Löwdin Transform on FCC Optimized UWB Pulses

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Outline

Signal Model

For UWB Impulse Radio strategies one usually uses **M-ary PPM** or PAM transmission.

$$s(t) = \sqrt{\mathcal{E}} \sum_n a_n \, p(t - nT_s - d_n T) \quad , \quad d_n \in \{0, 1, \ldots, M-1\} \tag{1}$$

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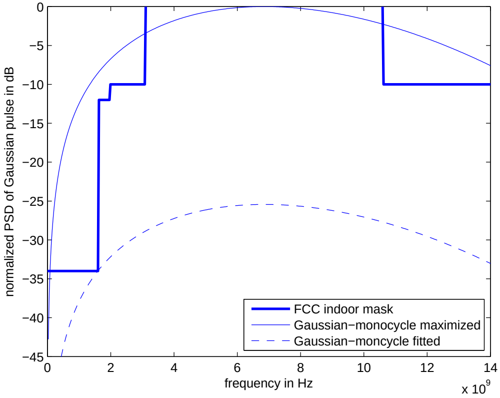

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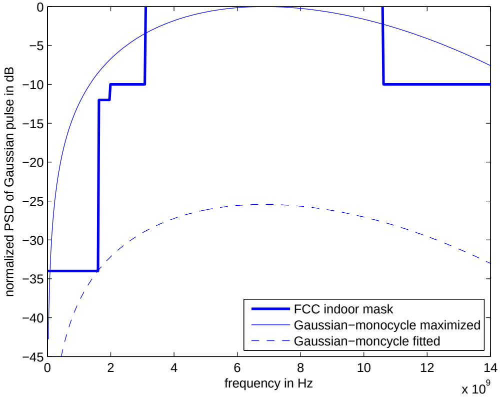

- **PPM** with matched filter receiver
- $T_s \geq (M-1)T + T_p$ and $T \geq T_p \Rightarrow$ ISI free **orthogonal** signals with energy $\mathcal{E}$

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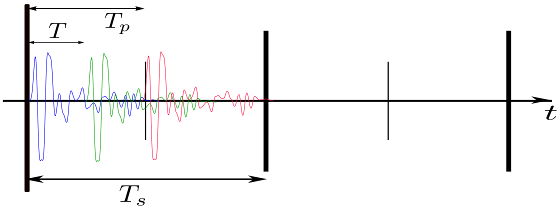

Optimizing Pulse Shape by FIR Prefiltering

Define an objective for the SNR to maximize the power in the passband $F_p = [3.1, 10.6]$GHz, called the NESP value (Luo et al., 2003):

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Idea: FIR prefiltering of q

If we use uniformly shifted translates of a generating pulse q we can express the linear combination p by a (real) FIR filter $\mathbf{g} = \{g_k\}_{k=0}^{L-1}$ with clock rate $T_0 = \frac{1}{2 \cdot 14\text{GHz}} \approx 36$ps.

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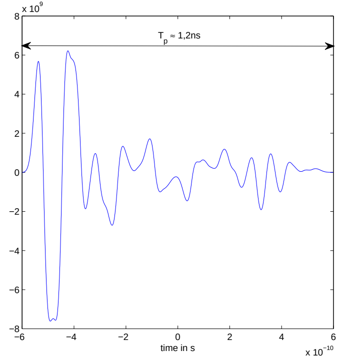
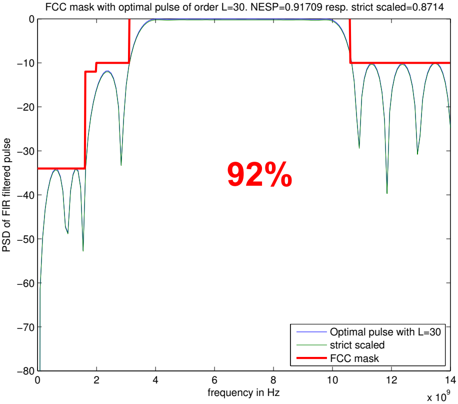
$$\begin{matrix} \max\limits_{\mathbf{g} \in \mathbb{R}^L} \int_{F_p} |\hat{\mathbf{g}}(f) \cdot \hat{q}(f)|^2 \, df \\ |\hat{\mathbf{g}}(f) \cdot \hat{q}(f)|^2 \leq S_{\text{FCC}}(f) \quad , |f| \leq 14Ghz \end{matrix} \quad \Rightarrow \quad \begin{matrix} \max\limits_{\mathbf{r} \in \mathbb{R}^L} \sum_{n=0}^{L-1} r_n \cdot c_n(q) \\ 0 \leq \hat{\mathbf{r}}(f) \leq M(f) \quad , |f| \leq 14Ghz \end{matrix} \tag{4}$$

This **convex** problem is equivalent to a semi-definite-program (Berger et al., 2006)

Optimal Pulse With SeDuMi, L=30





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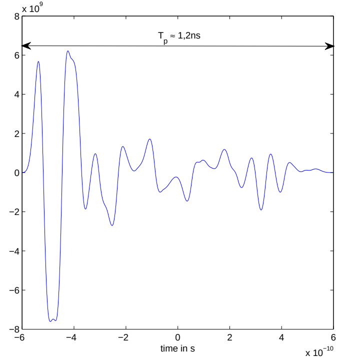

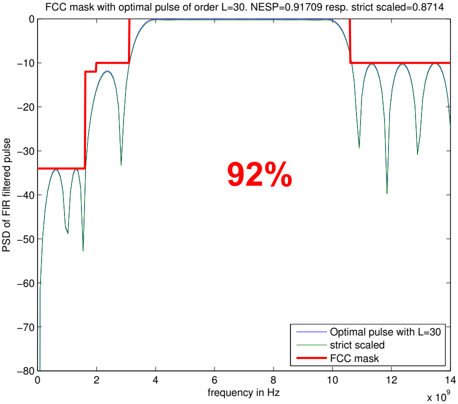

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Good, but . . .

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The Löwdin's orthogonalization on this pulse set is then given by (Löwdin, 1950)

$$p_m^{\circ}(t) = \sum_{k=-M}^{M} \left[\mathbf{G}_M^{-\frac{1}{2}}\right]_{mk} p_k(t) \quad , \quad m \in \{-M, \ldots, M\} \,. \tag{5}$$

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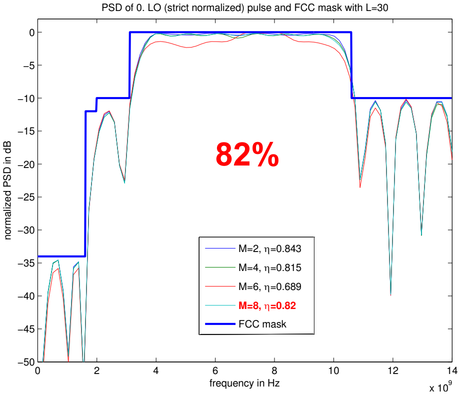
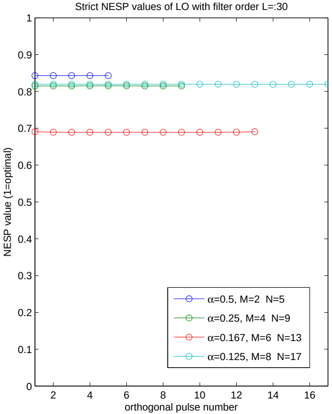
Properties:

1. All orthogonal pulses p_n° have same energy and support in $[-MT - \frac{T_p}{2}, MT + \frac{T_p}{2}]$
2. The Löwdin pulses posses minimal variation to the normed optimal pulse p, i.e. (Aiken et al., 1980)

$$\{p_n^{\circ}\} = \arg \min_{\{p_n'\} ONB} \sum_n \|p_n - p_n'\|_2^2 \tag{6}$$

Results With Löwdin Method





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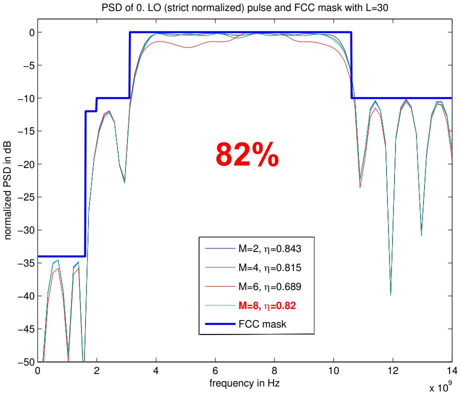

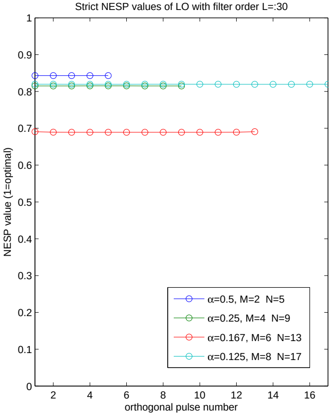

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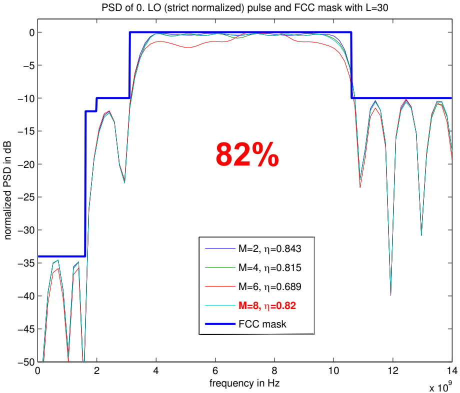

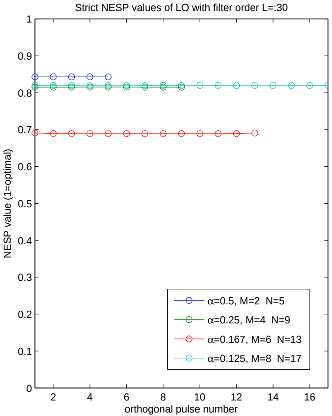

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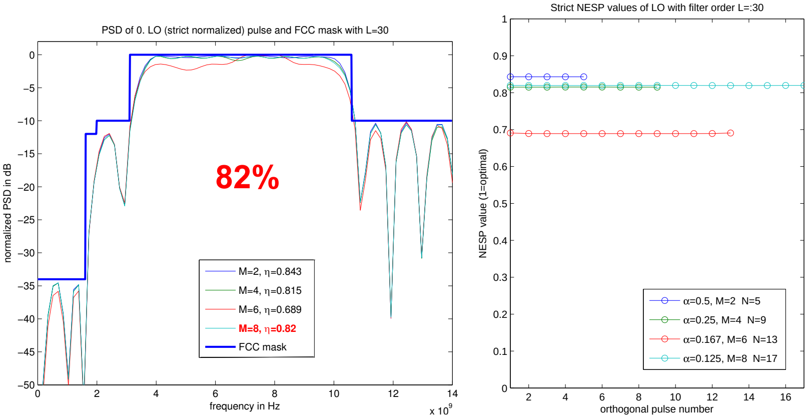

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- **Our approach:** 17 **Löwdin pulses with** 82% **vs.** 3 **standard pulses**
- Genetic Algorithm on B-Splines (Wang et al., 2008): 95%,
  but: high complexity, not realizable, **no PPM**,

Shift-Invariant-Systems

A **Nyquist-pulse** is a shift-orthonormal pulse which is given in frequency by

$$\hat{p}^{\circ}(\nu) = \frac{\hat{p}(\nu)}{\sqrt{\sum_k |\hat{p}(\nu+k)|^2}} \quad \Rightarrow \quad \Phi_p(\nu) := \sum_k |\hat{p}^{\circ}(\nu+k)|^2 = 1 \text{ a.e.} \qquad (7)$$

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Set $T := 1$ and choose $K \in \mathbb{N}$ with $T_p = KT = K$.
If p is smooth and decay fast, the set $\{p_n\}$ is a Riesz-basis for $V(p) := \overline{\text{span}\{p_n\}}$ iff

$$0 < A \leq \Phi_p(\nu) = (\mathbf{Z}r_p)(0,\nu) \leq B < \infty \tag{8}$$

$$\text{with Zak Transform} \quad (\mathbf{Z}p)(t,\nu) := \sum_{k\in\mathbb{Z}} p_k(t) e^{2\pi i k\nu}. \tag{9}$$

For this set, the Löwdin orthogonalization is a tight-frame construction with $\min_{p'} \|p - p'\|$ (Janssen and Strohmer, 2002)

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- The samples at n/N of the spectral function Φ_p are the eigenvalues of the circulant extension $\tilde{\mathbf{G}}_M$ of the Gram Matrix
- can be calcualted efficiently with the DFT

Stability for Shift-Invariant-Systems

Theorem (Stability of Löwdin Orthogonalization)

Let p be a continuous bounded pulse s.t. there exists $K \in \mathbb{N}$ with $\operatorname{supp}(p) \subset [-\frac{K}{2}, \frac{K}{2}]$, the shift-sequence $\{p_n\}_{n\in\mathbb{Z}}$ is a Riesz-basis for the ℓ^2–closure of its span $V(p)$ and $\hat{p} \in W(\mathbb{R})$. Then the limit of the Löwdin orthogonal pulses $\{p_m^\circ\}$ can be approximated by a set of ***approximative Löwdin orthogonal*** *(ALO) pulses $\{\tilde{p}_m^{\circ,M}\}_{m=-M}^{M}$, which can be represented point wise for $M > K$ and each $m \in \mathbb{Z}_M$ using the Zak Transform in the following way, $N = 2M + 1$,*

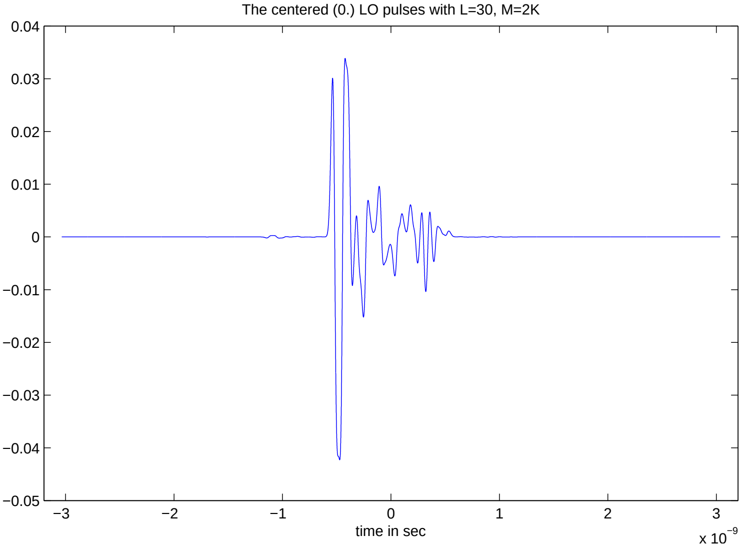
$$\tilde{p}_m^{\circ,M}(t) \equiv \begin{cases} \sum_{n=0}^{N-1} \frac{e^{-2\pi i \frac{mn}{N}} (\mathbf{Z}p)(t, \frac{n}{N})}{\sqrt{(\mathbf{Z}r_p)(0, \frac{n}{N})}} & , |t| \leq M + \frac{K}{2} \\ 0 & , else \end{cases} , \tag{10}$$

such that for each $m \in \mathbb{Z}$

$$p_m^\circ(t) = \lim_{M\to\infty} \tilde{p}_m^{\circ,M}(t) \quad , \quad t \in \mathbb{R} \tag{11}$$

converges point wise and defines an orthonormal generator $p^\circ := p_0^\circ$ for $V(p)$.

Löwdin Pulses in Time Domain for Decreasing T With Duration $T_{p^\circ} = 5T_p$



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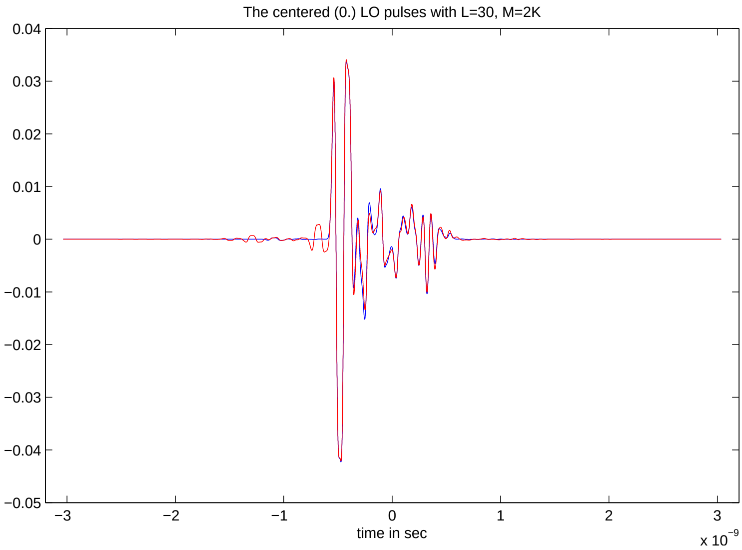

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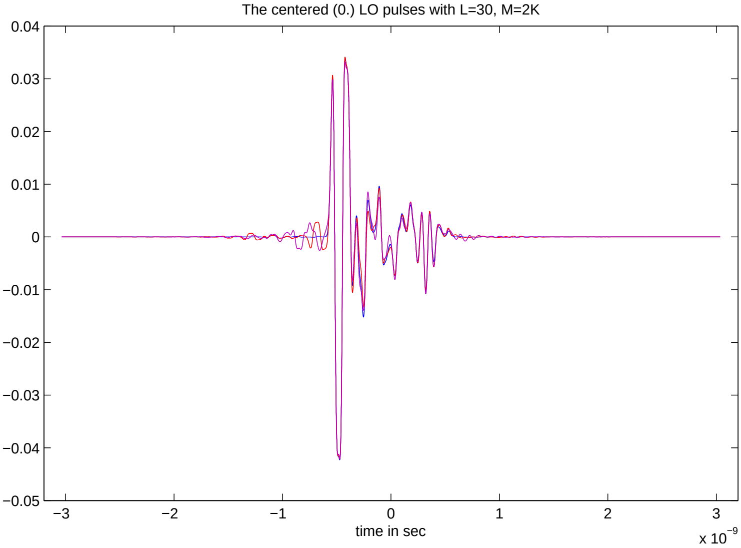

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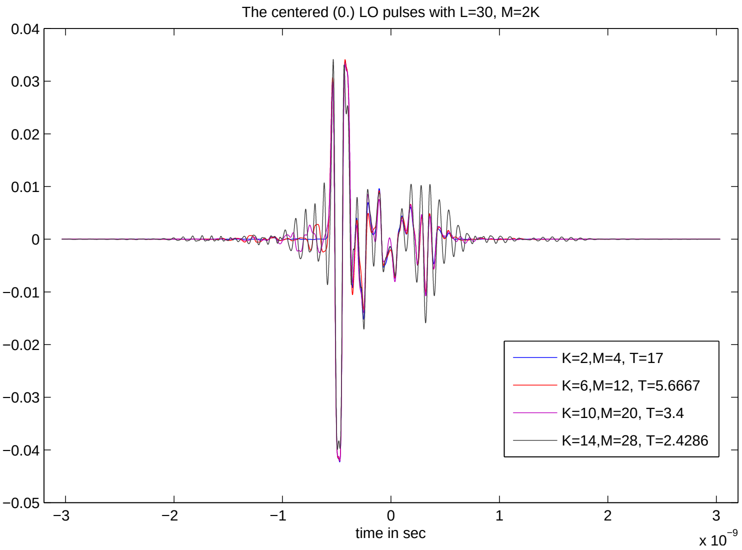
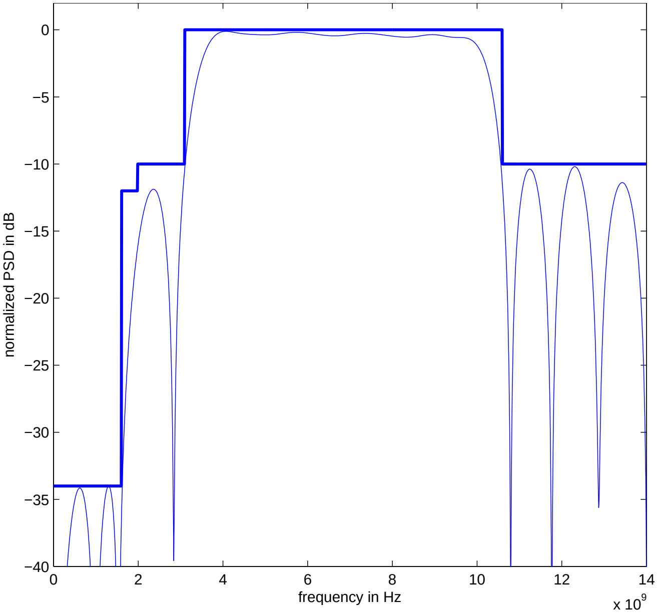


LO Spectral Results for $M = 2K$



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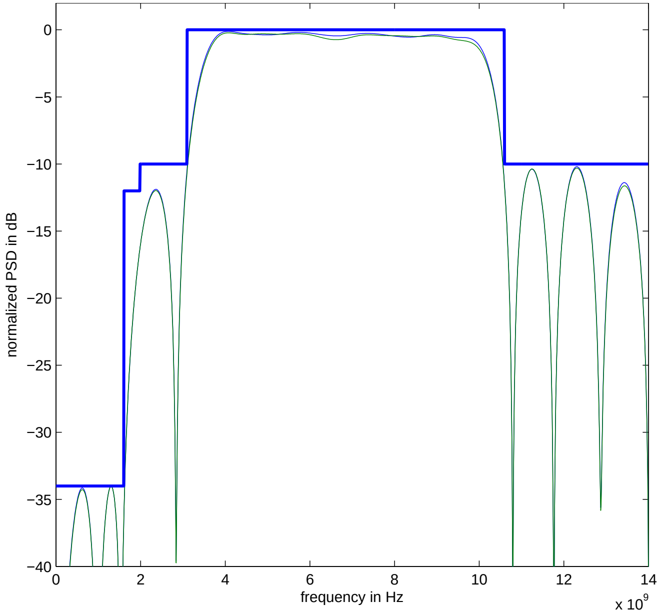

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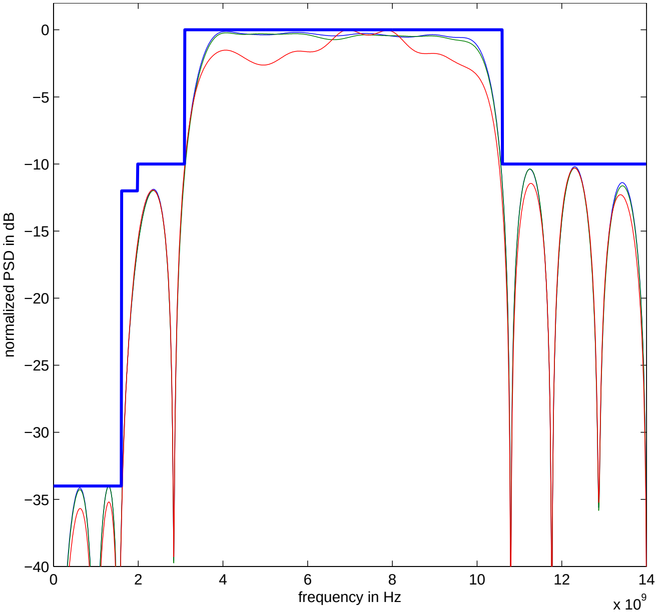

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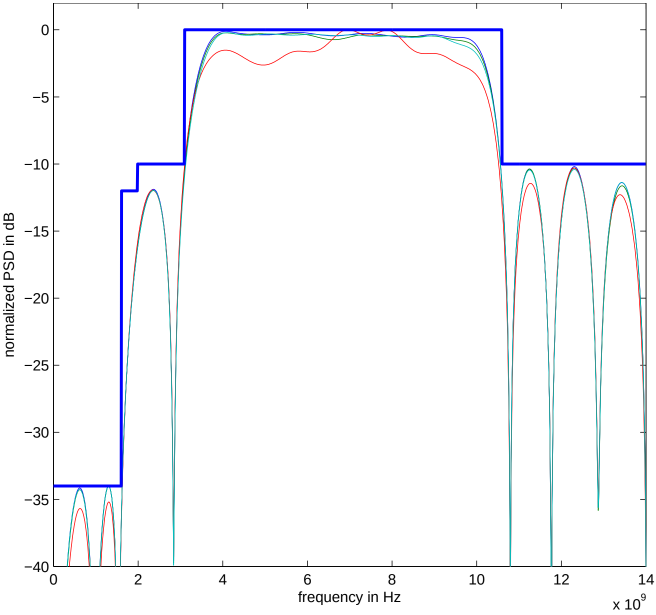

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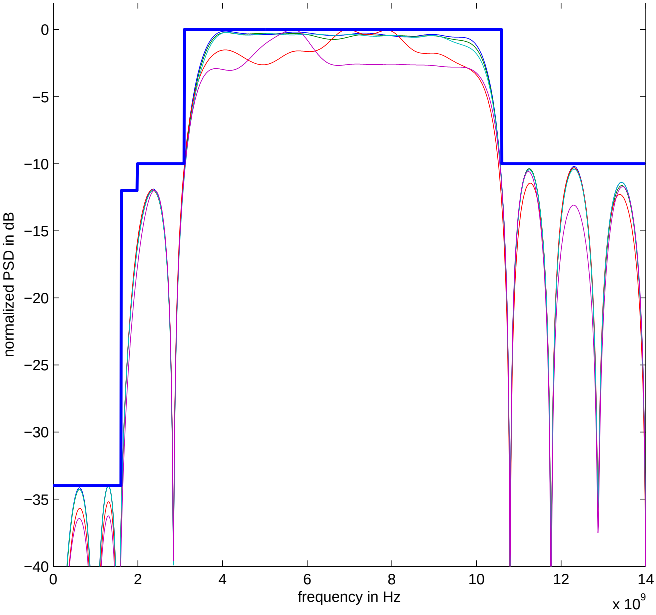

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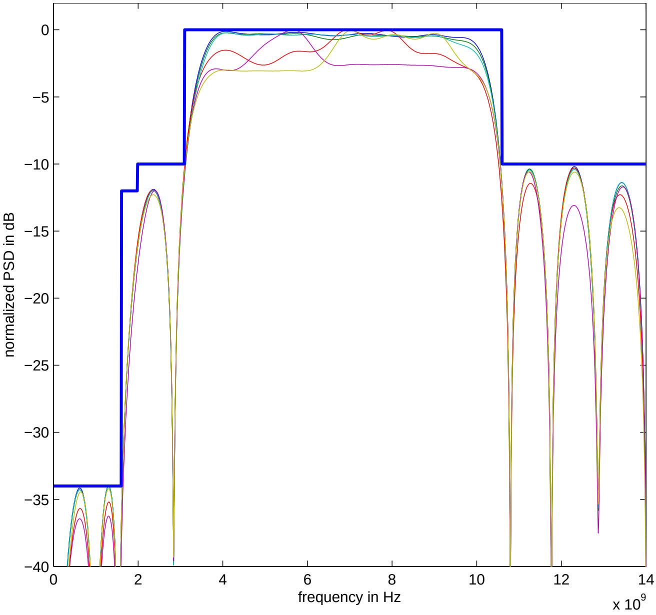

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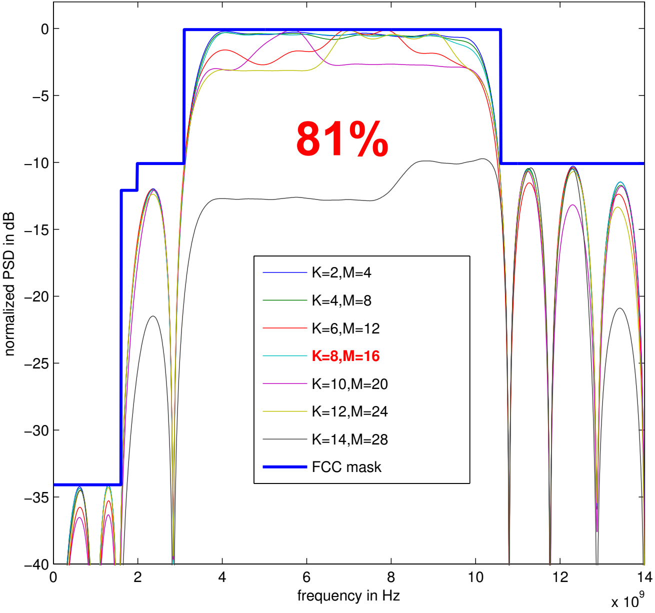

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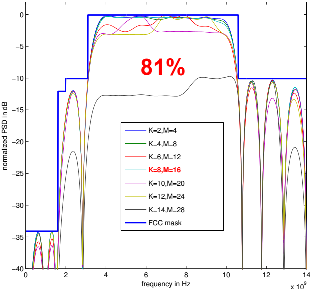
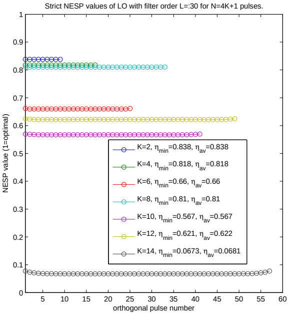
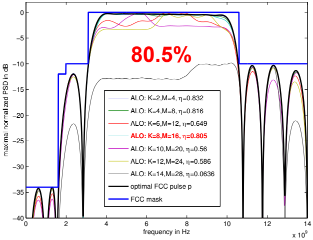


1.41 LO PSD



1.42 LO NESP



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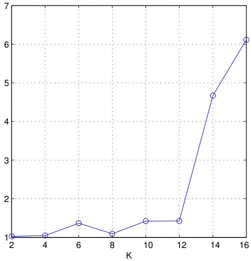
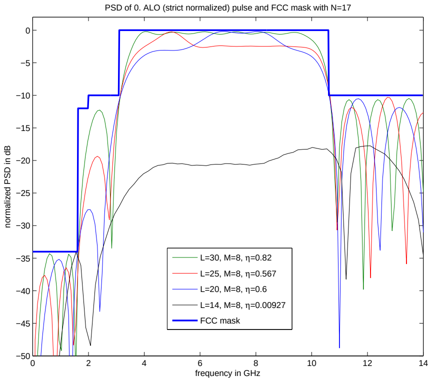
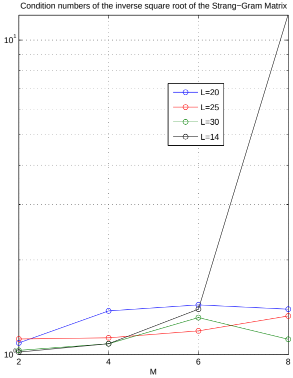


1.44 Condition Numbers of $\tilde{\mathbf{G}}_M^{-\frac{1}{2}}$

Dependence of Filter Order L



1.45 ALO PSD's



1.46 Condition Number

Observation:

- Increasing L decrease the condition number of $\mathbf{G}_M$ and $\tilde{\mathbf{G}}_M$
- Tighter Riesz basis → closer to an ONB → less distortion in time-frequency

Present and Future Work

Observation:

- For high filter order $L \longrightarrow$ shifts of the optimal pulse p are almost orthogonal.
- Condition number $\leq 3 \longrightarrow$ distortion in Frequency is minimal, and FCC optimization is well preserved.
- The Löwdin transform in the limit corresponds to an IIR filter $\mathbf{h}$ with clock rate $1/T$.

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Goals:

- Optimize problem 1 and 2 simultaneously such that the pulse shape is capable for an orthogonal PPM transmission.
- Find a condition on the optimized pulse which ensure tight Riesz Bounds A, B, hence a condition number close to 1.
- Investigate robustness against channel and hardware effects

Thank you for your attention.

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Signal Model

For UWB Impulse Radio strategies one usually uses **M-PPM or PAM transmission**.

$$s(t) = \sqrt{\mathcal{E}} \sum_n a_n p(t - nT_s - d_n T) \quad , \quad d_n \in \{0, 1, \ldots, M-1\} \qquad (12)$$

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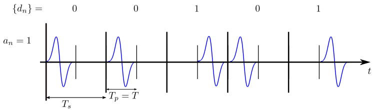

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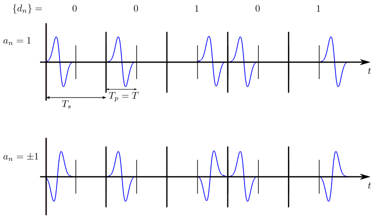

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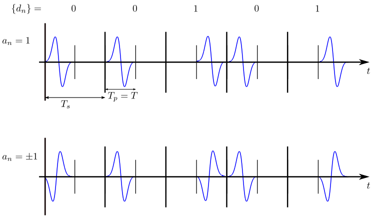

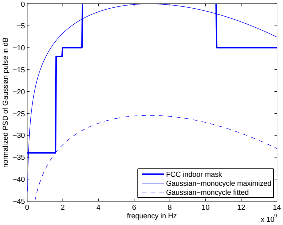

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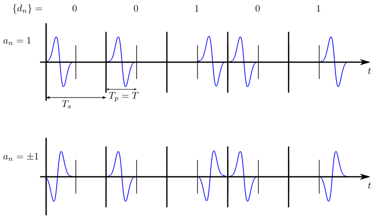

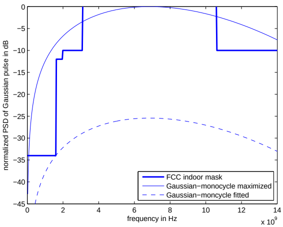

Optimal receiver for coherent memoryless AWGN channels: **matched filter**.

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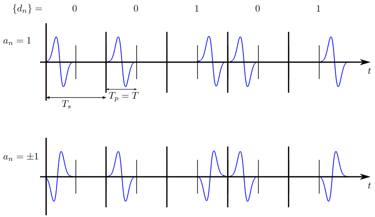

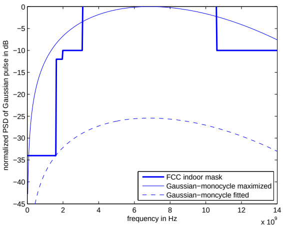

Optimal receiver for coherent memoryless AWGN channels: **matched filter**.

Single User: $$R_b \simeq \frac{1}{T_s} \cdot \log M \quad , \quad P_s(\mathcal{E}) \leq (M-1) Q\left(\sqrt{\frac{\mathcal{E}}{N_0}}\right) \tag{13}$$

Approach for Optimization (Wu, Tian, Davidson and Giannakis, 2003)

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Problem Formulation:

Define an objective for the SNR to maximize the power in the passband $F_p = [3.1, 10.6]$GHz, called the NESP value (normalized efficient signal power):

$$\eta := \frac{\int_{F_p} |\hat{p}(f)|^2 df}{\int_{F_p} S_{\text{FCC}}(f) df} \quad , \quad \tilde{\eta} := \int_{F_p} |\hat{p}(f)|^2 df. \tag{14}$$

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Idea: FIR prefiltering of q

If we use uniformly shifted translates of a generating pulse q we can express the linear combination p by a (real) FIR filter $\mathbf{g} = \{g_k\}_{k=0}^{L-1}$ with clock rate $T_0 = \frac{1}{2 \cdot 14\mathrm{GHz}} \approx 36$ps.

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If we fix the generating pulse q and the time-shift T_0, we get

$$\max_{\mathbf{g} \in \mathbb{R}^L} \int_{F_p} |\hat{\mathbf{g}}(f) \cdot \hat{q}(f)|^2 \, df \quad , \text{ s.t. } |\hat{p}(f)|^2 \leq S_{\text{FCC}}(f) \quad , \quad f \in [0, 14GHz], \tag{17}$$

which is a **semi-infinite non-convex optimization** problem, since $\tilde{\eta}$ is quadratic in $\mathbf{g}$.

Autocorrelation With LMI as a SDP Problem

1. Reformulate $\tilde{\eta}$ and the constraints in the autocorrelation $r_n = \sum_k g_k g_{k+n}$ of filter **g**
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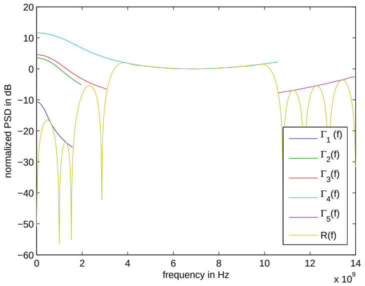

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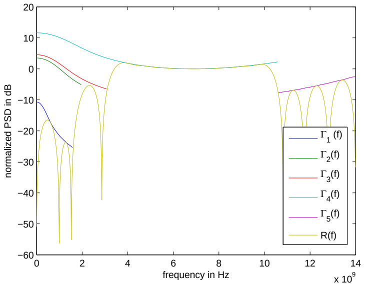

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$$\Leftrightarrow$$

$$K_I(\theta_i) = \left\{ \mathbf{r} \,\middle|\, \sum(\gamma_n^i - r_n)\tilde{\phi}_n(f) \geq 0, f \in \left[\theta_i, \frac{1}{2T_0}\right] \right\},$$
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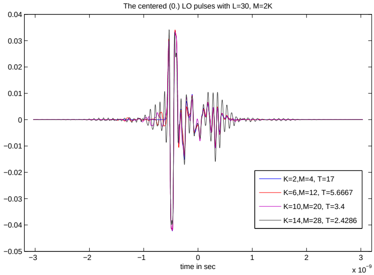
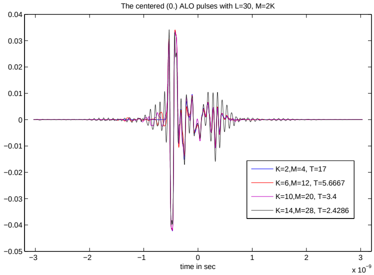
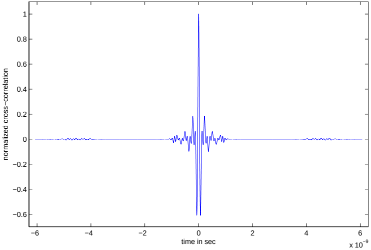
LO and ALO Pulses in Time



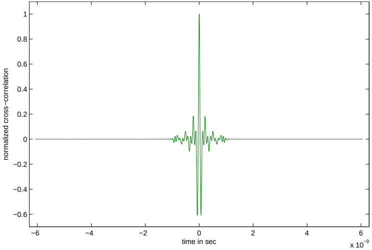
1.63 ALO pulses



1.64 Löwdin pulses



1.65 ALO Cross-correlation



1.66 LO Auto-correlation

Approach for orthogonalization

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Existing orthogonalization methods:

- Standard: non-overlapping pulses in PPM $\longrightarrow$ small Bit-Rates R_b
- FSK modulation (complex), yields same energy and cross-correlation properties $\longrightarrow$lower energy of single pulse
- Gram-Schmidt orthogonalization with respect to minimize L^2 distance (Tian et al., 2006) $\longrightarrow$NESP decreases fast in M
  - SEQ-UWB pulse selection, intelligent way to choose the pulses with largest NESP out of the orthogonal set.
- Hermite polynomials pulses (Quertani et. al 2005) $\longrightarrow$NESP optimization is hard
- Prolate spherical wave functions: strict time-limited, best frequency concentration $\tilde{\eta}/E_p \longrightarrow$ eigenfunctions of integral equation, only numerical solvable $\longrightarrow$ digital construction $\longrightarrow$ filter clock rate $>$ 64Ghz, bad NESP. (Parr et al., 2003)
- B-splines as pulses (Wang et al., 2008) $\longrightarrow$ high complexity, rectangle pulses

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Goal:

1. Orthogonalize the FCC optimized pulse p such that all generated orthogonal pulses p_n° are still close to FCC optimal.
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New orthogonalization method:

The *nonorthogonal problem* of overlapping linear independent functions was solved by Löwdin who extend in 1950 the results of (Landshoff, 1936) to the general case, which is well-known in the wavelet community (Schweiner, Wigner 1970), (Janssen and Strohmer, 2002) under the name **Löwdin orthogonalization** or symmetrical orthogonalization, which in fact is an orthonormalization.

Orthogonal Construction

1. Determine an optimal pulse p of design problem (18)
2. Consider $2M+1$ time translates $p_n(t) := p(t-nT)$ with time-shift $T := T_p/M$, s.t. all pulses have support in $[-T_s/2, T_s/2]$ with $T_s = 3T_p = 2MT + T_p =: T_{p^\circ}$. The data-rate $R = \log 3/T_s$ is hence multiplied by $\log(2M+1)/\log 3$.
3. Obtain the Gram-Matrix $\mathbf{G}_M$ with $[\mathbf{G}_M]_{nm} = (p_m, p_n)$ for $n, m \in \{-M, \dots, M\}$ with Matrix dimension $N = 2M+1$.
4. Derive the inverse square root $\mathbf{G}_M^{-\frac{1}{2}}$ via singular value decomposition

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$$p_m^\circ(t) = \sum_{k=-M}^{M} \left[\mathbf{G}_M^{-\frac{1}{2}}\right]_{mk} p_k(t) \quad , \quad m \in \{-M, \dots, M\}\,. \qquad (19)$$

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Properties:

1. All orthogonal pulses p_n° have energy $\|p_n^\circ\| = 1$ (normalization)
2. The p_n° are i.g. not translates of p_0°, hence differ in frequency domain which can violate FCC mask, a rescaling results hence in different energies $\mathcal{E}_n$
   $\rightarrow$ If $b_n > 0$ is the maximal scaling factor b s.t. $|b\hat{p}_n^\circ(f)|^2 \leq S_{\text{FCC}}(f)$, then $\sqrt{\mathcal{E}} := \min\{b_n\}$ is the valid scaling factor and energy for all pulses ($\|p_n^\circ\| = 1$).
3. The Löwdin pulses posses minimal variation to the normed optimal pulse p, i.e. (Aiken et al., 1980)

$$\{p_n^\circ\} = \arg \min_{\{p_n'\}ONB} \sum_n \|p_n - p_n'\|_2^2 \tag{20}$$

Shift-Invariant-Systems

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Set $T := 1$ and choose $K \in \mathbb{N}$ with $T_p = KT = K$. Then for any $M \geq K$ and $N := 2M + 1$, then the Toeplitz-Gram-Matrix $\mathbf{G}_M$ can be extended to a Cyclic-Matrix as a Strang preconditioner Circulant Matrix $\tilde{\mathbf{G}}_M$. If $\tilde{\mathbf{G}}_M$ is positive, the inverse square root can be efficiently calculated by the DFT. If M runs to infinity, we yield a shift-invariant-system

$$V(p) := \overline{\text{span}\{p_n\}} \subset L^2(\mathbb{R}) \qquad \{p_n\} \text{ is a Frame of Translates} \tag{21}$$

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If p is continuous, the set $\{p_n\}$ is a Riesz-basis for $V(p)$ iff

$$0 < A \leq \sum_{j \in \mathbb{Z}} |\hat{p}(\nu + k)|^2 = \mathbf{Z}(p * \overline{p}_-)(0, \nu) \leq B < \infty \quad \nu \text{ a.e.} \tag{22}$$

$$\text{with Zak Transform} \quad (\mathbf{Z}p)(t, \nu) := \sum_{k \in \mathbb{Z}} p_k(t) e^{2\pi i k \nu} \quad \text{and} \quad p_-(t) = p(-t). \tag{23}$$

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If $\hat{p}$ is in the Wiener space $W(\mathbb{R})$, then this holds point wise, moreover the samples in ν at n/N are the eigenvalues of $\tilde{\mathbf{G}}_M$ For this set, the Löwdin orthogonalization is a tight-frame construction with $\min_{p'} \|p - p'\|$ (Janssen and Strohmer, 2002)

Stability for shift-invariant-systems

Theorem (Stability of Löwdin Orthogonalization)

Let p be a continuous bounded pulse s.t. there exists $K \in \mathbb{N}$ with $\operatorname{supp}(p) \subset [-\frac{K}{2}, \frac{K}{2}]$, the shift-sequence $\{p_n\}_{n\in\mathbb{Z}}$ is a Riesz-basis for the ℓ^2–closure of its span $V(p)$ and $\hat{p} \in W(\mathbb{R})$. Then we can approximate the limit of the Löwdin orthogonalization $\{p^\circ_m\}$ by the sequence $\{\tilde{p}^{\circ,M}_m\}_{m=-M}^{M}$, which can be represented point wise for $M > K$ and each $m \in \mathbb{Z}_M$ using the Zak Transform in the following way, $N = 2M + 1$,

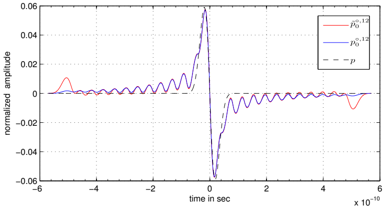
$$\tilde{p}^{\circ,M}_m(t) \equiv \begin{cases} \sum_{n=0}^{N-1} \frac{e^{-2\pi i \frac{mn}{N}} (\mathbf{Z}p)(t, \frac{n}{N})}{\sqrt{(\mathbf{Z}(p * \overline{p}_-))(0, \frac{n}{N})}} & , |t| \leq M + \frac{K}{2} \\ 0 & , else \end{cases} , \quad (24)$$

such that for each $m \in \mathbb{Z}$

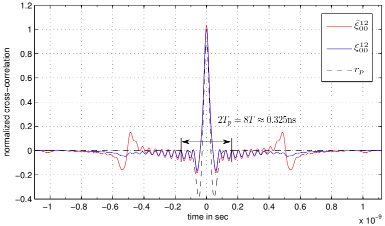
$$p^\circ_m(t) = \lim_{M\to\infty} \tilde{p}^{\circ,M}_m(t) \quad , \quad t \in \mathbb{R} \quad (25)$$

converges point wise and defines an orthonormal generator $p^\circ := p^\circ_0$ for $V(p)$.

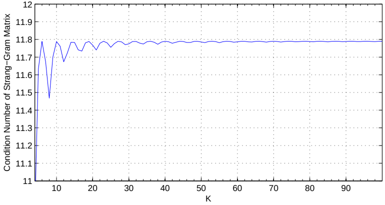
ALO and LO for Gaussian Monocylce



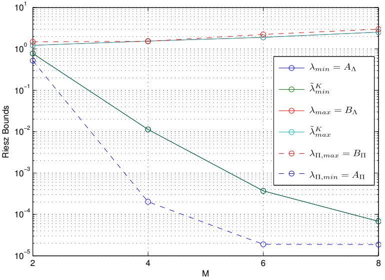
1.67 Pulse in time for $K = 12$ and $M = 4$



1.68 Auto–correlation $K = 12$ and $M = 4$



1.69 Convergence of the condition–number of $\tilde{\mathbf{G}}_K$



1.70 Calculated Riesz Bounds A, B