Compressed Sensing on the Image of Bilinear Maps

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Motivation

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Combined Sparsity in Communication Systems

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K-sparse signal
(+ noise)

Receiver
$\mathbf{z} + \mathbf{n} \in \mathbb{R}^N$

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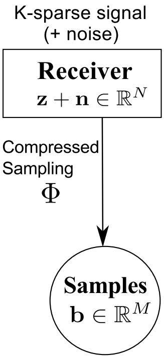

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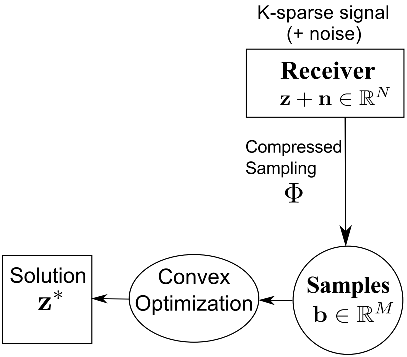

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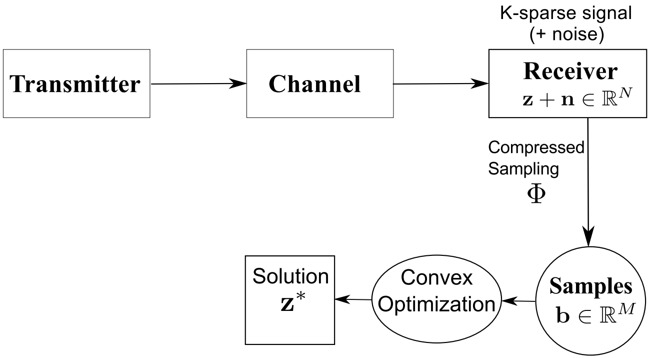

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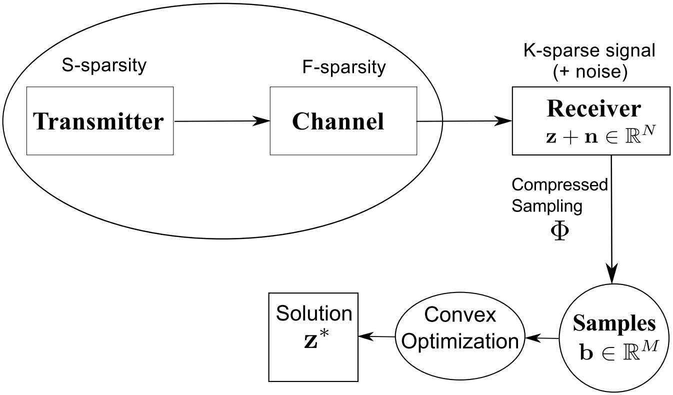

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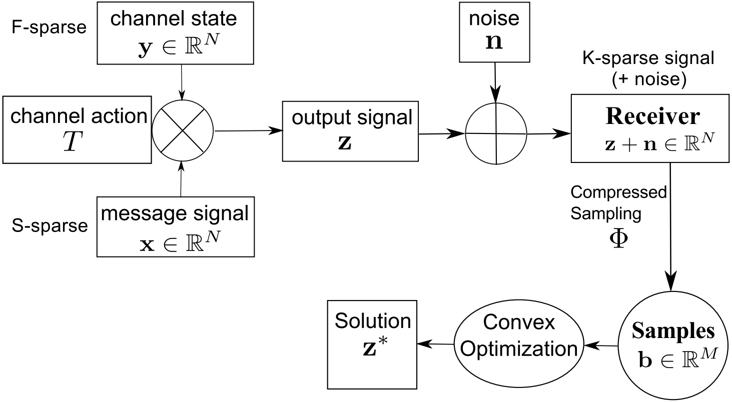

Mathematical Framework

Channel action

$$T : \underset{\text{message}}{\underset{\uparrow}{\mathbb{R}^n}} \times \underset{\text{state}}{\underset{\uparrow}{\mathbb{R}^n}} \to \mathbb{R}^n \quad \text{bilinear map} \tag{1}$$

Any Product on $\mathbb{R}^n$:

- Point Product in Time $T = \odot$, i.e. $\forall i \in \{0, \ldots, n-1\} : (\mathbf{x} \odot \mathbf{y})_i = x_i y_i$
- Point Product in Frequency (Circular Convolution $T = \circledast$)

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$$\mathbf{y} \in \Sigma_f \subset \mathbb{R}^n \tag{2}$$

is an unknown f-sparse configuration vector which can describe

- Fading effects
- Memory
- Jamming, Interference (ISI, ICI)
- Multi-Antenna

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Message Signal

$$\mathbf{x} \in \Sigma_s \subset \mathbb{R}^n \tag{3}$$

is an unknown s-sparse signal vector carrying the message, which can be

- sensor, network of sensors
- sparse Pictures: Astrophysics , Medical, etc.

Channel Output

What we want?

Find the “best” compressible sensing matrix Φ which allows a stable (noise) reconstruction of any channel action output $\mathbf{z} = T(\mathbf{x}, \mathbf{y})$.

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Assume T is known and fixed. Let $\{\mathbf{e}_i\}_{i=0}^{n-1}$ be the canonical basis in $\mathbb{R}^n$.

$$T(\Sigma_s, \Sigma_f) = \bigcup_{i=1}^{\binom{n}{s}} \bigcup_{j=1}^{\binom{n}{f}} \underbrace{T(X_i, Y_j)}_{\substack{=:Z_{i,j} \\ \text{non-linear}}} \tag{4}$$

where $X_i = \text{span}\{\mathbf{e}_{i_1}, \ldots, \mathbf{e}_{i_s}\}$, $Y_j = \text{span}\{\mathbf{e}_{j_i}, \ldots, \mathbf{e}_{j_f}\}$ are s resp. f dim. subspaces.

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How “sparse” is Z and $T(\Sigma_s, \Sigma_f)$?

At least sf-sparse, i.e. $T(\Sigma_s, \Sigma_f) \subset \Sigma_{sf}$.

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Intuition say yes, since every output is given by an $s+f$ parameter set. But the properties of Z depend on properties of T and X, Y.

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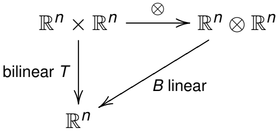
Linearization

Tensor Product $(\mathbb{R}^n \otimes \mathbb{R}^n, \otimes)$ is an n^2 dimensional linear space, given as the convex hull of the bilinear map $\otimes$ defined by $\mathbf{x} \otimes \mathbf{y} = (x_0 \mathbf{y}^T, \ldots, x_{n-1} \mathbf{y}^T)^T$ (Greub, 1967).



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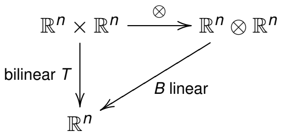

Any bilinear map T acting on $X \times Y$ can be described by a linear map B acting on U:

$$\mathbf{z} = T(\mathbf{x}, \mathbf{y}) = B(\mathbf{x} \otimes \mathbf{y}) \quad , \quad (\mathbf{x}, \mathbf{y}) \in X \times Y \subset \mathbb{R}^n \times \mathbb{R}^n \tag{5}$$

where $U := \{\mathbf{x} \otimes \mathbf{y} \mid (\mathbf{x}, \mathbf{y}) \in X \times Y\}$ is a set of *simple tensor products* and $Z = B(U)$.

Toy Example: Circular Convolution

Using the discrete Fourier transform (Fourier matrix) **F** we can describe the circular convolution $\circledast$ as the point product $\odot$ in the frequency domain

$$T(\mathbf{x}, \mathbf{y}) := \mathbf{x} \circledast \mathbf{y} = \sqrt{N}\mathbf{F}^*(\mathbf{F}\mathbf{x} \odot \mathbf{F}\mathbf{y}) \tag{6}$$

or pointwise by the modulation $(l \oplus k) := l + k \mod n$ as

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Idea:

Use the properties of $\circledast$ to transport $(s,f)-$sparsity in the Cartesian product $X \times Y$ to the output. This is possible for certain canonical subspace pairs which are maximal separated, i.e. if the image of the canonical basis $\mathcal{B}_X = \operatorname{span}\{\mathbf{e}_i \mid i \in I\}$ and $\mathcal{B}_Y = \operatorname{span}\{\mathbf{e}_j \mid j \in J\}$ given by

$$\mathcal{B}_Z := \{\mathbf{e}_k \mid k = i \oplus j \text{ for any } (i,j) \in I \times J\} \tag{8}$$

has cardinality equal to sf, (Hegde and Baraniuk, 2011April).

Results

For the circular convolution $B^{\circledast}$:

$n^2 > n$ for any $n > 1 \quad \Rightarrow \quad$ nullspace $\mathcal{N}(B^{\circledast}) \neq \{\mathbf{0}\}$ (non-trivial).

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There exist maximal separated cano. subspace pairs (X, Y) with dim. s resp. f, s.t.

$$\mathcal{N}(B^{\circledast}) \cap U = \{\mathbf{0}\} \tag{9}$$

Hence any $\mathbf{z}$ is uniquely represented by directions in X and Y (except by scalar multiple). Now since $Z = B^{\circledast}(U)$ we have

$$Z \subset B^{\circledast}(X \otimes Y) \qquad \text{subspace of } \mathbb{R}^n \text{ with dim. } sf \tag{10}$$

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Surprisingly, perfect reconstruction with high probability is possible from only $M = \mathcal{O}(s + f)$ measurements, (Hegde and Baraniuk, 2011April).

Restricted Norm Multiplicativity Property

We could derive a sufficient condition on T, X, Y for a δ-stable embedding of $Z \subset \mathbb{R}^n$ in an $m = \mathcal{O}(s + f)$ dimension subspace.

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Since in general, T has a non-trivial null-space in $X \times Y$ it could exist a representation set $O \subset X \times Y$ for Z s.t.

$$\alpha \|\mathbf{x}\| \|\mathbf{y}\| \leq \|T(\mathbf{x}, \mathbf{y})\| \leq \beta \|\mathbf{x}\| \|\mathbf{y}\| \quad , \quad (\mathbf{x}, \mathbf{y}) \in O \tag{11}$$

with $0 < \alpha < \beta < \infty$.

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Definition (Restricted norm multiplicativity property)

Let $X, Y \subset \mathbb{R}^n$. Then the bilinear map $T\colon X \times Y \to \mathbb{R}^n$ has the *restricted norm multiplicativity property* (RNMP), if

$$0 < \alpha(X, Y) := \sup_{\substack{O \subset X \times Y \\ T(O) = T(X,Y)}} \inf_{(\mathbf{x},\mathbf{y}) \in O \setminus \mathcal{N}} \frac{\|T(\mathbf{x}, \mathbf{y})\|}{\|\mathbf{x}\| \|\mathbf{y}\|}. \tag{12}$$

Moreover, we define the universal upper bound by

$$\beta(X, Y) := \sup_{(\mathbf{x},\mathbf{y}) \in X \times Y} \frac{\|T(\mathbf{x}, \mathbf{y})\|}{\|\mathbf{x}\| \|\mathbf{y}\|}. \tag{13}$$

RIP on $T(X, Y)$

Theorem

Let $s, f, n, m \in \mathbb{N}$ with $1 \leq s \leq f \leq sf \leq n$ and $X, Y \subset \mathbb{R}^n$ s resp. f dim. convex cones. If the bilinear map $T\colon X \times Y \to \mathbb{R}^n$ has the restricted norm multiplicativity property with bounds α and β, then a realization of a sub-Gaussian matrix $\Phi\colon \mathbb{R}^n \to \mathbb{R}^m$ with $m \leq n$ and $[\Phi]_{ij} \sim \mathcal{N}(0, 1/m)$ fulfills for every $\mathbf{z} \in T(X, Y)$

$$(1 - \delta) \|\mathbf{z}\| \leq \|\Phi\mathbf{z}\| \leq (1 + \delta) \|\mathbf{z}\| \tag{14}$$

for any $\delta \in (0, 1)$ with probability

$$\geq 1 - 2N(X^1, X^{\delta/d})N(Y^1, Y^{\delta/d})e^{-c(\delta)m} \tag{15}$$

and constants

$$d = d(\alpha, \beta) := \begin{cases} 7\frac{\beta}{\alpha}(2 + \sqrt{\alpha}) & , \quad \alpha \neq \beta \\ 12 & , \quad \alpha = \beta \end{cases} \quad \text{and} \quad c := \frac{6\delta^2 - \delta^3}{368}. \tag{16}$$

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Example for $T = \circledast$

For any pairs (X, Y) which are positive convex cones s.t. $\dim \overline{\mathrm{co}} X = s, \dim \overline{\mathrm{co}} Y = f$, we have $\alpha = 1$ and $\beta = \sqrt{s}$, s.t.

$$p \geq 1 - 2\left(\frac{378}{\delta}\sqrt{s}\right)^{s+f} e^{-c(\delta)m}. \tag{17}$$

Proof Idea

Extending proof technique in (Baraniuk et al., 2008January)

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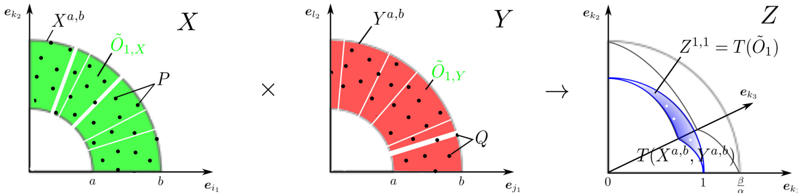

Figure: Net construction in the shells for covering the sphere in Z.

- The RNMP allow a representation of the sphere $Z^{1,1}$ in Z by the shells $X^{a,b}$ and $Y^{a,b}$ with

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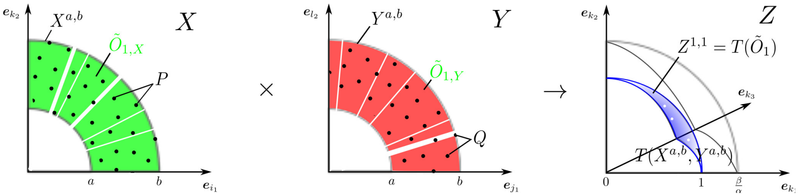

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- construct a net $R = T(P, Q)$ for $Z^{1,1}$ by ϵ-nets P in $X^{a,b}$ and Q in $Y^{a,b}$.

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The Probabilistic Part

1. Measure concentration phenomenon of Gaussian matrices:
   For every $\mathbf{r} \in \mathbb{R}^n$ and any $\delta \in (0, 1)$ it holds

$$|\|\Phi\mathbf{r}\| - \|\mathbf{r}\|| \leq \delta \|\mathbf{r}\| \tag{19}$$

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3. Any realization of Φ is a linear map on a finite dimensional normed space $\mathbb{R}^n$ and hence bounded, i.e. there exist $A \geq -1$ such that

$$\|\Phi \mathbf{z}\| \leq (1 + A) \|\mathbf{z}\| \quad , \quad \mathbf{z} \in Z \subset \mathbb{R}^n \tag{21}$$

where $1 + A \geq 0$ denotes the smallest upper bound.

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   For every $\mathbf{r} \in \mathbb{R}^n$ and any $\delta \in (0, 1)$ it holds

$$|\|\Phi\mathbf{r}\| - \|\mathbf{r}\|| \leq \delta \|\mathbf{r}\| \tag{19}$$

   with probability

$$> 1 - 2e^{-c_o(\delta/2)m}. \tag{20}$$

The Algebraic Part

3. Any realization of Φ is a linear map on a finite dimensional normed space $\mathbb{R}^n$ and hence bounded, i.e. there exist $A \geq -1$ such that

$$\|\Phi\mathbf{z}\| \leq (1 + A) \|\mathbf{z}\| \quad , \quad \mathbf{z} \in Z \subset \mathbb{R}^n \tag{21}$$

   where $1 + A \geq 0$ denotes the smallest upper bound.
4. Show that $A \leq \delta$ with high probability, by using the net and the measure concentration above. (This gives the upper bound in (14))

Using Modified Proof Technique in (Baraniuk et al., 2008January)

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5. Use the upper bound to show with the inverse triangular inequality the lower bound in (14), with same probability.

Circular Convolution Inequality

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Theorem (Reverse Young Inequality for sparse circular convolution)

Let s, f, n be integers s.t. $1 \leq s \leq f \leq sf < n$, then there exist $a = a(s, f) > 0$, s.t. for all $\mathbf{x} \in \Sigma_s$ and $\mathbf{y} \in \Sigma_f$ it holds

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Positive Symmetric Toeplitz Matrix vs. Positive Harmonic Functions

To find the lower bound $a(s,f)$ is an **NP-hard** problem. It is equivalent to a problem of Cathedory and Fejer (Caratheodory and Fejer, 1911)

$$a(f,f) := \min_{\mathbf{y} \in S^{f-1}} \lambda(\mathbf{B}_{\mathbf{y}}^{(f)}) = \min_{\mathbf{y} \in S^{f-1}} \min_{r<1} \min_{\omega \in [0,2\pi)} \left(1 + 2 \sum_{k=1}^{f-1} b_k(\mathbf{y}) r^k \cos(k\omega) \right) . \tag{23}$$

where the Laurent Polynomial coefficients are

$$b_k(\mathbf{y}) = \sum_{l=0}^{f-k-1} y_l y_{l+k} \tag{24}$$

Subexponential Decay

Numerically, we could show that $a(s,s)$ decays super exponentially as s^{-s}.

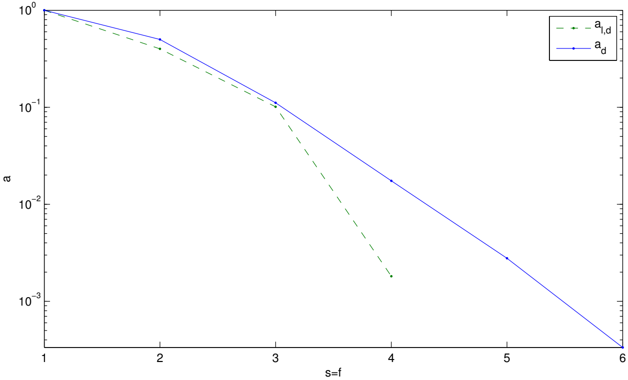

Figure: Approximation results of a for $s = f$.

Conclusions

Our Theorem shows a stable embedding of a signal set $\mathcal{M}$ if all differences $\Delta = \{\mathbf{z}_1 - \mathbf{z}_2 \mid \mathbf{z}_1, \mathbf{z}_2 \in \mathcal{M}\}$ are contained in the union of images of positive convex cones $X^+ \circledast Y^+$.

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  $\hookrightarrow$ Leads into an embedding (immersion theory) of each Z. I.e. we need to characterize the nullspace of T on each $X \times Y$.
- Find additional Restrictions to X and Y such that an useful $a(s, f)$ exists, e.g. demand decaying laws of the signals.

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Thank You!

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- By using the triangle inequality and using a zero addition $\mathbf{p} - \mathbf{p}$ and $\mathbf{q} - \mathbf{q}$ we have for an arbitrary $T(\mathbf{p}, \mathbf{q}) \in Q$ that all $\mathbf{z} \in T(X^\epsilon(\mathbf{p}), Y^\epsilon(\mathbf{q})) \cap \partial Z^1$ satisfy:

$$\left| \|\Phi \mathbf{z}\| - \|\Phi T(\mathbf{p}, \mathbf{q})\| \right| \leq \|\Phi(T(\mathbf{x} - \mathbf{p}, \mathbf{y} - \mathbf{q}))\| + \|\Phi(T(\mathbf{x} - \mathbf{p}, \mathbf{q}))\| + \|\Phi(T(\mathbf{p}, \mathbf{y} - \mathbf{q}))\| .$$

3.Step: Using Upper bounds for T and Φ

With the the universal bound $1 + A$ for Φ in (21) we obtain

$$\leq (1 + A)\big(\, \|T(\mathbf{p}, \mathbf{y} - \mathbf{q})\| + \|T(\mathbf{x} - \mathbf{p}, \mathbf{q})\| + \|T(\mathbf{x} - \mathbf{p}, \mathbf{y} - \mathbf{q})\|\,\big) \tag{26}$$

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The upper bound β of the RNMP for T in (12) gives

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4.Step: Using net properties

Since $\mathbf{p} \in X^R$ and $\mathbf{q} \in Y^R$ are ϵ-net points, i.e. $\|\mathbf{x} - \mathbf{p}\| \leq \epsilon$ and $\|\mathbf{y} - \mathbf{q}\| \leq \epsilon$, we get

$$\leq \beta(1 + A)\left(R\epsilon + R\epsilon + \epsilon^2\right) \tag{27}$$

$$\leq \beta(1 + A)(2R + \epsilon)\epsilon \overset{\epsilon \leq 1}{\leq} (1 + A)\beta(2R + 1)\epsilon. \tag{28}$$

If we define the constant

$$c = c(\alpha, \beta) := \beta(2R + 1) = \beta\left(2/\sqrt{\alpha} + 1\right) > 1, \tag{29}$$

we obtain the upper bound

$$\|\Phi \mathbf{z}\| \leq (1 + A)c\epsilon + \|\Phi T(\mathbf{p}, \mathbf{q})\|. \tag{30}$$

2.Step: Measure Concentration

Unfortunately, the nesting $r/R \leq \|T(\mathbf{p},\mathbf{q})\| \leq R/r$ is independent of $\epsilon(\delta)$ and hence not useful for establishing a δ-RIP. To obtain an useful nesting, we can use the continuity property (bilinarity) of T to upper and lower bound $\|T(\mathbf{p},\mathbf{q})\|$ in terms of ϵ for every Cartesian product of two convex covering sets $X^\epsilon(\mathbf{p})$, $Y^\epsilon(\mathbf{q})$.

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$$Z^{-1}(\mathbf{q},\mathbf{p}) := \{\, (\mathbf{x},\mathbf{y}) \in X^\epsilon(\mathbf{p}) \times Y^\epsilon(\mathbf{q}) \mid \|T(\mathbf{x},\mathbf{y})\| = 1 \}\,.$$

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$$\|T(\mathbf{p},\mathbf{q})\| = \|T(\mathbf{x}-\mathbf{c},\mathbf{y}-\mathbf{d})\| = \|T(\mathbf{x},\mathbf{y}) - T(\mathbf{x},\mathbf{d}) - T(\mathbf{c},\mathbf{y}) + T(\mathbf{c},\mathbf{d})\|. \quad (31)$$

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Let us discuss the discontinuity of this norm estimation. If we have $\alpha = \beta$, hence norm multiplicativity, then we would get $c = 3$. But in fact, this is to bad, since the shells are now unit spheres and every $\mathbf{p}, \mathbf{q}$ is normalized and hence by the norm multiplicativity $T(\mathbf{p}, \mathbf{q})$. But this gives $c = 0$. To respect this fact we define $\tilde{c}$ and get for all point pairs

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4.Step: Upper Bound for RIP

Then we can use the measure concentration in (19) to obtain with probability larger than in (20)

$$\|\Phi \mathbf{z}\| \leq (1 + A)c\epsilon + (1 + \delta/2)(1 + \tilde{c}\epsilon) \tag{33}$$
$$= 1 + Ac\epsilon + c\epsilon + \tilde{c}\epsilon + \delta(\tilde{c}\epsilon + 1)/2. \tag{34}$$

Now there exist a maximal $\mathbf{z}' \in \partial Z^1$ s.t. equality in (21) is achieved. Hence we get

$$A(1 - c\epsilon) \leq \frac{2c\epsilon + 2\tilde{c}\epsilon + \delta(\tilde{c}\epsilon + 1)}{2} \tag{35}$$
$$\Leftrightarrow A \leq \frac{2c\epsilon + \tilde{c}\epsilon(2 + \delta) + \delta}{2(1 - c\epsilon)}. \tag{36}$$

Let us proceed by case distinction. If $\alpha = \beta$ then $\tilde{c} = 0, c = 3$ and

$$A \leq \frac{3\epsilon + \frac{\delta}{2}}{1 - 3\epsilon}. \tag{37}$$

Defining $\epsilon = \frac{\delta}{12} \leq 1$ with $\delta \in (0, 1)$ we get

$$A \leq \frac{\frac{\delta}{4} + \frac{\delta}{2}}{1 - \frac{\delta}{4}} \overset{\delta<1}{\leq} \frac{\frac{3}{4}\delta}{\frac{3}{4}} = \delta. \tag{38}$$

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This upper bound holds with probability larger than

$$> 1 - 2N(X^R, X^{\frac{\delta}{d}})N(Y^R, Y^{\frac{\delta}{d}})e^{-c_0(\delta/2)M} \quad ,$$

with constant

$$\tilde{d} := \tilde{d}(\alpha, \beta) = \begin{cases} 7\beta(2/\sqrt{\alpha} + 1) & , \quad \alpha \neq \beta \\ 12 & , \quad \alpha = \beta \end{cases}. \tag{41}$$

5.Step: Lower Bound for RIP

The lower bound $1-\delta$ follows from this with

$$\|\Phi \mathbf{z}\| \geq \|\Phi T(\mathbf{p},\mathbf{q})\| - (1+A)c\epsilon \tag{42}$$

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Considering all $\mathbf{z} \in \partial Z^1$ we get by inserting (40) and (32) with same probability (41)

$$\|\Phi \mathbf{z}\| \geq \left(1 - \frac{\delta}{2}\right)\left(1 - \tilde{c}\frac{\delta}{\tilde{d}(\alpha, \beta)}\right) - (1 + \delta)\frac{c\delta}{\tilde{d}(\alpha, \beta)}. \tag{43}$$

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Case $\alpha=\beta$: Then $\tilde{c}=0$, $c=3$ and $\tilde{d}=12$. This gives

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Case $\alpha \neq \beta$: Then $\tilde{c} = c, \tilde{d} = 7c$ and

$$\|\Phi \mathbf{z}\| \geq 1 - \frac{\delta - c\epsilon\delta}{2} - c\epsilon - \frac{2c\delta}{7c} \geq 1 - \frac{1 - \frac{\delta}{7}}{2}\delta - \frac{\delta}{7} - \frac{2\delta}{7}$$
$$\overset{\delta > 0}{\geq} 1 - (7+6)\delta/14 \geq 1 - \delta.$$

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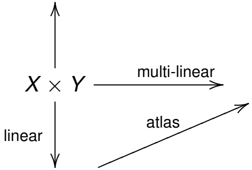
Since the covering number $N(X^R, X^\epsilon)$ remains the same if we scale both sets X^R, X^ϵ by $1/R=\sqrt{\alpha}$, (?Pis89, Lemma 4.16), we have finally granted the RIP with probability

$$> 1-2N(X^1, X^{\frac{\delta}{d}})N(Y^1, Y^{\frac{\delta}{d}})e^{-c_0(\frac{\delta}{2})M} \quad , \quad d := \sqrt{\alpha}\tilde{d} \qquad \square$$

Possible Structures for $X \times Y$



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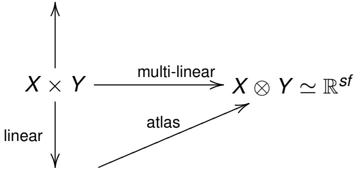

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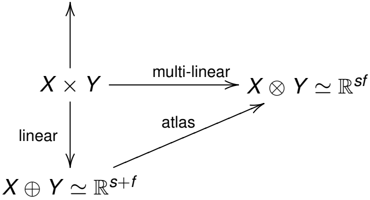

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