Distillation of secret-key from a class of compound memoryless quantum sources

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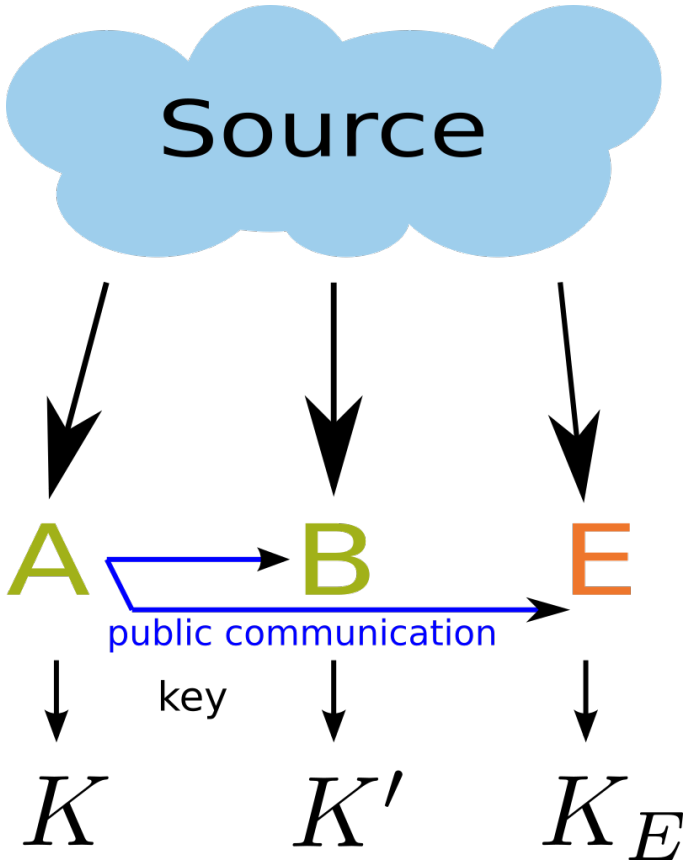
Classical and Quantum IT - Dictionary of Correspondences

Classical Theory	Quantum Theory
Alphabet $\mathcal{X}$	Hilbert space $\mathcal{H}$
Probability distribution $p \in \mathfrak{P}(\mathcal{X})$	Quantum state / Density matrix, $\rho \in \mathcal{L}(\mathcal{H}),\ \rho$ p.s.d, $\mathrm{tr}\rho = 1$
Shannon entropy $H(X)$	von Neumann entropy $S(\rho) := -\mathrm{tr}(\rho \log \rho)$
Classsical mutual informtation $I(X;Y) := H(X) + H(Y) - H(XY)$	Quantum mutual information $I(A;B,\rho_{AB}) := S(\rho_A) + S(\rho_B) - S(\rho_{AB})$
Decision rule $\{D_m\}_{m\in M} \subset 2^{\mathcal{X}}, \bigcup_{m\in M} D_m = \mathcal{X}$	Positive operator valued measure (POVM) $\{E_m\}_{m\in M}, 0 \leq E_m \leq \mathbb{1}_{\mathcal{H}}, \sum_{m\in M} E_m = \mathbb{1}$

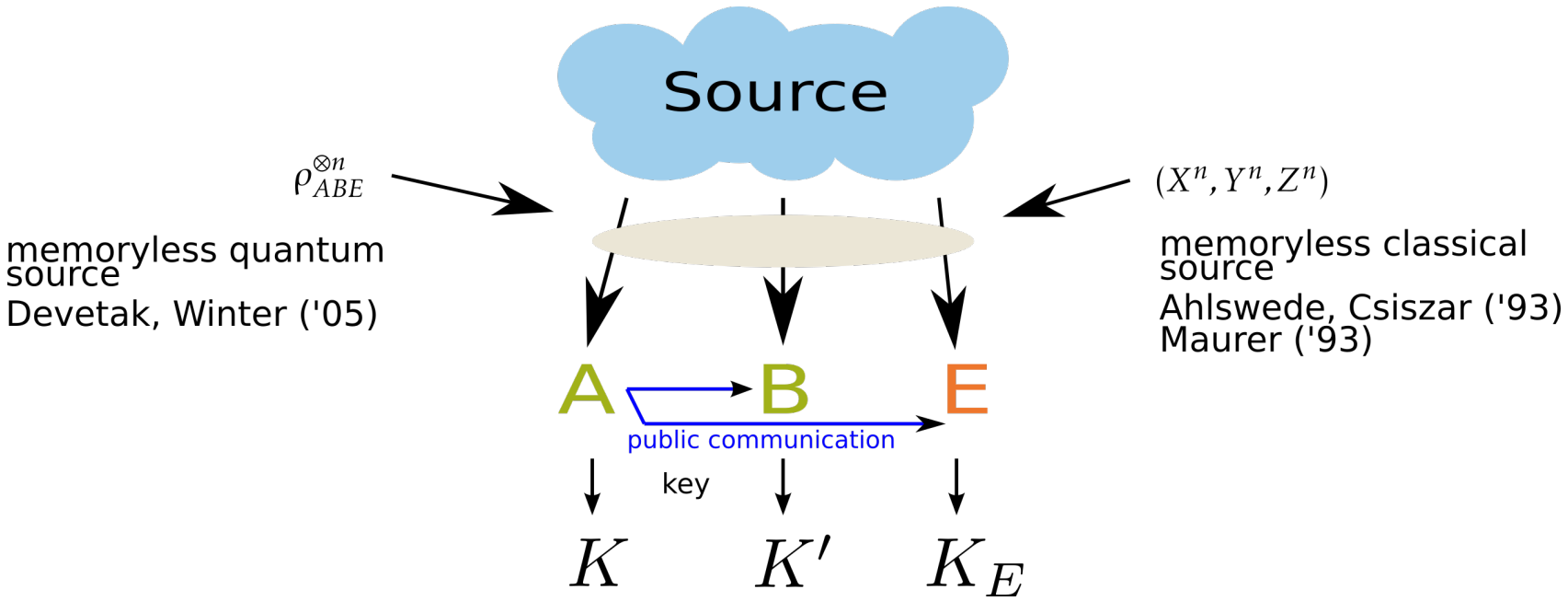
Introduction

- Common randomness shared by users being secure against eavesdropping third parties is a valuable resource in information theory.
- A possibility to obtain secret-keys is to generate it from noisy&insecure correlations distributed by sources.
- We allow public forward communication of classical messages.



Known results

- In case of perfectly known memoryless sources, the asymptotic key capacities where determined



- **Assumption of perfect knowledge of the generating p.d. or density matrix is hardly fulfilled in reality → Need for robust protocols in case of system uncertainty.**

Source model: Compound cqq sources

- n outputs of a compound quantum source $\mathfrak{I} = \{\rho_s\}_{s\in S}$ are described by a density matrix

$$\rho_s^{\otimes n} := \underbrace{\rho_s \otimes \cdots \otimes \rho_s}_{n \text{ times}} \qquad (n \in \mathbb{N})$$

  where s is any of S (unresolved to A and B).

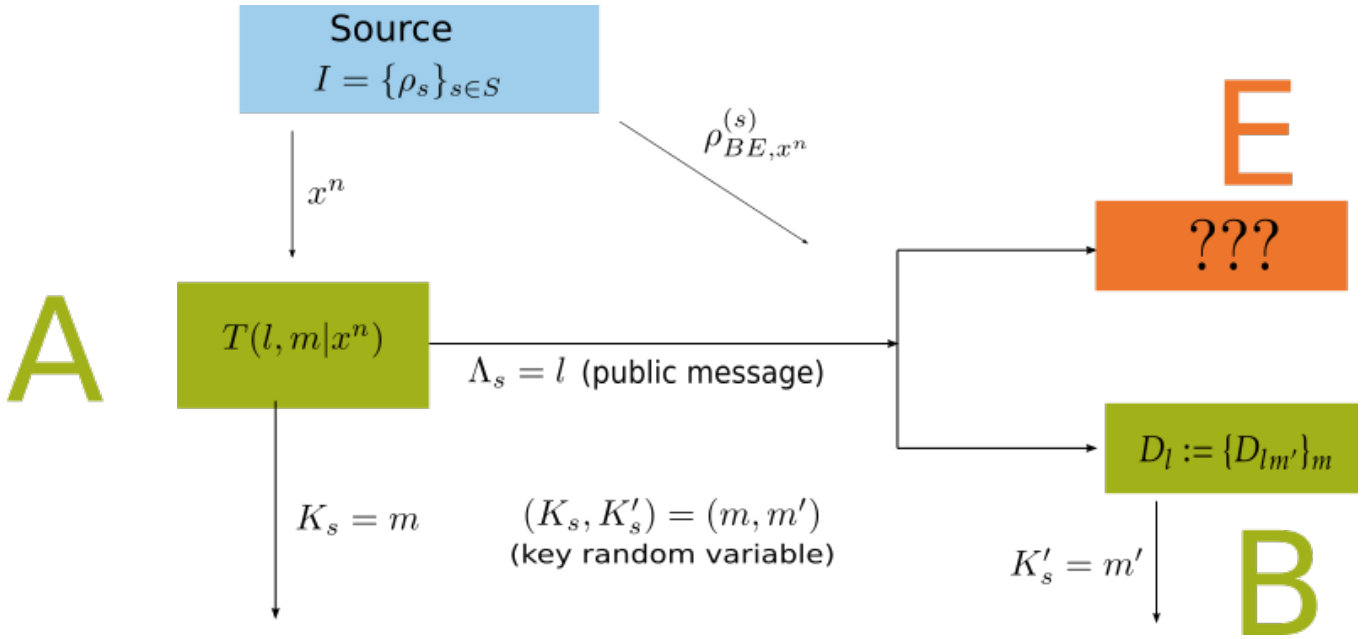
We do not restrict ourselves to $|S| < \infty$ or S countable!

- A compound classical-quantum-quantum (cqq) source is described by a set $\mathfrak{I} := \{\rho_s\}_{s\in S}$ of density matrices on $\mathcal{H}_{ABE}$ such that

$$\rho_s = \sum_{x\in\mathcal{X}} p_s(x)\ |e_x^A\rangle\langle e_x^A| \otimes \rho_{BE,x}^{(s)}$$

  with p_s being a p.d. on $\mathcal{X}$, and $\rho_{BE,x,s}$ a density matrix on $\mathcal{H}_{BE}$ for each $x \in \mathcal{X}$.

Forward secret-key distillation protocols



An (n, M, L, ϵ)-**protocol** for $\mathbb{I} = \{\rho_s\}_{s\in S}$ is a pair (T, D) with

- $(T(l,m|x^n))_{l\in[L],m\in[M],x^n\in\mathcal{X}^n}$ a stochastic matrix
- $D = \{\mathcal{D}_l := \{D_{lm}\}_{m\in[M]}\}_{l\in[L]}$ a collection of POVMs.

such that for all $s \in S$

1. $\Pr(K_s \neq K'_s) \leq \epsilon$, and
2. $\log M - H(K_s) + I(K; E^n\Lambda, \rho_{\Lambda K E^n, s}) \leq \epsilon$.

Operational Interpretation of the performance criteria

- The expression

$$\log M - H(K_s) + I(K; E^n \Lambda, \rho_{\Lambda K E^n, s})$$

quantifies equidistribution and security of the key.

$\rightarrow$ Quantum version of the **security index**.

- Operational significance:

$$I(K; E^n \Lambda, \rho_{\Lambda K E^n, s}) \geq I(K_s; \hat{K}_{E,s})$$

for each eavesdropper's estimate $\hat{K}_E$ of the key random variable ("Holevo bound", Holevo '73).

Definitions

$R \geq 0$ is called an **achievable forward secret-key distillation rate** for $\mathfrak{I}$, if there exists a sequence of $(n, M_n, L_n, \epsilon_n)$ secret-key distillation protocols with

1. $\liminf_{n\to\infty} \frac{1}{n} \log M_n \geq R$,
2. $\limsup_{n\to\infty} \frac{1}{n} \log L_n < \infty$
3. $\lim_{n\to\infty} \epsilon_n = 0$

The **forward secret-key capacity** of $\mathfrak{I}$ is defined by

$$K_{\rightarrow}(\mathfrak{I}) := \sup\{R \geq 0 : \ R \text{ achievable forward secret-key distillation rate}\}$$

Regularity condition

- Define for a set $\mathfrak{I}$ of cqq density matrices

$$\begin{aligned}\mathcal{P}_{\mathfrak{I}} &:= \{p : \ p \text{ marginal p.d. on the } A \text{ system}\}\\ \mathfrak{I}_p^{AE} &:= \{\rho_{AE} : \ \rho \in \mathfrak{I} \ \wedge \ \rho_A = p\}\\ \mathfrak{I}_p^{AB} &:= \{\rho_{AB} : \ \rho \in \mathfrak{I} \ \wedge \ \rho_A = p\}\end{aligned}$$

- **Oberservation**: Some compound cqq sources resist general protocol structures.

  $\rightarrow$ This happens, if members of $\mathfrak{I}$ with nearby A-marginals differ much regarding the sets of AB and AE marginals.

Definition

A set $\mathfrak{I}$ of cqq density matrices is called **regular**, if it fulfills

$$\forall p,q \in \mathcal{P}_{\mathfrak{I}} \ \forall \epsilon > 0 \exists \delta > 0 : \quad \|p-q\|_1 \leq \delta \ \Rightarrow \ d_H(\mathfrak{I}_p^{AB}, \mathfrak{I}_q^{AB}) + d_H(\mathfrak{I}_p^{AE}, \mathfrak{I}_q^{AE}) < \epsilon$$

with $d_H(X,Y)$ being the Hausdorff distance of sets X, Y.

Main result

Theorem

Let $\mathfrak{I}$ be a regular set of cqq density matrices in $\mathcal{H}_{ABE}$. It holds

$$K_{\rightarrow}(\mathfrak{I}) \;=\; \lim_{k\to\infty} \frac{1}{k}\, K_{\rightarrow}^{(1)}(\mathfrak{I}^{\otimes k}),$$

where

$$K_{\rightarrow}^{(1)}(\mathfrak{I}^{\otimes k}) := \inf_{p\in\mathcal{P}_{\mathfrak{I}}} \sup_{\Gamma:=T\leftarrow U\leftarrow p^k} \left(\inf_{\sigma\in\mathfrak{I}_p^{AB}} I(U;B^k|T,\sigma_{k,\Gamma}) - \sup_{\sigma\in\mathfrak{I}_p^{AE}} I(U;E^k|T,\sigma_{k,\Gamma}) \right)$$

with the maximization being over all Markov chains $T \leftarrow U \leftarrow p^k$ resulting from application of Markov transition matrices $P_{T|U}$, $P_{U|Y}$ and

$$\sigma_{k,\Gamma} := \sum_{x^k\in\mathcal{X}^k} \sum_{t\in\mathcal{T}} \sum_{u\in\mathcal{U}} P_{T|U}(t|u)\, P_{U|X^k}(u|x^k)\, p^k(x^k)\; |t\rangle\langle t| \otimes |u\rangle\langle u| \otimes \sigma_{x^k}$$

$I(X;YZ|T) := \sum_{t\in\mathcal{T}} P_T(t) I(X;Y|T=t)$ conditional quantum mutual information of cqq state.

Operational significance of regularity

- Regularity of cqq sources is not only a technical issue.
- If A has additional perfect knowledge of his/her distribution p, regularity plays no role.
- Define $K_{\rightarrow,SMI}$ to be the forward secret-key capacity with sender marginal knowledge.

Theorem

For each set $\mathfrak{I}$ of cqq density matrices, it holds

$$K_{\rightarrow,SMI}(\mathfrak{I}) = \lim_{k\rightarrow\infty} \frac{1}{k} K_{\rightarrow}^{(1)}(\mathfrak{I}^{\otimes k}).$$

Consequently

$$K_{\rightarrow}(\mathfrak{I}) = K_{\rightarrow,SMI}(\mathfrak{I}),$$

if $\mathfrak{I}$ is regular.

Advantage of SMI - Example

We present, with $\mathcal{H}_A = \mathcal{H}_B = \mathcal{H}_E = \mathbb{C}^2 \otimes \mathbb{C}^2$ example of a compound cqq source $\mathfrak{I}$ with

$$0 = K_{\rightarrow}(\mathfrak{I}) < K_{\rightarrow,SMI}(\mathfrak{I}) = \log \dim \mathcal{H}_A.$$

Define with π being the bit equidistribution, and $\Pi := \frac{\mathbb{1}_{\mathbb{C}^2}}{2}$ the maximally mixed qubit density matrix

$$\rho_p := \begin{cases} \sum_{x,y=1}^{2} \pi(x) \cdot \pi(y) \cdot |x,y\rangle\langle x,y|_A \otimes |x\rangle\langle x|_B \otimes \Pi_B \otimes \Pi_E \otimes |y\rangle\langle y|_E & \text{if } p = \pi \\ \sum_{x,y=1}^{2} \pi(x) \cdot p(y) \cdot |x,y\rangle\langle x,y|_A \otimes \Pi_B \otimes |y\rangle\langle y|_B \otimes |x\rangle\langle x|_E \otimes \Pi_E & \text{otherwise,} \end{cases}$$

The compound cqq source generated by $\mathfrak{I} := \{\rho_p\}_{p \in \mathcal{P}(\{0,1\})}$ has the stated properties.

References

H. Boche, G. Janßen.
"Distillation of secret-key from a class of compound memoryless quantum sources",
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(or on the purple USB Stick)