# Stable Embedding of Sparse Convolutions

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**L**inear **T**ime **I**nvariant System: $\mathbf{x} * \mathbf{y}$

| When is the LTI System stable? ($\Rightarrow$ injectivity) |
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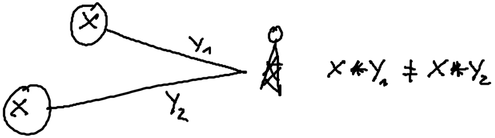

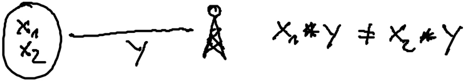

Possible if convolution has the **R**estricted **N**orm **M**ultiplicativity **P**roperty, i.e.

$$\alpha \|\mathbf{x}\| \|\mathbf{y}\| \leq \|\mathbf{x} * \mathbf{y}\| \leq \beta \|\mathbf{x}\| \|\mathbf{y}\|$$

for some $\alpha, \beta > 0$.

Let $G = (G, +)$ be a *torsion free*, discrete, abelian group.

- Set $G$ with group action $+$ (addition)
- $g_1, g_2 \in G$: $g_1 + g_2 = g_2 + g_2 \in G$
- Exists identity $0 \in G$
- Exists inverse element $-g$ for each $g \in G$ s.t. $g - g = 0$.
- $ng \neq 0$ for all $g \in G \setminus \{0\}, n \in \mathbb{Z} \setminus \{0\}$

Examples

- $\mathbb{Z}$
- $\mathbb{Q}$

We define for an integer $s \leq |G|$ the set of $s$–sparse sequences:

$$\ell_s^2(G) := \left\{ \mathbf{x} : G \to \mathbb{C} \;\middle|\; \|\mathbf{x}\|^2 := \sum_{i \in G} |x_i|^2 < \infty, |\operatorname{supp} \mathbf{x}| \leq s \right\}. \tag{1}$$

For $\mathbf{x} \in \ell_s^2(G)$ and $\mathbf{y} \in \ell_f^2(G)$ its *convolution* is given element wise as:

$$(\mathbf{x} * \mathbf{y})_j = \sum_{i \in G} x_i y_{j-i} \quad \text{for all} \quad j \in G. \tag{2}$$

# Hermitian Toeplitz Matrices

- For $A \in [0, n-1]_s = \{A \subset [0, n-1] \mid |A| = s\}$ we define the projection

$$\mathbf{P}_A : \mathbb{C}^n \to \mathbb{C}^s. \tag{3}$$

- From any $n \times n$–matrix $\mathbf{B}$ we get a $s \times s$ principal submatrix

$$\mathbf{B}^A = \mathbf{P}_A \mathbf{B} \mathbf{P}_A^* \tag{4}$$

- Further, we denote by $\mathbf{B}_\mathbf{a}$ an $n \times n$– *Hermitian Toeplitz matrix* generated by $\mathbf{a} \in \Sigma_k^n$ with *symbol* for $\omega \in [0, 2\pi)$

$$b(\mathbf{a}, \omega) = \sum_{k=1-n}^{n-1} b_k(\mathbf{a}) e^{\imath k\omega} = 1 + \sum_{k=1}^{n-1} (\mu_k \cos(k\omega) + \nu_k \sin(k\omega)) \tag{5}$$

with

$$\mu_k := 2\Re(b_k(\mathbf{a})), \quad \nu_k := -2\Im(b_k(\mathbf{a})) \quad \text{and} \quad b_k(\mathbf{a}) := \sum_{i=0}^{n-1} a_i a_{i+k}$$

FEJÉR-RIESZ factorization:
*non-negative trigonometric polynomial of order not larger than n.*

**Theorem ([W. & Jung,'13])**

*Let s and f be natural numbers and G a torsion-free, discrete, abelian group. Then there exist constants* $0 < \alpha(s,f) \leq \beta(s,f) < \infty$ *depending solely on s and f, s.t. for all* $\mathbf{x} \in \ell_s^2(G)$ *and* $\mathbf{y} \in \ell_f^2(G)$ *it holds:*

$$\alpha(s,f) \|\mathbf{x}\| \|\mathbf{y}\| \leq \|\mathbf{x} * \mathbf{y}\| \leq \beta(s,f) \|\mathbf{x}\| \|\mathbf{y}\| . \tag{6}$$

*Moreover, we have* $\beta^2(s,f) = \min\{s,f\}$ *and with* $n = \lfloor 2^{2(s+f-2)\log(s+f-2)} \rfloor + 1$ *:*

$$\alpha^2(s,f) = \min\left\{ \min_{\substack{\tilde{\mathbf{y}} \in \Sigma_f^n, \|\tilde{\mathbf{y}}\|=1 \\ I \in [0,n-1]_s}} \lambda(\mathbf{B}_{\tilde{\mathbf{y}}}^I), \min_{\substack{\tilde{\mathbf{x}} \in \Sigma_s^n, \|\tilde{\mathbf{x}}\|=1 \\ J \in [0,n-1]_f}} \lambda(\mathbf{B}_{\tilde{\mathbf{x}}}^J) \right\}, \tag{7}$$

*which is a decreasing sequence in s and f. If* $\beta(s,f) = 1$ *we get equality with* $\alpha(s,f) = 1$.

# Proof

- Upper bound trivial, CAUCHY-SCHWARZ inequality
- Lower bound, given by a ***NP**-hard bi-quadratic optimization problem*:

$$\alpha(s,f) := \min_{\substack{\mathbf{x}\in\ell_s^2(G)\\ \mathbf{y}\in\ell_f^2(G)}} \frac{\|\mathbf{x} * \mathbf{y}\|}{\|\mathbf{x}\|\,\|\mathbf{y}\|} = \min_{\substack{\mathbf{x}\in\ell_s^2(G),\mathbf{y}\in\ell_f^2(G)\\ \|\mathbf{x}\|=\|\mathbf{y}\|=1}} \|\mathbf{x} * \mathbf{y}\| \tag{8}$$

- For each $(\mathbf{x},\mathbf{y})$ it exists $I, J \subset G$, s.t. $\operatorname{supp}\mathbf{x} \subseteq I, \operatorname{supp}\mathbf{y} \subseteq J$ and $|I| = s, |J| = f$.
- Let $I = \{i_0, \ldots, i_{s-1}\}$ and $J = \{j_0, \ldots, j_{f-1}\}$ then $(\mathbf{x},\mathbf{y})$ can be represented by $\mathbf{u} \in \mathbb{C}^s$ and $\mathbf{v} \in \mathbb{C}^f$ component-wise as:

$$x_i = \sum_{\theta=0}^{s-1} u_\theta \delta_{i,i_\theta} \quad , \quad y_j = \sum_{\gamma=0}^{f-1} v_\gamma \delta_{j,j_\gamma} \quad \text{for all} \quad i,j \in G \tag{9}$$

Ok, let's start!

$$\|\mathbf{x}*\mathbf{y}\|^2 = \sum_{j\in G}\left|\sum_{i\in G} x_i y_{j-i}\right|^2 = \sum_{j\in G}\left|\sum_{i\in G}\left(\sum_{\theta=0}^{s-1} u_\theta \delta_{i,i_\theta}\right)\left(\sum_{\gamma=0}^{f-1} v_\gamma \delta_{j-i,j_\gamma}\right)\right|^2$$

$$= \sum_{j\in G}\left|\sum_{\theta=0}^{s-1}\sum_{\gamma=0}^{f-1}\left(\sum_{i\in G} u_\theta \delta_{i,i_\theta} v_\gamma \delta_{j,j_\gamma+i}\right)\right|^2$$

$$(i \to i+i_0) \to = \sum_{j\in G}\left|\sum_{\theta}\sum_{\gamma}\left(\sum_{i\in G} u_\theta \delta_{i+i_0,i_\theta} v_\gamma \delta_{j,j_\gamma+i+i_0}\right)\right|^2$$

$$(j \to j+i_0+j_0) \to = \sum_{j\in G}\left|\sum_{\theta}\sum_{\gamma}\left(\sum_{i\in G} u_\theta \delta_{i,i_\theta-i_0} v_\gamma \delta_{j,j_\gamma-j_0+i}\right)\right|^2$$

Therefore we can allways assume for the support $I, J \subset G$ that $i_0 = j_0 = 0$.

$$= \sum_{j\in G}\left|\sum_{\theta}\sum_{\gamma}\left(u_\theta v_\gamma \delta_{j,j_\gamma+i_\theta}\right)\right|^2 = \sum_{j\in G}\sum_{\theta,\theta'}\sum_{\gamma,\gamma'} u_\theta \overline{u_{\theta'}} v_\gamma \overline{v_{\gamma'}} \delta_{j,j_\gamma+i_\theta}\delta_{j,j_{\gamma'}+i_{\theta'}}$$

$$= \sum_{\theta,\theta'}\sum_{\gamma,\gamma'} u_\theta \overline{u_{\theta'}} v_\gamma \overline{v_{\gamma'}} \underbrace{\delta_{j_\gamma+i_\theta,j_{\gamma'}+i_{\theta'}}}_{\text{Fourth order tensor } \mathcal{A}_{I,J}} =: b_{I,J}(\mathbf{u},\mathbf{v})$$

$$\mathcal{A}_{I,J} = (\delta_{i_\theta + j_\gamma, i_{\theta'} + j_{\gamma'}}) \in \{0,1\}^{sf \times sf}$$
$$\Downarrow$$
$$\min_{\substack{I,J \subset G \\ |I|=s, |J|=f}} \min_{\substack{\mathbf{u} \in \mathbb{C}^s, \mathbf{v} \in \mathbb{C}^f \\ \|\mathbf{u}\| = \|\mathbf{v}\| = 1}} b_{I,J}(\mathbf{u}, \mathbf{v})$$

- Each $(I, J)$ generates an NP-hard problem [Ling et al., '09]
- Find the minimum over all these NP hard problems (countable many)!

Wow, is that maybe somehow easier? Are there finite many problems?

$(sf)^2$ elements each equal 0 or 1 $\Rightarrow$ Not more than $2^{(sf)^2}$ problems

**Problem:** How can the additive structure be represented by finite many numbers?

Let us consider a mapping $\phi$ of the indices. For $I, J \subset G$ with $0 \in I \cap J$ an injective map:

$$\phi : I + J \to \mathbb{Z} \tag{10}$$

which additional satisfies (preserves additive structure of the indices):

$$\forall i, i' \in I, j, j' \in J : i + j = i' + j' \underset{\Leftrightarrow}{\Rightarrow} \phi(i) + \phi(j) = \phi(i') + \phi(j') \tag{11}$$

is called a *Freiman homomorphism on $I, J$* resp. a *Freiman isomorphism*

| |
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| Show for any $I, J \subset G$ with $\lvert I\rvert = s, \lvert J\rvert = f$ the existence of a Freiman isomorphism $\phi$ such that $\phi(I), \phi(J) \subset [0, n-1] = \{0, 1, \ldots, n-1\}$ with $n = n(s, f)$. |

Indeed, for $A = I \cup J$ with $\lvert A\rvert \leq s + f - 1$ one may find:

**Conjecture ([Konyagin and Lev, '00])**

*Let $A \subset \mathbb{Z}$ with $\lvert A\rvert = m$ then there exist an Freiman isomorphism $\phi$ s.t. $\phi(A) \subset [0, 2^{m-2}]$.*

Still unsolved!

## Lemma ([Grynkiewicz, '13])

*Let $G$ be a torsion-free additive abelian group and $A \subset G$ be finite sets containing zero with $m := |A|$ and Freiman dimension $d = \dim^+(A+A)$. Then there exists an injective Freiman homomorphism:*

$$\phi : A + A \to \mathbb{Z}$$

*such that*

$$diam(\phi(A)) \leq d!^2 \left(\frac{3}{2}\right)^{d-1} 2^{m-2} + \frac{3^{d-1}-1}{2}.$$

- $A = I \cup J$ with $\operatorname{diam}(\Phi(A)) = \max(\phi(A)) - \min(\phi(A))$.
- Using a result of [Tao & Vu, '06] to find $d \leq |A| - 2 \leq m - 2$
- $\phi$ bijective Freiman homomorphism on $A + A \Rightarrow$ Freiman isomorphism on $A$
- Setting $\phi' = \phi - c^*$ (still Freiman) with

$$c^* := \min_{a \in I \cup J} \phi(a). \tag{12}$$

- $\tilde{I} := \phi'(I)$ and $\tilde{J} := \phi'(J)$ with $|\tilde{I} \cup \tilde{J}| \leq s + f - 1$
- Using some log estimates gives finally

$$\operatorname{diam}(\phi(A)) < \lfloor 2^{2(s+f-2)\log(s+f-2)} \rfloor + 1 =: n \tag{13}$$

- Hence we can represent the addition by subsets $0 \in \tilde{I} \cup \tilde{J} \subset [0, n-1]$.

$$b_{I,J}(\mathbf{u},\mathbf{v}) = \sum_{\theta,\theta'} \sum_{\gamma,\gamma'} u_\theta \overline{u_{\theta'}} v_\gamma \overline{v_{\gamma'}} \delta_{\tilde{i}_\theta + \tilde{j}_\gamma, \tilde{j}_{\gamma'} + \tilde{i}_{\theta'}} . \tag{14}$$

Define **new** embedding of $\mathbf{u}, \mathbf{v}$ into $\mathbb{C}^n$ by:

$$\tilde{x}_i = \sum_{\theta=0}^{s-1} u_\theta \delta_{i,\tilde{i}_\theta} \ , \ \tilde{y}_j = \sum_{\gamma=0}^{f-1} v_\gamma \delta_{j,\tilde{j}_\gamma} \quad \text{for all} \quad i,j \in [0, n-1]. \tag{15}$$

which implies the projection identities

$$u_\theta = \sum_{i=0}^{n-1} \tilde{x}_i \delta_{i,\tilde{i}_\theta} \quad , \quad v_\gamma = \sum_{j=0}^{n-1} \tilde{y}_j \delta_{j,\tilde{j}_\gamma} , \tag{16}$$

And going backwards, i.e.

$$b_{I,J}(\mathbf{u},\mathbf{v}) = \sum_{\theta,\theta'} \sum_{i,i'=0}^{n-1} u_\theta \overline{u_{\theta'}} \delta_{i,\tilde{i}_\theta} \delta_{i',\tilde{i}_{\theta'}} \sum_{\gamma,\gamma'} \sum_{j,j'=0}^{n-1} v_\gamma \overline{v_{\gamma'}} \delta_{j,\tilde{j}_\gamma} \delta_{j',\tilde{j}_{\gamma'}} \delta_{\tilde{j}_\gamma + (\tilde{i}_\theta - \tilde{i}_{\theta'}),\tilde{j}_{\gamma'}} \tag{17}$$

$$(15) \rightarrow = \sum_{i,i'=0}^{n-1} \tilde{x}_i \overline{\tilde{x}_{i'}} \sum_{j,j'=0}^{n-1} \tilde{y}_j \overline{\tilde{y}_{j'}} \delta_{j+(i-i'),j'} \tag{18}$$

$$= \sum_{i,i'=0}^{n-1} \tilde{x}_i \overline{\tilde{x}_{i'}} \underbrace{\sum_{j=0}^{n-1} \tilde{y}_j \overline{\tilde{y}_{j+(i-i')}}}_{=:\overline{(\mathbf{B}_{\tilde{\mathbf{y}}})_{i',i}}} = \langle \tilde{\mathbf{x}}, \mathbf{B}_{\tilde{\mathbf{y}}} \tilde{\mathbf{x}} \rangle \tag{19}$$

- $\mathbf{B}_{\tilde{\mathbf{y}}}$ is a $n \times n$ Hermitian Toeplitz matrix with first row $(\mathbf{B}_{\tilde{\mathbf{y}}})_{0,k} = \sum_{j=0}^{n-k} \overline{\tilde{y}_j} \tilde{y}_{j+k} =: b_k(\tilde{\mathbf{y}})$ resp. first column $(\mathbf{B}_{\tilde{\mathbf{y}}})_{k,0} =: b_{-k}$ for $k \in [0, n-1]$ and *symbol* $b(\tilde{\mathbf{y}}, \omega)$ given by (5), see e.g. [Böttcher & Grudsky, ’05]
- $b(\tilde{\mathbf{y}}, \omega)$ is *normalized non-negative trigonometric polynomial of order* $n-1$.
- For fixed $\tilde{\mathbf{y}} \in \mathbb{C}^n$: smallest eigenvalue of $\mathbf{B}_{\tilde{\mathbf{y}}}$, quadratic optimization (SDP) Problem:

$$\lambda(\mathbf{B}_{\tilde{\mathbf{y}}}) := \min_{\tilde{\mathbf{x}} \in \mathbb{C}^n, \|\tilde{\mathbf{x}}\|=1} \langle \tilde{\mathbf{x}}, \mathbf{B}_{\tilde{\mathbf{y}}} \tilde{\mathbf{x}} \rangle . \tag{20}$$

# Eigenvalue Problem

$0 \leq \min_\omega b(\tilde{\mathbf{y}}, \omega) \Rightarrow$ By the spectral theory of Toeplitz matrices we then have $\lambda(\mathbf{B}_{\tilde{\mathbf{y}}}) > 0$.
Hence $\mathbf{B}_{\tilde{\mathbf{y}}}$ is invertible and the *determinant* $\det(\mathbf{B}_{\tilde{\mathbf{y}}}) \neq 0$.
Using:

$$\frac{1}{\lambda(\mathbf{B}_{\tilde{\mathbf{y}}})} = \left\| \mathbf{B}_{\tilde{\mathbf{y}}}^{-1} \right\| \tag{21}$$

we can estimate the smallest eigenvalue (singular value) by the determinant as:

$$\lambda(\mathbf{B}_{\tilde{\mathbf{y}}}) \geq |\det(\mathbf{B}_{\tilde{\mathbf{y}}})| \frac{1}{\sqrt{n}(\sum_k |b_k(\tilde{\mathbf{y}})|^2)^{(n-1)/2}}. \tag{22}$$

The $\ell^2$–norm of the sequence $b_k(\tilde{\mathbf{y}})$ can be upper bounded for $n > 1$ by the CAUCHY-SCHWARZ inequality (instead one may also utilize the upper bound of the theorem):

$$\sum_k |b_k(\tilde{\mathbf{y}})|^2 \le 1 + 2\sum_{k=1}^{n-1} |\sum_{j=0}^{n-1} \tilde{y}_j \overline{\tilde{y}}_{j+k}|^2 \le 1 + 2\sum_{k=1}^{n-1} \|\tilde{\mathbf{y}}\|^4 = 1 + 2(n-1) < 2n, \tag{23}$$

which is independent of $\tilde{\mathbf{y}} \in \mathbb{C}^n$ with $\|\tilde{\mathbf{y}}\| = 1$!
Since the determinant is a continuous function in $\tilde{\mathbf{y}}$ over a compact set, the minimum is attained and is denoted by $0 < d_n := \min_{\tilde{\mathbf{y}}} |\det(\mathbf{B}_{\tilde{\mathbf{y}}})|$. Note, that $d_n$ is a decreasing sequence, since we extend the minimum to a larger set by increasing $n$. Hence we get:

$$\min_{\tilde{\mathbf{y}} \in \mathbb{C}^n, \|\tilde{\mathbf{y}}\|=1} \left( |\det(\mathbf{B}_{\tilde{\mathbf{y}}})| \frac{1}{\sqrt{n}(2n)^{(n-1)/2}} \right) = \frac{\sqrt{2}}{(2n)^{n/2}} d_n. \tag{24}$$

This is a valid lower bound by (22) for the smallest eigenvalue of all $\mathbf{B}_{\tilde{\mathbf{y}}}$. Hence we have

$$\min_{\tilde{\mathbf{y}} \in \mathbb{C}^n, \|\tilde{\mathbf{y}}\|=1} \lambda(\mathbf{B}_{\tilde{\mathbf{y}}}) > \sqrt{2}(2n)^{-\frac{n}{2}} d_n > 0. \tag{25}$$

Now, bringing the support back into play, we see that $\tilde{\mathbf{x}}$ and $\tilde{\mathbf{y}}$ are fully realized by the Freiman isomorphism as $\tilde{I} = \phi'(I), \tilde{J} = \phi'(J)$, where $\tilde{\mathbf{x}}$ cuts out (in a symmetrical way) for a fixed $\tilde{\mathbf{y}} \in \mathbb{C}^n$ an $s \times s$ Hermitian matrix $\mathbf{B}_{\tilde{\mathbf{y}}}^{\tilde{I}} = \mathbf{P}_{\tilde{I}} \mathbf{B}_{\tilde{\mathbf{y}}} \mathbf{P}_{\tilde{I}}^*$ (principal submatrix, actually also Toeplitz) given by the green elements (here we have re-ordered $I$ such that $\tilde{I}$ is ordered)

$$\mathbf{B}_{\tilde{\mathbf{y}}} := \begin{pmatrix} b_0 & \cdots & b_{\tilde{i}_0} & \cdots & b_{\tilde{i}_1} & \cdots & b_{\tilde{i}_{s-1}} & \cdots & b_{n-1} \\ \vdots & \ddots & \vdots & & \vdots & & \vdots & & \vdots \\ b_{\tilde{i}_0} & \cdots & b_{\tilde{i}_0-\tilde{i}_0} & \cdots & b_{\tilde{i}_1-\tilde{i}_0} & \cdots & b_{\tilde{i}_{s-1}-\tilde{i}_0} & \cdots & b_{n-1-\tilde{i}_0} \\ \vdots & & \vdots & \ddots & \vdots & & \vdots & & \vdots \\ b_{\tilde{i}_1} & \cdots & b_{\tilde{i}_1-\tilde{i}_0} & \cdots & b_{\tilde{i}_1-\tilde{i}_1} & \cdots & b_{\tilde{i}_{s-1}-\tilde{i}_1} & \cdots & b_{n-\tilde{i}_1} \\ \vdots & & \vdots & & \vdots & \ddots & \vdots & & \vdots \\ b_{\tilde{i}_{s-1}} & \cdots & b_{\tilde{i}_{s-1}-\tilde{i}_0} & \cdots & b_{\tilde{i}_1-\tilde{i}_{s-1}} & \cdots & b_{\tilde{i}_{s-1}-\tilde{i}_{s-1}} & \cdots & b_{n-\tilde{i}_{s-1}} \\ \vdots & & \vdots & & \vdots & & \vdots & \ddots & \vdots \\ b_{n-1} & \cdots & b_{n-1-\tilde{i}_0} & \cdots & b_{n-1-\tilde{i}_1} & \cdots & b_{n-1-\tilde{i}_{s-1}} & \cdots & b_0 \end{pmatrix}.$$

Minimizing over all $\mathbf{u} \in \mathbb{C}^s$ we have by CAUCHY's Interlacing Theorem, see e.g. [6, Prop.9.19], for all $s \le n \in \mathbb{N}$

$$\lambda(\mathbf{B}_{\tilde{\mathbf{y}}}^{\tilde{I}}) \ge \lambda(\mathbf{B}_{\tilde{\mathbf{y}}}) > 0 \quad , \quad \tilde{\mathbf{y}} \in \mathbb{C}^n, \tilde{I} \in [n]_s. \tag{26}$$

Hence, this also holds for $\tilde{\mathbf{y}} \in \Sigma_f^n$ and we get for our problem in (8)

$$\begin{aligned}\alpha^2(s,f) &= \min_{\substack{\mathbf{x}\in\ell_s^2,\mathbf{y}\in\ell_f^2\\ \|\mathbf{x}\|=\|\mathbf{y}\|=1}} \|\mathbf{x} * \mathbf{y}\| \ge \min\left\{ \min_{\tilde{I}\in[0,n-1]_s} \min_{\substack{\tilde{\mathbf{y}}\in\Sigma_f^n\\ \|\tilde{\mathbf{y}}\|=1}} \lambda(\mathbf{B}_{\tilde{\mathbf{y}}}^{\tilde{I}}), \min_{\tilde{J}\in[0,n-1]_f} \min_{\substack{\tilde{\mathbf{x}}\in\Sigma_s^n\\ \|\tilde{\mathbf{x}}\|=1}} \lambda(\mathbf{B}_{\tilde{\mathbf{y}}}^{\tilde{I}}) \right\} \\ &\ge \min_{\substack{\mathbf{a}\in\Sigma_{\beta^2}^n\\ \|\mathbf{a}\|=1}} \lambda(\mathbf{B}_{\mathbf{a}}) \ge \min_{\substack{\mathbf{a}\in\mathbb{C}^n\\ \|\mathbf{a}\|=1}} \lambda(\mathbf{B}_{\mathbf{a}}) =: \alpha_{n(s,f)}^2.\end{aligned}$$

- We know, that if $I \cup J$ is a *Sidon set*, then we need indeed $n = 2^{s+f-3} + 1$ natural numbers to express the combinatoric of the convolution (Konyagin-Lev Conjecture holds). Nevertheless, the set over which we minimze is much larger then the combinatorics of the supports. Hence this is only an lower bound for $\alpha^2(s,f)$.
- Unfortunately, the combinatoric can only be removed by using the CAUCHY Interlacing theorem, which obtains only a lower bound $\alpha_n$ for $\alpha(s,f)$.

# Application: Zero-Padded Circular Convolution

Consider the (cyclic, torsion) group $\mathbb{Z}/n\mathbb{Z}$ then the (circular) convolution is given by

$$(\mathbf{x} \circledast \mathbf{y})_j = \sum_{i=0}^{n-1} x_i y_{j \ominus i} \quad , \quad j \in [0, n-1] \tag{27}$$

Appending $n-1$ zeros to $\mathbf{x}, \mathbf{y}$ circular convolution equals regular convolution

$$((\mathbf{x}, \mathbf{0}) \circledast (\mathbf{y}, \mathbf{0}))_j = \sum_{i=0}^{2n-2} x_i y_{j \ominus i} \quad , \quad j \in [0, 2n-2] \tag{28}$$

$$= \begin{cases} \sum_{i=0}^{n-1} x_i y_{2n-1-j-i} & , \quad j \in [0, n-1] \\ \sum_{i=0}^{n-1} x_i y_{j-i} & , \quad j \in [n, 2n-2] \end{cases} \tag{29}$$

### Corrolary (RNMP for Sparse ZP Circular Convolutions [W & Jung, '13] )

*Let $s, f, n \in \mathbb{N}$ with $\beta^2(s, f) \leq n$ and $n'(s, f, n) := \min\{\lfloor 2^{2(s+f-2)\log(s+f-2)} \rfloor + 1, n\}$. Then it exists $\alpha_{n'} > 0$ such that for all $\mathbf{x} \in \Sigma_s^n, \mathbf{y} \in \Sigma_f^n$ it holds*

$$\alpha_{n'} \|\mathbf{x}\| \|\mathbf{y}\| \leq \|(\mathbf{x}, \mathbf{0}) \circledast (\mathbf{y}, \mathbf{0})\| \leq \beta \|\mathbf{x}\| \|\mathbf{y}\| , \tag{30}$$

*where $(\mathbf{x}, \mathbf{0}), (\mathbf{y}, \mathbf{0}) \in \mathbb{C}^{2n-1}$ denotes the vectors padded by $n-1$ zeros.*

- Zero Padding : $\mathbf{x} \to (\mathbf{x}, \mathbf{0}) \in \mathbb{C}^{n'=2n-1}$
- Symmetrize (not complex–linear, but linear in $\mathbb{R}^{n'}$)

$$\mathbf{x} \to \mathcal{S}(\mathbf{x}) := (0, x_0, x_1, \dots, x_{n-1}, \bar{x}_{n-1}, \dots, \bar{x}_1, \bar{x}_0)^T \in \mathbb{C}^{2n'+1}$$
$$\Rightarrow \quad \mathcal{S}(\mathbf{x}) \circledast \mathcal{S}(\mathbf{y}) = \mathcal{S}(\mathbf{x}) \circledast \mathcal{S}(\mathbf{y})$$

- What if $\mathbf{x} = \mathbf{y}$?

$$A(\mathbf{x}) = \mathcal{S}(\mathbf{x}) \circledast \mathcal{S}(\mathbf{x}) = \mathcal{S}(\mathbf{x}) \circledast \mathcal{S}(\mathbf{x})$$
$$= \mathbf{F}^* (\mathbf{F}\mathcal{S}(\mathbf{x}) \odot \overline{\mathbf{F}\mathcal{S}(\mathbf{x})}) = \mathbf{F}^* |\mathbf{F}(\mathcal{S}(\mathbf{x})|^2$$
$$\Rightarrow \quad A(\mathbf{x}_1) - A(\mathbf{x}_2) = \mathcal{S}(\mathbf{x}_1 - \mathbf{x}_2) \circledast \mathcal{S}(\mathbf{x}_1 + \mathbf{x}_2)$$

**Theorem ([W & Jung, '13])**

*Let $n \in \mathbb{N}$, then $m = 4n - 1$ absolute-square Fourier measurements of ZP and symmetrized vectors are* ***stable up to a global sign*** *for $\mathbf{x} \in \mathbb{C}^n$, i.e. for all $\mathbf{x}_1, \mathbf{x}_2 \in \mathbb{C}^n$ it holds*

$$\left\| \left| \mathbf{F}\mathcal{S}\begin{pmatrix}\mathbf{x}_1 \\ \mathbf{0}\end{pmatrix} \right|^2 - \left| \mathbf{F}\mathcal{S}\begin{pmatrix}\mathbf{x}_2 \\ \mathbf{0}\end{pmatrix} \right|^2 \right\| \geq c \, \|\mathbf{x}_1 - \mathbf{x}_2\| \, \|\mathbf{x}_1 + \mathbf{x}_2\| \tag{31}$$

*with $c = c(m) = \frac{\alpha_m}{2\sqrt{m}} > 0$ and $\mathbf{F} = \mathbf{F}_m$. If $x_0 \in \mathbb{R}$ one can reduce to $m = 4n - 3$.*

# References

[1] P. Walk and P. Jung. “On a Reverse $\ell_2$–inequality for Sparse Circular Convolutions”. In: *IEEE International Conference on Acoustics, Speech, and Signal Processing*. 2013, pp. 4638 –4642.

[2] C. Ling et al. “Biquadratic Optimization Over Unit Spheres and Semidefinite Programming Relaxations”. In: *SIAM J. Optim.* 20 (2009), pp. 1286–1310.

[3] S. Konyagin and V. Lev. “Combinatorics and linear algebra of Freiman’s Isomorphism”. In: *Mathematika* 47 (2000), pp. 39–51.

[4] D. J. Grynkiewicz. *Structural Additive Theory*. Developments in Mathematics: Volume 30. Springer, 2013.

[5] T. Tao and V. Vu. *Additive Combinatorics*. Cambridge University Press, 2006.

[6] A. Böttcher and S. M. Grudsky. *Spectral Properties of Banded Toeplitz Matrices*. SIAM, 2005.

[7] P. Walk and P. Jung. “Stable Recovery from the Magnitude of Symmetrized Fourier Measurements”. In: *Arxiv* (2013).

$\Rightarrow$ http://arxiv.org/abs/1312.2222

## Thanks for Your Attention!