Phase Retrieval via Structured Modulations In Paley Wiener Spaces

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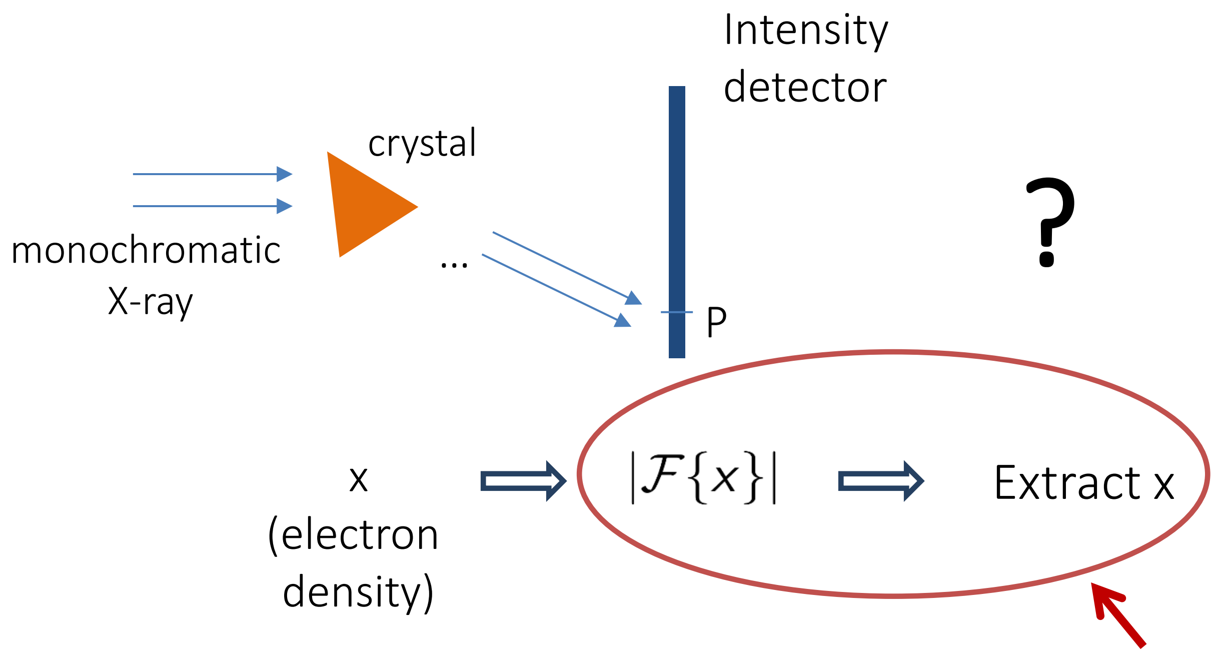
Overview

- Phase Retrieval
  - Applications
  - Finite dimensional
- Main results - ∞ dimensional
  - Measurement setup
  - Choice of modulation and sampling parameters
  - Signal reconstruction – Main Theorem
- Corollary
- Discussion and outlook

Phase Retrieval - Motivation

X-Ray crystallography



Phase retrieval problem

Phase Retrieval - Motivation

Signal processing approach: n-dimensional

Problem formulation	Given $c_m = \lvert\langle a_m, x\rangle\rvert^2$ for $m = 1, \dots, M$ Recover $x \in \mathbb{C}^n$

- Specific and multiple measurements

When is $x \mapsto \langle a_m, x\rangle$ injective?

- Choose $\{a_m\}_m$ as basis
- M = n (e.g. DFT)

When is $x \mapsto \lvert\langle a_m, x\rangle\rvert^2$ injective?

- Choose $\{a_m\}_m$ as MUB or as a 2-uniform M/n tight frame with $M = n^2$
- Generic frame: M ≥ 4n-2

[Balan et al. 2009]

- Efficient reconstruction algorithm: convex optimization techniques

R. Balan, B. G. Bodmann, P. G. Casazza, D. Edidin, „Painless reconstruction from magnitudes of frame coefficients ", J. Fourier Anal. Appl., vol. 15 (Aug. 2009), no. 4, 488-501.

Phase Retrieval - Motivation

n dimensions ⇒ ∞ dimensions ?

Phase Retrieval – ∞-dimensional

Time (spatially) limited signals $x \in \mathcal{L}^2(\mathbb{T})$ with $\mathbb{T} = \left[-\frac{T}{2}, \frac{T}{2}\right]$

⇒ Isomorphic „Fourier Transform“:

$$x \mapsto \hat{x}(z) := \int_{\mathbb{T}} x(t)\,\mathrm{e}^{\mathrm{i}tz}\,\mathrm{d}t\,, \forall z \in \mathbb{C}$$

Theorem (Paley-Wiener)

For all $x \in \mathcal{L}^2(\mathbb{T})$: $\int_{\mathbb{T}} x(t)\,\mathrm{e}^{\mathrm{i}tz}\,\mathrm{d}t =: \hat{x}(z) \in \mathcal{PW}_{T/2}$

⇔ $\hat{x}(z)$ is an entire function with $|\hat{x}(z)| \leq C\mathrm{e}^{\frac{T}{2}|z|}$

and $\int_{\mathbb{R}} |\hat{x}(\omega)|^2 \mathrm{d}\omega < \infty$

Phase Retrieval – ∞-dimensional

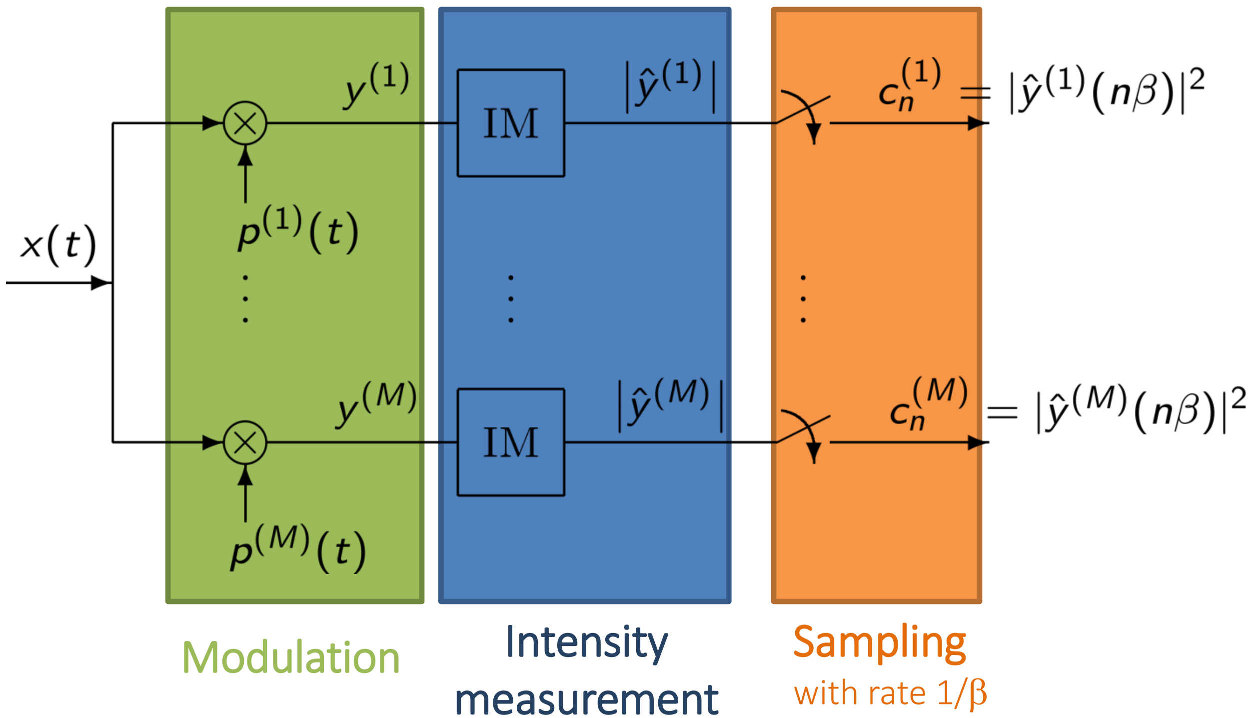
Problem formulation	Given $c_m = |\phi_m(\hat{x})|^2$ for $m = 1, \dots, M$ Recover $\hat{x} \in \mathcal{PW}_{T/2}$

⓪ Find $\phi_m(a_m, \lambda_n)$ such that ⇒ Measurement setup

① $\hat{x}_n \mapsto |\langle a_m, \hat{x}_n\rangle|^2$ injective for $\hat{x}_n \in \mathbb{C}^n$ ⇒ MUB, 2-uniform … a_i

② $\hat{x} \mapsto \{\hat{x}_n\}_{n\in\mathbb{Z}}$ injective ⇒ Sampling points λ_n

Our approach – Measurement setup



$$p^{(m)}(t) := \sum_{k=1}^{K} \overline{\alpha_k^{(m)}} e^{i\tilde{\lambda}_k t}\ , m = 1, \dots M\ , \tilde{\lambda}_k \in \mathbb{C}$$

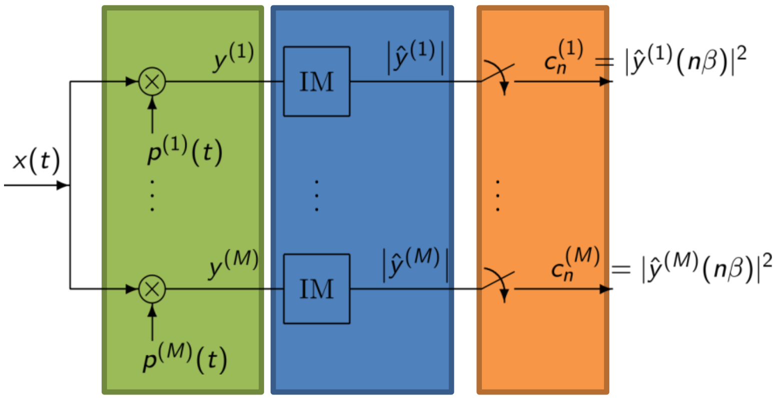
$$\alpha^{(m)} := \begin{pmatrix} \alpha_1^{(m)} \\ \vdots \\ \alpha_K^{(m)} \end{pmatrix} \quad \hat{x}_n := \begin{pmatrix} \hat{x}(n\beta + \tilde{\lambda}_1) \\ \vdots \\ \hat{x}(n\beta + \tilde{\lambda}_K) \end{pmatrix}$$

$$c_n^{(m)} := |\sum_{k=1}^{K} \overline{\alpha_k^{(m)}} \hat{x}(n\beta + \tilde{\lambda}_k)|^2 = |\langle \alpha^{(m)}, \hat{x}_n \rangle_{\mathbb{C}^K}|^2$$

For fix n as in k-dimensional phase retrieval!

Our approach – Measurement setup

$$\alpha^{(m)} := \begin{pmatrix} \alpha_1^{(m)} \\ \vdots \\ \alpha_K^{(m)} \end{pmatrix} \quad \hat{x}_n := \begin{pmatrix} \hat{x}(n\beta + \tilde{\lambda}_1) \\ \vdots \\ \hat{x}(n\beta + \tilde{\lambda}_K) \end{pmatrix}$$



Problem formulation	Given $c_n^{(m)} = \lvert\langle \alpha^{(m)}, \hat{x}_n \rangle\rvert^2$ for $m = 1, \ldots, M$ Recover $\hat{x} \in \mathcal{PW}_{T/2}$

① For each n solve $\{\lvert\langle \alpha^{(m)}, \hat{x}_n \rangle\rvert^2\}_{m=1,\ldots,M} \to \hat{x}_n$ ➡ Choice of vectors $\alpha^{(m)}$?

How big is M?

② $\hat{x}_n \overset{ⓐ}{\to} \{\hat{x}(\lambda_n)\}_{n\in\mathbb{Z}} \overset{ⓑ}{\to} \hat{x}$ ➡ Choice of $\tilde{\lambda}_k$?

Choice of modulator coefficients $\alpha^{(m)}$

① For each n solve

$$\{|\langle\alpha^{(m)}, \hat{x}_n\rangle|^2\}_{m=1,\dots,M} \to \hat{x}_n$$

➡ Choice of vectors $\alpha^{(m)}$?

How big is M?

- Sufficient condition: $\alpha^{(m)}$ is 2-uniform M/K-tight, $M = K^2$
- Instead of $\hat{x}_n$ obtain $Q_{\hat{x}_n} := \hat{x}_n\hat{x}_n^*$

A [Balan et al., 2009]

$$Q_{\hat{x}_n} = \frac{(K+1)}{K}\sum_{m=1}^{M} c_n^{(m)} Q_{\alpha^{(m)}} - \frac{1}{K}\sum_{m=1}^{M} c_n^{(m)} I$$

B [Candes et al., 2013]

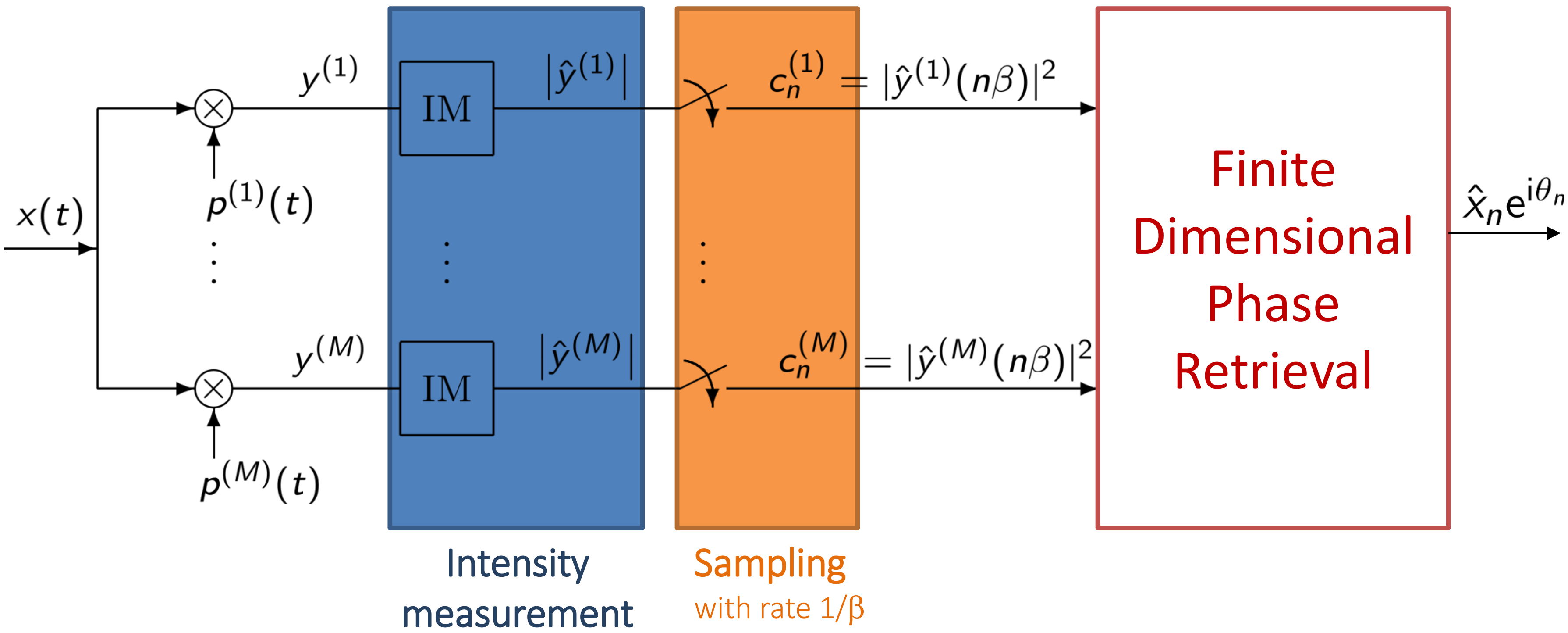
$$\min_x \ \text{trace}(Q_x = xx^*)$$
$$\text{s. t. trace}(\alpha^{(m)}\alpha^{(m)*} Q_x) = c_m \ \forall m = 1, \dots, M$$
$$Q_x \geq 0$$

- Eigenvectors of $Q_{\hat{x}_n} := \hat{x}_n\hat{x}_n^*$: $\hat{x}_n e^{i\theta} \ \forall\theta \in [-\pi, \pi]$

➡ **Finite dimensional signal recovery only up to constant phase**

R. Balan et al., „Painless reconstruction from magnitudes of frame coefficients “, J. Fourier Anal. Appl., vol. 15 (Aug. 2009), no. 4, 488-501.
E. J. Candès, Y. C. Eldar, T. Strohmer, V. Voroninski, „Phase Retrieval via Matrix Completion “, SIAM J. Imaging Sci. vol. 6 (2013), no. 1, 199-255.

Sampling in the complex plane

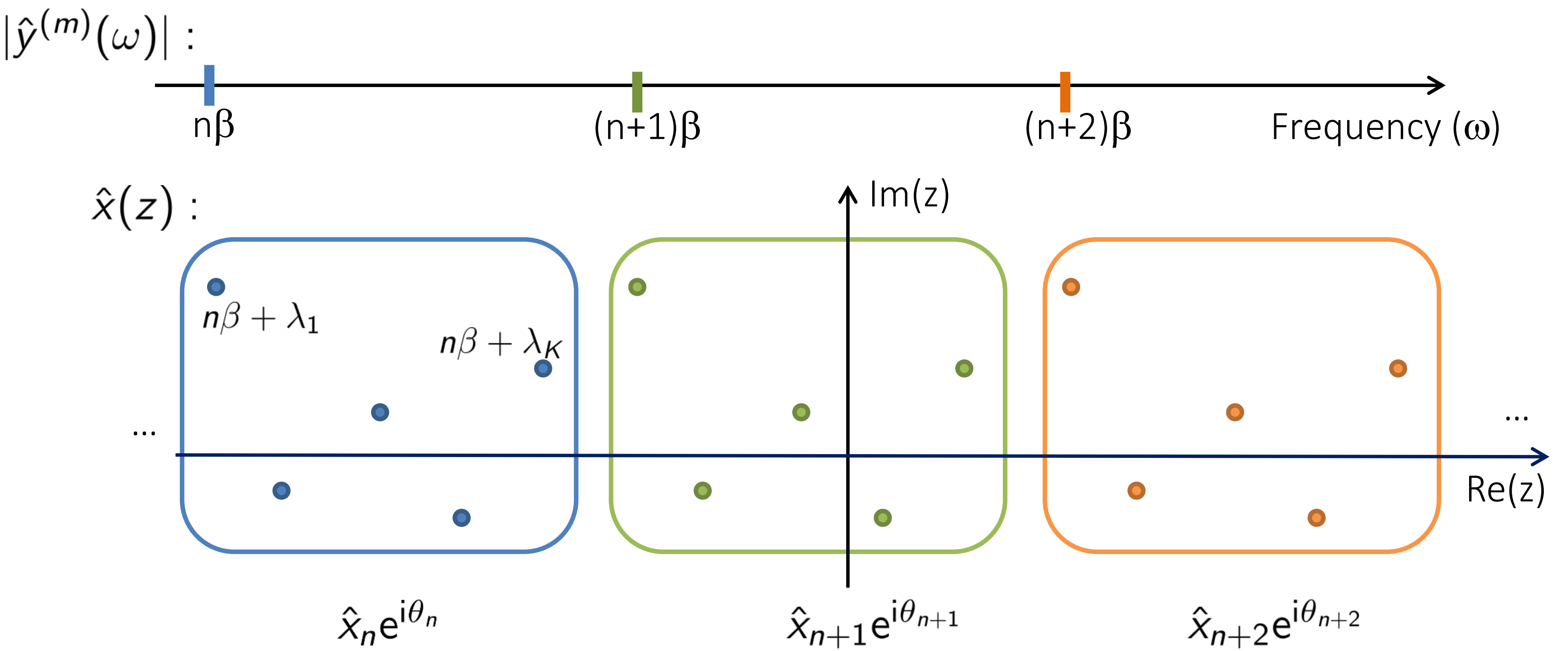


Sampling in the complex plane

$$|\hat{y}^{(m)}(\omega)| := \left|\sum_k \overline{\alpha_k^{(m)}}\, \hat{x}(\omega+\lambda_k)\right|$$

$$\hat{x}_n e^{i\theta_n} := \begin{pmatrix} \hat{x}(n\beta+\lambda_1) \\ \vdots \\ \hat{x}(n\beta+\lambda_K)\end{pmatrix} e^{i\theta_n}$$

$$|\hat{y}^{(m)}(n\beta)|^2 := \left|\sum_k \overline{\alpha_k^{(m)}}\, \hat{x}(n\beta+\lambda_k)\right|^2 = |\langle \alpha^{(m)}, \hat{x}_n\rangle|^2$$

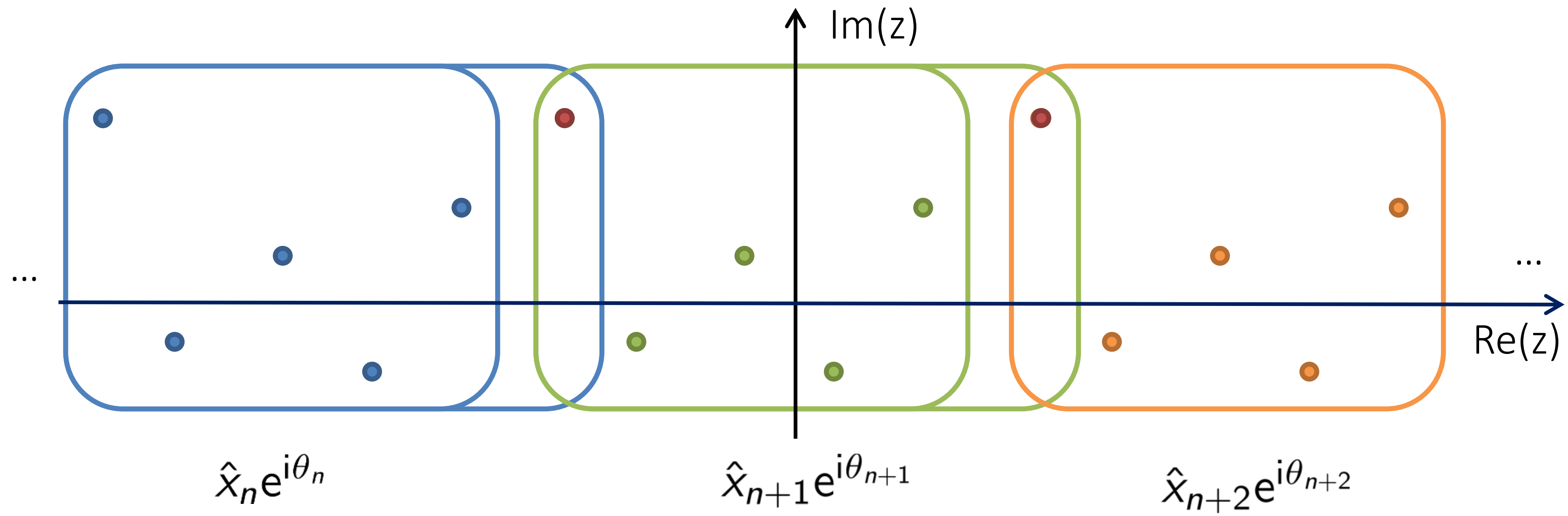


Phase propagation

② $\{\hat{x}(n\beta + \tilde{\lambda}_k)e^{i\theta_n} : n \in \mathbb{Z}, k = 1, \dots, K\} \overset{ⓐ}{\to} \{\hat{x}(\lambda_n)\}_{n\in\mathbb{Z}} \to \hat{x}$

PROBLEM The constant phases are generally different!

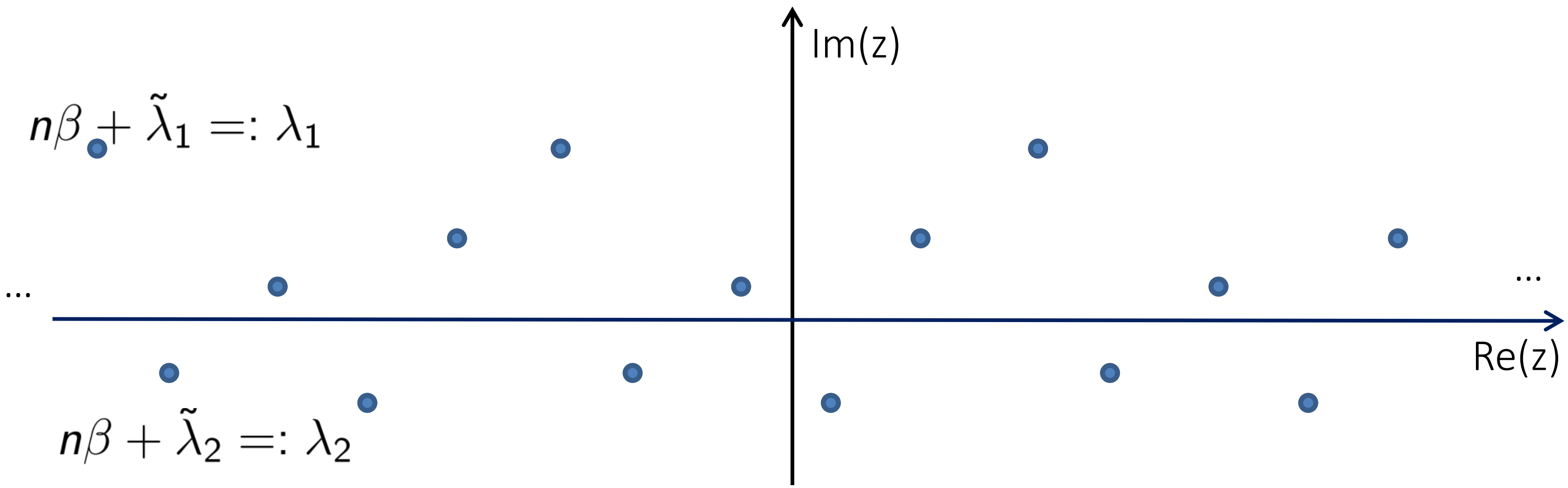
⇒ $\theta_n \overset{!}{=} \theta_{n'} \; \forall n' \neq n \in \mathbb{Z}$



Choice of $\tilde{\lambda}_k$:

- Consecutive finite blocks should have at least one overlap
- $\hat{x} \overset{!}{\neq} 0$ at these overlaps!

Sampling in the complex plane



Relabelling all sampling points $\Lambda = \{\lambda_n\}_{n \in \mathbb{Z}}$

Choice of λ_n

Problem formulation	Given $c_n^{(m)} = |\langle \alpha^{(m)}, \hat{x}_n \rangle|^2$ for $m = 1, \dots, M$ Recover $\hat{x} \in \mathcal{PW}_{T/2}$

① For each n solve

$$\{|\langle \alpha^{(m)}, \hat{x}_n \rangle|^2\}_{m=1,\dots,M} \to \hat{x}_n$$

→ Choice of vectors $\alpha^{(m)}$?

How big is M?

② $\hat{x}_n \xrightarrow{(a)} \{\hat{x}(\lambda_n)\}_{n\in\mathbb{Z}} \xrightarrow{(b)} \hat{x}$

→ Choice of λ_n ?

Choice of λ_n

- Interpolation condition: $\hat{x}_a(\lambda_n) = \hat{x}(\lambda_n)\ \forall n \in \mathbb{Z}$

  Solved by $\hat{x}_a(z) = \sum_{n\in\mathbb{Z}} \hat{x}(\lambda_n)\hat{\psi}_n(z)$

  with Lagrange interpolator $\hat{\psi}_n(z) = \prod_{m\neq n} \frac{z-\lambda_m}{\lambda_n-\lambda_m}$

- Choice of λ_n for perfect reconstruction such that

  $\hat{x}_a(z) = \hat{x}(z)\ \forall z \in \mathbb{C}$

  ⟺ $\hat{x}_a(\lambda_n) = \hat{x}(\lambda_n)\ \forall n \in \mathbb{Z}$ solved uniquely

  ($\{\lambda_n\}_{n\in\mathbb{Z}}$ is a complete interpolating sequence)

Nice result for Paley Wiener spaces

Choice of λ_n

For $\hat{x} \in \mathcal{PW}_{T/2}$

Theorem (Complete Interpolating Sequence)

Let $\hat{x} \in \mathcal{PW}_{T/2}$ and $\{c_n\}_{n\in\mathbb{Z}} \in \ell^2$.
The interpolation problem $\hat{x}(\lambda_n) = c_n \ \forall n \in \mathbb{Z}$ has a unique solution
($\{\lambda_n\}_{n\in\mathbb{Z}}$ is a complete interpolating sequence)
$\Longleftrightarrow$ $\{e^{i\lambda_n t}\}_{n\in\mathbb{Z}}$ is a Riesz basis for $\mathcal{L}^2(\mathbb{T})$

One example: Zeros of sine-type functions of type ≥T/2

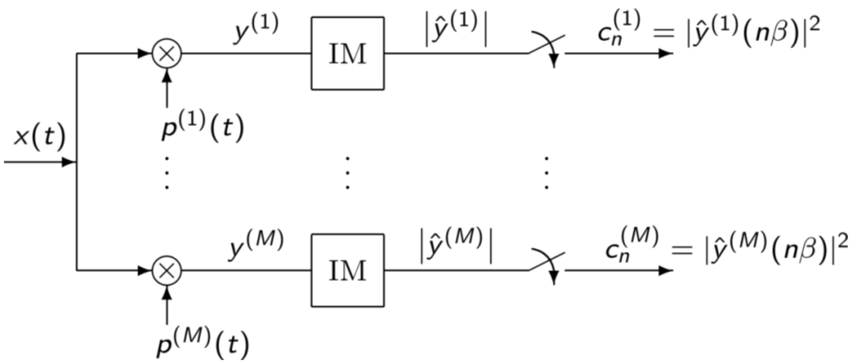
Definition (Sine-type functions)

An entire function of exponential type T/2 $S(z) = P.V.\prod_{n\in\mathbb{Z}}(1 - \frac{z}{\lambda_n})$ is called sine-type of type T/2 if it has simple and separated zeros λ_n, for which there exist A, B, H s.t.

$$A\,e^{\frac{T}{2}|\eta|} \le |S(\xi + i\eta)| \le B\,e^{\frac{T}{2}|\eta|}, \quad \text{for } |\eta| > H .$$

B. Y. Levin, „Lectures on entire functions“, American Mathematical Society, Providence, RI, 1997.

Main Theorem



$$p^{(m)}(t) := \sum_{k=1}^{K} \overline{\alpha_k^{(m)}} e^{i\tilde{\lambda}_k t}, \; m = 1, \dots M, \; \tilde{\lambda}_k \in \mathbb{C}$$

$$\alpha^{(m)} := \begin{pmatrix} \alpha_1^{(m)} \\ \vdots \\ \alpha_K^{(m)} \end{pmatrix} \qquad \hat{x}_n := \begin{pmatrix} \hat{x}(n\beta + \tilde{\lambda}_1) \\ \vdots \\ \hat{x}(n\beta + \tilde{\lambda}_K) \end{pmatrix}$$

Main Theorem

Given the measurement setup and $p^{(m)}$ as above.
Then $\hat{x} \in \mathcal{PW}_{T/2}$ can be perfectly recovered from $c_n^{(m)} = |\langle \alpha^{(m)}, \hat{x}_n \rangle|^2$ for $m = 1, \dots, M$ whenever

1. $\alpha^{(m)}$ constitute a 2-uniform M/K tight frame with $M = K^2$ ⟶ $\hat{x}_n \mapsto c_n^{(m)}$ injective

2a. $\tilde{\lambda}_k$ s.t. consecutive blocks have at least one overlap with $x \neq 0$ ⟶ $\hat{x}_n \mapsto \{\hat{x}(\lambda_n)\}_{n\in\mathbb{Z}}$ well-defined

2b. $\{\lambda_n\}_{n\in\mathbb{Z}}$ is an complete Interpolating sequence ⟶ $\{\hat{x}(\lambda_n)\}_{n\in\mathbb{Z}} \mapsto \hat{x}$ unique

Corollary

How can we ensure that $\hat{x}(\lambda) \neq 0$ at the overlapping sampling points?

Corollary

Let the maximal energy of x be known $\|x\|_{\mathcal{L}^2(\mathbb{T})} \leq W_0$.
By the Plancherel Pólya Theorem

$$\exists M \; \forall x: \; |\hat{x}(z)| \leq M W_0 e^{\frac{T}{2}|\eta|}$$

Now consider the following function as our signal in the Fourier domain with T'≥ T

$$\hat{v}(z) = D\cos(\tfrac{T'}{2}z) - \hat{x}(z)$$

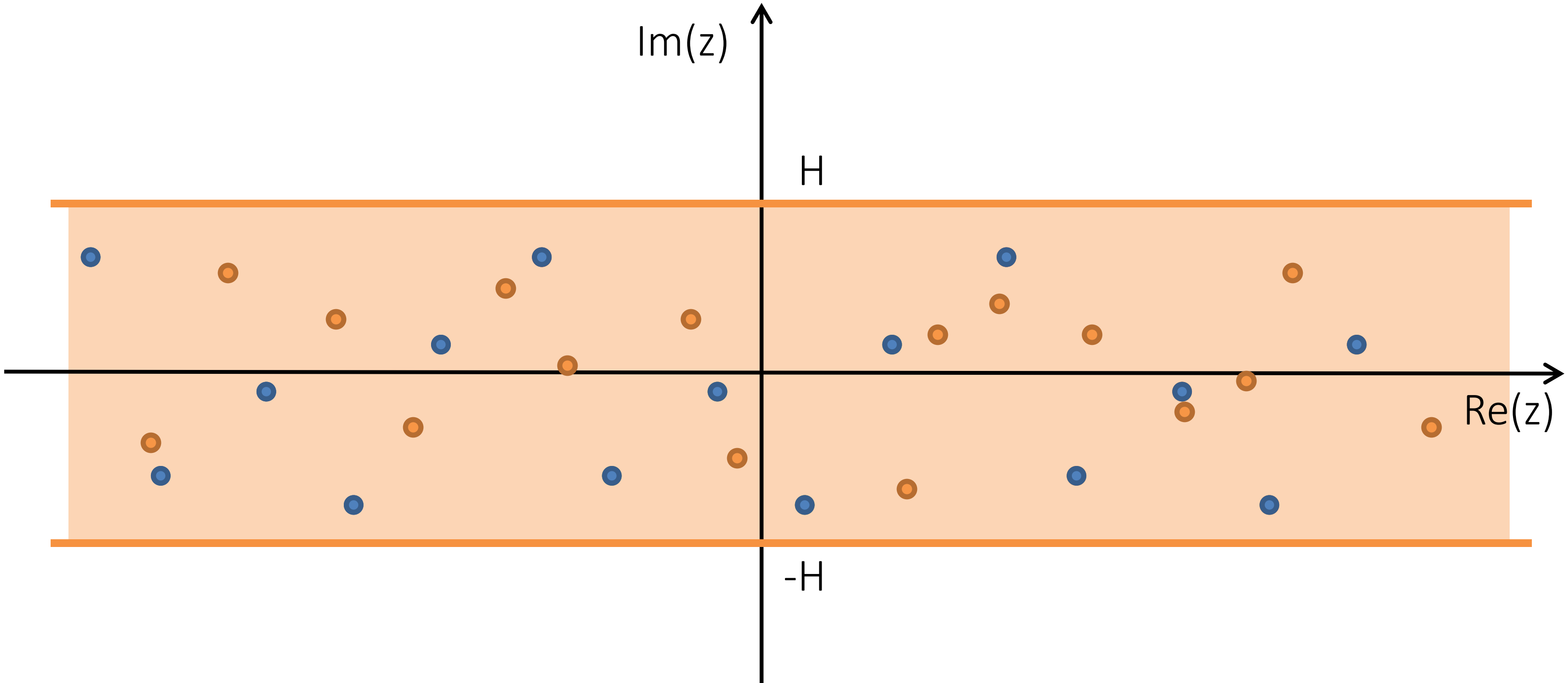
Then the zeros are concentrated in a strip $|\eta| > H$, such that given H, one can construct a complete interpolating sequence which enables perfect reconstruction from the samples $c_n^{(m)} = |\langle \alpha^{(m)}, \hat{v}_n \rangle|^2$ up to a constant phase.

Example construction of feasible λ_n ⟶

Proof- Zeros of sine-type functions

Theorem (Levin)

By shifting zeros of a sine-type function (e.g. such that $|k| > H$ for all k), the corresponding function $S(z) = P.V. \prod_{n\in\mathbb{Z}} (1 - \frac{z}{\lambda_n})$ remains to be a sine- type function, i.e. the resulting zeros are still a complete interpolating sequence.



B. Y. Levin, „Lectures on entire functions“, American Mathematical Society, Providence, RI, 1997.

Summary and outlook

- Perfect signal reconstruction from magnitude measurements of the Fourier Transform for $x \in \mathcal{L}^2(\mathbb{T}) \leftrightarrow \hat{x} \in \mathcal{PW}_{T/2}$ using the special structure of the modulators

$$p^{(m)}(t) := \sum_{k=1}^{K} \overline{\alpha_k^{(m)} e^{i\lambda_k t}} \; , m = 1, \dots M \; , \lambda_k \in \mathbb{C}$$

- Overlap condition unnecessary when maximal energy $\|x\|_{\mathcal{L}^2(\mathbb{T})} \leq W_0$ of the signal is given

- For K = 2 and a =1, we obtain the minimal overall sampling rate $R = 4 R_{\mathrm{Ny}}$ where R_{Ny} is the Nyquist rate.
  - Compare to n-dimensional case: 4n-2 and 4n-4 (sufficient)

Remaining questions:

➡ How does robustness compare to existing algorithms?

➡ How can we extend this formalism to continuous functions on a 2-dimensional case?