unknown and an open problem. It is an interesting observation that a local random source at the transmitter allows characterization of the analytical behavior of the Shannon capacity.

Achievable ϵ-Rate

- Shannon ϵ-capacity captures the case of a non-vanishing probability of decoding error
- We need a slight adaptation of the definition of an achievable rate as follows

Definition (Achievable ϵ-Rate and Capacity)

Let $0 < \epsilon < 1$ be fixed. A rate $R > 0$ is an *achievable ϵ-rate* for an AVC $\mathfrak{W}$ if for every $\tau > 0$ there exists an $n(\tau) \in \mathbb{N}$ and a sequence of codes such that for all $n \geq n(\tau)$ we have

$$\frac{1}{n} \log M_n \geq R - \tau$$

$$\max_{s^n \in \mathcal{S}^n} \bar{e}(s^n) \leq \epsilon.$$

The *Shannon ϵ-capacity* $C^\epsilon(\mathfrak{W})$ of an AVC $\mathfrak{W}$ is given by the supremum of all achievable ϵ-rates R.

Remarks

- For a fixed DMC W we have

$$C^{\epsilon}(W) = C_1(W) = C_L(W), \quad L \in \mathbb{N}$$

- Strong converse: there is no gain in rate by allowing a small but non-vanishing decoding error instead of a vanishing decoding error

- Already for compound channels, the Shannon ϵ-capacity is usually greater than the Shannon capacity with vanishing decoding error

R. Ahlswede, "Certain results in coding theory for compound channels," in *Proc. Colloquium Inf. Th.* Debrecen, Hungary: Bolyai Mathematical Society, 1967, pp. 35–60

ϵ-Capacity

- For AVCs we obtain the following result for the ϵ-capacity

Theorem

Let $0 < \epsilon < 1/2$ be arbitrary. Then the ϵ-capacity $C^\epsilon(\mathfrak{W})$ of a finite AVC $\mathfrak{W}$ satisfies

$$C^\epsilon(\mathfrak{W}) = C_1(\mathfrak{W}).$$

- Similarly to the previous discussion, we immediately further obtain

Corollary

For $0 < \epsilon < 1/2$, the continuity and super-additivity behavior of $C^\epsilon(\cdot)$ is completely characterized by the list-L capacity $C_L(\cdot)$ for $L = 1$.

Ahlswede's Conjecture

- This characterizes the behavior of the ϵ-capacity $C^{\epsilon}(\cdot)$ in the range $0 < \epsilon < 1/2$
- For $1/2 \leq \epsilon < 1$ we cannot say anything

The problem is that, to date, there is nothing known about $C^{\epsilon}(\cdot)$ in this range for finite AVCs which are symmetrizable. Ahlswede conjectured in his 2006 Shannon lecture that *"for a finite symmetrizable AVC $\mathfrak{W}$, the ϵ-capacity always satisfies $C^{\epsilon}(\mathfrak{W}) = 0$ for all $0 < \epsilon < 1$."* If this conjecture is true, our results in this work immediately yield a complete characterization of the continuity and super-additivity behavior of $C^{\epsilon}(\cdot)$ for all $0 < \epsilon < 1$.

Outline

Conclusions

- It has been demonstrated by Haemers and Alon that the zero-error capacity is super-additive, disproving a previous conjecture of Shannon
- Despite such explicit examples of super-additivity, there is surprisingly little known in general for non-trivial channels
- In this work, the AVC under list decoding has been studied and a complete theory has been developed including characterizations of
  - Super-additivity behavior and discontinuity points of the list capacity
  - Capacity under the maximum error criterion and randomized encoding
  - ϵ-capacity of finite AVCs under the average error criterion

Thank you for your attention!

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