Entanglement-assisted classical capacity of compound quantum channels

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Introduction

- Channel uncertainty is a basic feature of real-world communication systems.

- **Channel model:** *Compound (memoryless) quantum channel* generated by a set $\mathfrak{I} \subset \mathcal{C}(\mathcal{H}_A, \mathcal{H}_B)$ of c.p.t.p. maps.

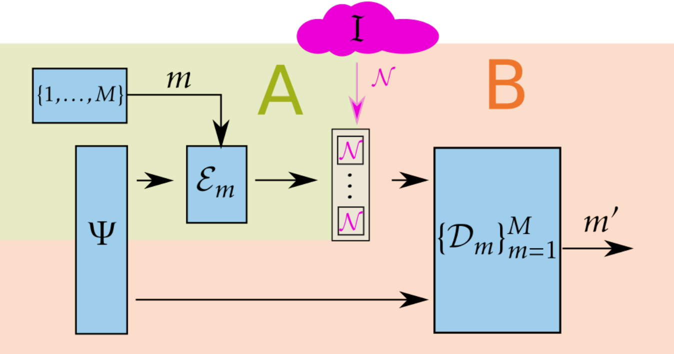
  Channel map for n channel uses

  $$\underbrace{\mathcal{N} \otimes \cdots \otimes \mathcal{N}}_{n-\text{times}},$$

  where $\mathcal{N}$ can be *any* member of $\mathfrak{I}$.

- All coding procedures and protocols must be universal.

Entanglement-assisted message transmission



If the generating c.p.t.p. map is **perfectly known**, it holds

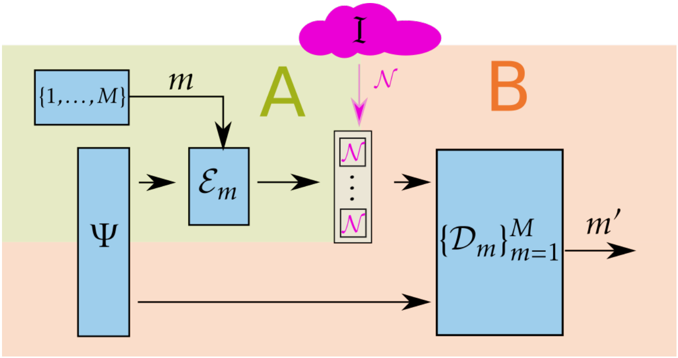
Theorem (Bennett et. al. ’02, Holevo ’02)

For each $\mathcal{N} \in \mathcal{C}(\mathcal{H}_A, \mathcal{H}_B)$ *it holds*

$$C_{EA}(\mathcal{N}) = \sup_{\rho \in \mathcal{S}(\mathcal{H}_A)} I(\rho, \mathcal{N})$$

with $I(\rho, \mathcal{N}) := S(\rho) + S(\mathcal{N}(\rho)) - S(\mathcal{N} \otimes \mathrm{id}(\Psi))$, Ψ *any purification of* ρ.

Entanglement-assisted message transmission

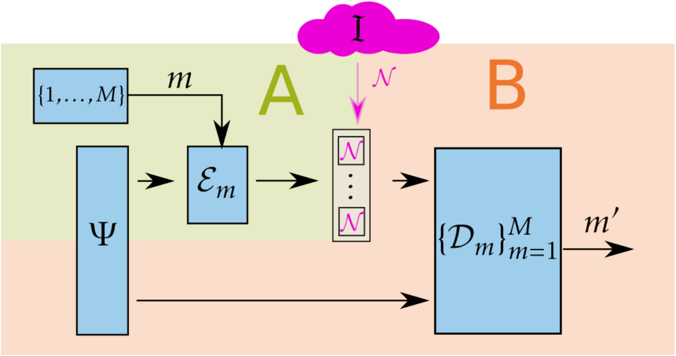


Definition

An (n, M)-*code for entanglement-assisted (EA) message transmission* over $\mathfrak{I}$ is a family $\mathcal{C} := (\Psi, \mathcal{E}_m, D_n)_{m=1}^M$, with

- $\Psi \in \mathcal{S}(\mathcal{K}_A \otimes \mathcal{K}_B)$ a pure quantum state shared by A,B,
- $\mathcal{E}_m \in \mathcal{C}(\mathcal{K}_A, \mathcal{H}_A^{\otimes n})$ for all $m \in [M]$,
- $\{D_m\}_{m=1}^M \subset \mathcal{L}(\mathcal{H}_B^{\otimes n} \otimes \mathcal{K}_B)$ a POVM

Entanglement-assisted message transmission



$$\bar{e}(n,\mathcal{C},\mathfrak{I}) := \sup_{\mathcal{N}\in\mathfrak{I}} \frac{1}{M}\sum_{m=1}^{M} \mathrm{tr}\left\{(\mathbb{1}-D_m)(\mathcal{N}^{\otimes n}\circ\mathcal{E}_m\otimes\mathrm{id}_{\mathcal{K}_B})(\Psi)\right\} \qquad \text{(average error)},$$

$$e(n,\mathcal{C},\mathfrak{I}) := \sup_{\mathcal{N}\in\mathfrak{I}} \max_{m\in[M]} \mathrm{tr}\left\{(\mathbb{1}-D_m)(\mathcal{N}^{\otimes n}\circ\mathcal{E}_m\otimes\mathrm{id}_{\mathcal{K}_B})(\Psi)\right\} \qquad \text{(maximal error)}.$$

Definition

A number $R \geq 0$ is called an *achievable rate for EA message transmission over the compound quantum channel* $\mathfrak{I}$ *under average error criterion,* if there is a sequence $\{\mathcal{C}_n\}_{n=1}^{\infty}$ of (n, M_n)-codes for EA message transmission over $\mathfrak{I}$ which has the properties

1. $\liminf_{n\to\infty} \frac{1}{n} \log M_n \geq R$, and
2. $\lim_{n\to\infty} \overline{e}(n, \mathcal{C}_n, \mathfrak{I}) = 0$

We call

$$\overline{C}_{EA}(\mathfrak{I}) := \sup\{R \geq 0 : \ R \text{ achievable rate for av. error EA message transmission over } \mathfrak{I}\}$$

$$C_{EA}(\mathfrak{I}) := \sup\{R \geq 0 : \ R \text{ achievable rate for max. error EA message transmission over } \mathfrak{I}\}$$

Theorem

Let $\mathfrak{I} \subset \mathcal{C}(\mathcal{H}_A, \mathcal{H}_B)$ *be a set of c.p.t.p. maps. It holds*

$$\overline{C}_{EA}(\mathfrak{I}) = C_{EA}(\mathfrak{I}) \ = \ \sup_{\rho \in \mathcal{S}(\mathcal{H}_A)} \inf_{\mathcal{N} \in \mathfrak{I}} I(\rho, \mathcal{N}).$$

Proof sketch

- Achievability

  - **Ingredient 1**: Good Encodings from **[Hsieh et al. '08]**:

$$\exists \{\mathcal{E}_x\}_{x\in\mathcal{X}}, \Psi\ \forall \mathcal{N} : |\chi(q_*, V_{\mathcal{N}}) - k\cdot I(\mathrm{tr}_B\Psi, \mathcal{N})| \leq 2\dim\mathcal{H}_A \cdot \log(k+1)$$

    with q_* being the equidistribution on $\mathcal{X}$, and $V_{\mathcal{N}}$ the cq-channel

$$x \mapsto \mathcal{N}^{\otimes k} \circ \mathcal{E}_x \otimes \mathrm{id}^{\otimes k}(\Psi^{\otimes k}).$$

  - **Ingredient 2:** Good Codes for compound memoryless classical-quantum channels from **[Boche, Bjelaković '08]**:

    $\exists (n, M) -$ code $\tilde{\mathcal{C}}$ for $\{V_{\mathcal{N}}\}_{\mathcal{N}\in\mathrm{I}}$:

$$R \geq \inf_{\mathcal{N}\in\mathcal{I}} \chi(q_*, V_{\mathcal{N}}) - \delta \qquad \wedge \qquad \inf_{\mathcal{N}\in\mathrm{I}} e(n, \tilde{\mathcal{C}}, V_{\mathcal{N}}^{\otimes n}) \leq 2^{-\sqrt[16]{n}c}$$

- Converse: Standard techniques + Code symmetrization argument.

Strong converse bounds

Strong converse: Even allowing any asymptotical transmission error threshold $\lambda \in (0,1)$ does not allow higher transmission rates.

$$\overline{N}_{EA}(n,\mathfrak{I},\lambda) := \max\{M : \exists (n,M)-\text{EA code } \mathcal{C} \text{ for } \mathfrak{I} \text{ such that } \overline{e}(n,\mathcal{C},\mathfrak{I}) \leq \lambda\}.$$
$$N_{EA}(n,\mathfrak{I},\lambda) := \max\{M : \exists (n,M)-\text{EA code } \mathcal{C} \text{ for } \mathfrak{I} \text{ such that } e(n,\mathcal{C},\mathfrak{I}) \leq \lambda\}.$$

A strong converse holding for average (maximal) transmission error means

$$\forall \lambda \in (0,1): \quad \limsup_{n\to\infty} \frac{1}{n} \log \overline{N}_{EA}(n,\mathfrak{I},\lambda) \leq \sup_{\rho \in \mathcal{S}(\mathcal{H}_A)} \inf_{\mathcal{N}\in\mathfrak{I}} I(\rho,\mathcal{N}), \text{ and} \tag{1}$$
$$\forall \lambda \in (0,1): \quad \limsup_{n\to\infty} \frac{1}{n} \log N_{EA}(n,\mathfrak{I},\lambda) \leq \sup_{\rho \in \mathcal{S}(\mathcal{H}_A)} \inf_{\mathcal{N}\in\mathfrak{I}} I(\rho,\mathcal{N}). \tag{2}$$

Theorem

Neither (1) nor (2) do hold in general.

An Application: EA message transmission over AVQC

The results proven can be used to determine the entanglement assisted classical capacities of arbitrarily varying quantum channels.

Definition

The arbitrarily varying quantum channel generated by a set $\mathfrak{I} := \{\mathcal{N}_s\}_{s \in S} \subset \mathcal{C}(\mathcal{H}_A, \mathcal{H}_B)$ is determined by the possible transmission maps

$$\{\mathcal{N}_{s^n} : s^n \in S^n\}_{n \in \mathbb{N}},$$

where

$$\mathcal{N}_{s^n} := \mathcal{N}_{s_1} \otimes \cdots \otimes \mathcal{N}_{s_n} \qquad (s^n := (s_1, \ldots, s_n) \in S^n).$$

The robustification methods from **[Ahlswede '78]** can be used in combination with the constructed codes for the compound quantum channel generated by $\mathrm{conv}(\mathfrak{I})$.

An Application: EA message transmission over AVQC (cont.)

Theorem

Let $\mathfrak{I} \subset \mathcal{C}(\mathcal{H}_A, \mathcal{H}_B)$ be a set of c.p.t.p. maps. It holds

$$\overline{C}_{EA}^{AV}(\mathfrak{I}) = C_{EA}^{AV}(\mathfrak{I}) = C_{EA}(\mathrm{conv}(\mathfrak{I})) = \sup_{\rho \in \mathcal{S}(\mathcal{H}_A)} \inf_{\mathcal{N} \in \mathrm{conv}(\mathfrak{I})} I(\rho, \mathcal{N}).$$

Using the strong converse for (i.e. $|\mathfrak{I}| = 1$) **[Bennett et. al. '14]**, **[Gupta, Wilde '14]**, we obtain

Corollary

The EA classical capacity of each AVQC $\mathfrak{I}$ obeys a strong converse in case of average and maximal error criterion.

References

Long version:

H. Boche, G. Janßen, S. Kaltenstadler.
"Entanglement-assisted classical capacities of compound and arbitrarily varying quantum channels",
arXiv:1606.09314 (2016).

Related work:

M.Berta, H. Gharibyan, M. Walter.
"Entanglement-assisted capacities of compound quantum channels",
arXiv:1603.02282 (2016).