Strong Divergence of the Shannon Sampling Series for an Infinite Dimensional Signal Space

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Adaptive vs. Non-Adaptive Approximation

Non-Adaptive Approximation:
Shannon sampling series

$$(S_N f)(t) = \sum_{k=-N}^{N} f(k) \frac{\sin(\pi(t-k))}{\pi(t-k)}$$

Approximation $S_N f$ uses all samples $\{f(k)\}_{k=-N}^{N}$.

Adaptive Approximation:
Selection of specific samples $f(k)$ for the approximation.

Consequences:

- Non-Adaptive Approximation: sequence of linear operators. Convergence analysis with Banach–Steinhaus theorem (uniform boundedness principle).
- Adaptive Approximation: leads to non-linear operators.

Signal Spaces

Definition (Paley–Wiener Space)

For $1 \leqslant p \leqslant \infty$ we denote by $\mathcal{PW}_\sigma^p$ the Paley-Wiener space of functions f with a representation $f(z) = \frac{1}{2\pi} \int_{-\sigma}^{\sigma} g(\omega)\, e^{iz\omega}\, d\omega$, $z \in \mathbb{C}$, for some $g \in L^p[-\sigma, \sigma]$.

The norm for $\mathcal{PW}_\sigma^p$ is given by $\|f\|_{\mathcal{PW}_\sigma^p} = \left(\frac{1}{2\pi} \int_{-\sigma}^{\sigma} |\hat{f}(\omega)|^p \, d\omega \right)^{1/p}$.

Properties:

- $\mathcal{PW}_\sigma^p \supset \mathcal{PW}_\sigma^s$ for $1 \leqslant p < s \leqslant \infty$
- $\|f\|_\infty \leqslant \|f\|_{\mathcal{PW}_\sigma^1}$
- $\mathcal{PW}_\sigma^2$ is the space of bandlimited functions with finite $L^2(\mathbb{R})$-norm (finite energy).

Without loss of generality, we can restrict to $\sigma = \pi$.

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$$\limsup_{N\to\infty} \left(\max_{t\in\mathbb{R}} \sum_{k=-N}^{N} f(k) \frac{\sin(\pi(t-k))}{\pi(t-k)} \right) = \infty, \tag{1}$$

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  that is the peak value diverges as N tends to infinity.
- Recently strengthened in [BF14] : There exists a signal $f \in \mathcal{PW}_\pi^1$ such that

$$\lim_{N\to\infty} \left(\max_{t\in\mathbb{R}} \sum_{k=-N}^{N} f(k) \frac{\sin(\pi(t-k))}{\pi(t-k)} \right) = \infty. \tag{2}$$

Important difference in the divergence behavior of (1) and (2)

[BF14] H. Boche and B. Farrell, “Strong divergence of reconstruction procedures for the Paley-Wiener space PW_π^1 and the Hardy space H^1,” *Journal of Approximation Theory*, vol. 183, pp. 98–117, Jul. 2014

Weak Divergence vs. Strong Divergence

We say a sequence $\{x_n\}_{n\in\mathbb{N}} \subset \mathbb{C}$

- diverges weakly if $\limsup_{n\to\infty}|x_n| = \infty$.
- diverges strongly if $\lim_{n\to\infty}|x_n| = \infty$.

Weak lim sup divergence:
Merely guarantees the existence of a subsequence $\{N_n\}_{n\in\mathbb{N}}$ for which we have $\lim_{n\to\infty} x_{N_n} = \infty$.
Leaves the possibility that there exist a different subsequences $\{N_n^*\}_{n\in\mathbb{N}}$ such that $\limsup_{n\to\infty} x_{N_n^*} < \infty$.

Strong lim divergence:
Divergence for all subsequences $\{N_n\}_{n\in\mathbb{N}}$.

adaptive techniques not convergent	$\Leftrightarrow$	strong divergence

Banach–Steinhaus Theorem, Weak Divergence, and Residual Sets

Divergence results as in

$$\limsup_{N\to\infty}\left(\max_{t\in\mathbb{R}}\sum_{k=-N}^{N} f(k)\frac{\sin(\pi(t-k))}{\pi(t-k)}\right)=\infty$$

are usually proved by using the Banach–Steinhaus theorem (uniform boundedness principle).

$\rightarrow$ The obtained divergence is in terms of the $\limsup$ (weak divergence) and not a statement about strong divergence.

$\rightarrow$ Strength of the Banach–Steinhaus theorem: the divergence statement holds for all functions from a residual set.

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- $\rightarrow$ Strength of the Banach–Steinhaus theorem: the divergence statement holds for all functions from a residual set.

We cannot use the Banach–Steinhaus theorem to prove strong divergence.

Historical Remarks

- Paul Erdős seems to be the first who noticed strong divergence for the Lagrange interpolation in 1941 [E41].
- He later observed that his proof was erroneous. The question remains unsolved to date.
- To the best of our knowledge the Shannon sampling result is the first discovery (up to now) of strong divergence for approximation processes [BF14].
- The result is surprising because the Shannon sampling series is locally uniformly convergent [B67].

[E41] P. Erdős, "On divergence properties of the Lagrange interpolation parabolas," *Annals of Mathematics*, vol. 42, no. 1, pp. 309–315, Jan. 1941

[BF14] H. Boche and B. Farrell, "Strong divergence of reconstruction procedures for the Paley-Wiener space $\mathcal{PW}_\pi^1$ and the Hardy space $\mathcal{H}^1$," *Journal of Approximation Theory*, vol. 183, pp. 98–117, Jul. 2014

[B67] J. L. Brown, Jr., "On the error in reconstructing a non-bandlimited function by means of the bandpass sampling theorem," *Journal of Mathematical Analysis and Applications*, vol. 18, pp. 75–84, 1967, Erratum, ibid, vol. 21, 1968, p. 699

Set of Functions Creating Divergence

We are interested in the the set

$$D_{\mathsf{str}} = \left\{ f \in \mathcal{PW}_{\pi}^{1} \colon \lim_{N\to\infty} \left(\max_{t\in\mathbb{R}} \sum_{k=-N}^{N} f(k) \frac{\sin(\pi(t-k))}{\pi(t-k)} \right) = \infty \right\}$$

- Does the set D_{str} have further interesting structural properties?
- Does the set D_{str} have a linear structure?

Basic Definition: Lineability and Spaceability

The zero function plays a special role.

Lineability:
A subset S of a Banach space X is said to be lineable if $S \cup \{0\}$ contains an infinite dimensional subspace.

Spaceability:
A subset S of a Banach space X is said to be spaceable if $S \cup \{0\}$ contains a closed infinite dimensional subspace of X.

Linear Structure

Difficult to show linear structure for D_{str}.

The set of convergence has always a linear structure, i.e., is a linear subspace:

- f_1, f_2 such that $S_N f_1$ and $S_N f_2$ converge
- $S_N(f_1 + f_2)$ converges

The set of divergence D_{str} has no linear structure:

- f_1 any function such that $S_N f_1$ converges
- g any function such that $S_N g$ diverges
- $g_1 = f_1 + g$, $g_2 = f_1 - g$
- $S_N g_1$ and $S_N g_2$ diverge
- But $S_N(g_1 + g_2) = S_N(2f_1)$ converges

Set of Functions Creating Divergence

$$\mathrm{D}_{\text{str}} = \left\{ f \in \mathcal{PW}_{\pi}^{1} \colon \lim_{N\to\infty} \left(\max_{t\in\mathbb{R}} \sum_{k=-N}^{N} f(k) \frac{\sin(\pi(t-k))}{\pi(t-k)} \right) = \infty \right\}$$

Theorem

The set D_{str} is spaceable, i.e., $\mathrm{D}_{str} \cup \{0\}$ contains an infinite dimensional closed subspace of $\mathcal{PW}_{\pi}^{1}$.

- The above theorem is stronger than the result proved in the proceedings.

An Infinite Dimensional Closed Subspace $\mathcal{U}$

In the proof we constructed an infinite dimensional closed subspace $\mathcal{U}$ of $\mathcal{PW}_{\pi}^{1}$ such that

$$\lim_{N\to\infty}\left(\max_{t\in\mathbb{R}}\sum_{k=-N}^{N} f(k)\frac{\sin(\pi(t-k))}{\pi(t-k)}\right)=\infty$$

for all $f \in \mathcal{U}$, $f \not\equiv 0$.

$\mathcal{U}$ has interesting properties

- $\mathcal{U}$ has an unconditional basis $\{\zeta_n\}_{n\in\mathbb{N}}$: For all $f \in \mathcal{U}$ there exists a unique sequence of coefficients $\{a_n(f)\}_{n\in\mathbb{N}}$ such that

$$\lim_{N\to\infty} \left\| f - \sum_{n=1}^{N} a_{\Pi(n)}(f)\zeta_{\Pi(n)} \right\|_{\mathcal{PW}_\pi^1} = 0$$

  for any permutation $\Pi\colon \mathbb{N} \to \mathbb{N}$.
- There exists a constant $C_1 > 0$ such that for all $f \in \mathcal{U}$ we have

$$C_1 \left(\sum_{n=1}^{\infty} |a_n(f)|^2 \right)^{\frac{1}{2}} \leqslant \|f\|_{\mathcal{PW}_\pi^1} \leqslant \left(\sum_{n=1}^{\infty} |a_n(f)|^2 \right)^{\frac{1}{2}}.$$

- $\mathcal{U}$ is isomorphic to the Hilbert space l^2.
- If we equip the space $\mathcal{U}$ with the norm $\|f\|_{\mathcal{U}} = \left(\sum_{n=1}^{\infty} |a_n(f)|^2\right)^{\frac{1}{2}}$ then it becomes a Hilbert space.
- $\{\zeta_n\}_{n\in\mathbb{N}}$ is a Riesz basis for the Hilbert space $(\mathcal{U}, \|\cdot\|_{\mathcal{U}})$.

Conclusions

- Weak divergence $\rightarrow$ full theory given by Banach–Steinhaus (residual set)
- Strong divergence $\Leftrightarrow$ adaptive techniques not convergent
- We proved that the set of signals in $\mathcal{PW}_{\pi}^{1}$ for which the Shannon sampling series diverges strongly is spaceable.
- That is, we have strong divergence for all signals (except the zero signal) from an infinite dimensional closed subspace.

Thank you!