On the Decay - and the Smoothness Behavior of the Fourier Transform, and the Construction of Signals Having Strong Divergent Shannon Sampling Series

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Theorem 1 (Riemann-Lebesgue Lemma)
Let $f \in L^1(\mathbb{R})$. The Fourier transform $\hat{f}$ of f is continuous. Furthermore, it vanishes at infinity, in the sense that $\lim_{|\omega|\to\infty} \hat{f}(\omega) = 0$.

- Fourier transform (FT) and inverse Fourier transform (IFT) of $f \in L^1(\mathbb{R})$:

$$\begin{aligned} \hat{f}(\omega) &= \int_{-\infty}^{+\infty} f\ e^{-i\omega t} \mathrm{d}t \quad \text{(FT)} \\ \check{f}(t) &= \frac{1}{2\pi} \int_{-\infty}^{+\infty} f\ e^{i\omega t} \mathrm{d}\omega \quad \text{(IFT).} \end{aligned}$$

- Riemann-Lebesgue Lemma asserts regularity behaviour of FT (or equivalently IFT), viz. continuity and vanishability at infinity.
- Riemann-Lebesgue Lemma for Fourier series: Fourier coefficients of a w.l.o.g. 2π-periodic signals integrable on $[-\pi, \pi]$ approach with increasing absolute index zero.

Typical vague conclusions:

- "Time-limited signals are approximatively band-limited.[1]"
- "Periodic integrable signals can be approximated very well by a few dominant Fourier coefficients of low indexes."

The following question remains however unanswered:

- How "regular" can the FT of a signal f in $L^1(\mathbb{R})$ be?
  - How slow can $\hat{f}$ approach zero?
  - How is the continuity behaviour of $\hat{f}$?
- Is one able to specify the regularity behaviour of the FT of integrable signals, without any further specific constraints (e.g. differentiability, vanishability at infinity (Test functions, Schwartz functions))?

The following answers shall be given in this presentation:

- **"Typically/Generically"**, the FT of integrable signals (concentrated essentially on $[-t_g, t_g]$, $t_g > 0$) decay arbitrarily slowly.
- **"Typically/Generically"**, the FT of integrable signals have arbitrarily weak continuity behaviour at a given point $\omega \in \mathbb{R}$.

[1] $f \in L^1(\mathbb{R})$ is approximatively band-limited means that:

$$\forall \epsilon > 0 : \exists \omega_g(\epsilon) \in \mathbb{R}^+ : \quad \left|\hat{f}(\omega)\right| \leqslant \epsilon, \quad \forall \left|\omega\right| > \omega_g.$$

Table of contents

- An operator is here always a linear mapping between vector spaces.
- $\mathcal{X}_1, \mathcal{X}_2$ normed spaces, $\mathrm{T} : \mathcal{X}_1 \to \mathcal{X}_2$ operator

$$\|\mathrm{T}\| := \sup_{\|x\|_{\mathcal{X}_1} \neq 0} \frac{\|\mathrm{T}x\|_{\mathcal{X}_2}}{\|x\|_{\mathcal{X}_1}} = \sup_{\|x\|_{\mathcal{X}_1} \leqslant 1} \|\mathrm{T}x\|_{\mathcal{X}_2} = \sup_{\|x\|_{\mathcal{X}_1} = 1} \|\mathrm{T}x\|_{\mathcal{X}_2}.$$

- $\mathrm{T} : \mathcal{X}_1 \to \mathcal{X}_2$ (an operator!) is said to be bounded, if $\|\mathrm{T}\| < \infty$.
- Boundedness is equivalent with continuity for operators.
- Lebesgue spaces (Banach spaces!): $L^p(\mathbb{R})$, $L^p([-\sigma, \sigma])$, $\sigma > 0$ (w.l.o.g. $L^p([-\pi, \pi])$!), $p \in [1, \infty]$.

Definition 2 (Nowhere Dense Set, Set of 1. Category, Residual Set)

Let $\mathcal{B}$ be a Banach space, and $\mathcal{M} \subset \mathcal{B}$.

- $\mathcal{M}$ is nowhere dense if the inner of the closure of $\mathcal{M}$ is empty.
- $\mathcal{M}$ is said to be of 1. category (or meagre) if $\mathcal{M}$ can be represented by countable union of nowhere dense sets.
- $\mathcal{M}$ is residual if $\mathcal{M}$ is a complement of a set of 1. category.

- Nowhere dense set: Set "being perforated by holes"[2]
- Set of 1. Category: Set approximable by sets "being perforated by holes" - Small set in the topological sense
- Residual set - Large set in the topological sense.

[2] J. Oxtoby. *Measure and Category : A Survey of the Analogies between Topological and Measure Spaces*. Graduate texts in mathematics. New York: Springer-Verlag, 1980.

- Baire Category Thm. – "Those categorization of sets into large - and small subsets is non-trivial":
  - A Banach space is not of first category –
    "The whole Banach space can not be small"
  - Residual sets are dense and closed under countable intersections –
    Denseness alone is not sufficient for largeness of a set!
  - Meagre sets are closed under countable union –
    "Impossible: the whole Banach space is approximable by such small sets"

Definition 3 (Generic property)

Let $\mathcal{B}$ be a Banach space. A property in $\mathcal{B}$ is said to hold generically for (elements of) $\mathcal{B}$, if it holds for all elements in a residual subset of $\mathcal{B}$.

Notice: A generic property might not holds for all elements of a Banach space, but for "typical" ones.

The Following Thm.[3] is a central result in functional analysis:

Theorem 4 (Banach-Steinhaus Theorem)

Given a (possibly uncountable) family Φ *of bounded operators between Banach spaces* $\mathcal{B}_1$ *and* $\mathcal{B}_2$*. Suppose that* Φ *is pointwise bounded on whole* $\mathcal{B}_1$*, i.e* $\sup_{T\in\Phi} \|Tx\|_{\mathcal{B}_2} < \infty$*. Then the set* Φ *is uniform bounded w.r.t. to the operator norm, i.e.* $\sup_{T\in\Phi} \|T\|_{\mathcal{B}_1\to\mathcal{B}_2} < \infty$

Corollary 5

Let $\mathcal{B}_1$ *and* $\mathcal{B}_2$ *be Banach spaces. Given a (possibly uncountable) family* Φ *of bounded operators between* $\mathcal{B}_1$ *and* $\mathcal{B}_2$*. If it holds* $\sup_{T\in\Phi} \|T\|_{\mathcal{B}_1\to\mathcal{B}_2} = \infty$*, then there exists* $x \in \mathcal{B}_1$*, for which* $\sup_{T\in\Phi} \|Tx\|_{\mathcal{B}_1} = \infty$ *holds.*

[3] S. Banach and H. Steinhaus. "Sur le principe de la condensation de singularités". In: *Fund. Math.* 9 (1927), pp. 50–61.

It is even possible to sharpen previous Corollary as follows:

Corollary 6 (The Principle of Condensation of Singularity)

Let $\mathcal{B}_1$ and $\mathcal{B}_2$ be Banach spaces. Given a (possibly uncountable) family Φ of bounded operators between $\mathcal{B}_1$ and $\mathcal{B}_2$. If it holds $\sup_{\mathrm{T}\in\Phi} \|\mathrm{T}\|_{\mathcal{B}_1\to\mathcal{B}_2} = \infty$, then the set D, for which it holds $\sup_{\mathrm{T}\in\Phi} \|\mathrm{T}x_*\|_{\mathcal{B}_1} = \infty, \quad \forall x_* \in D$, is a residual set.

The principle of condensation of singularity gives a powerful - and relatively-easy-to-handle tool for proving divergence results:

- Rewrite the problem s.t. Banach spaces $\mathcal{B}_1$ and $\mathcal{B}_2$ (mostly $\mathcal{B}_2 = \mathbb{C}, \mathbb{R}$) and a Family of bounded operators Φ between $\mathcal{B}_1$ and $\mathcal{B}_2$ occurs in its reformulation.
- Show that $\sup_{\mathrm{T}\in\Phi} \|\mathrm{T}\|_{\mathcal{B}_1\to\mathcal{B}_2} = \infty$. Usually easier than observing the behaviour of $\|\mathrm{T}x\|_{\mathcal{B}_2}$, for every $\mathrm{T} \in \Phi$, and $x \in \mathcal{B}_1$:
  - Mostly, it is sufficient to give only a lower bound for $\|\mathrm{T}\|$, $\mathrm{T} \in \Phi$.
  - Computing $\|\mathrm{T}\|$ is still easier than computing $\|\mathrm{T}x\|_{\mathcal{B}_2}$, for every $\mathrm{T} \in \Phi$, and $x \in \mathcal{B}_1$.
- One can immediately infer that there exists not only an $x \in \mathcal{B}_1$ s.t. the divergence result holds, rather it holds for <u>typical</u> elements of $\mathcal{B}_1$.

Theorem 7

Let $\mathrm{M} : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ be an arbitrary monotonically non-decreasing function, with $\lim_{\omega\to\infty} \mathrm{M}(|\omega|) = +\infty$. Then there exists a function $f_* \in L^1(\mathbb{R})$, with $f_*(t) = 0$, for a.e. $|t| > \pi$, such that it holds:

$$\limsup_{|\omega|\to\infty} \mathrm{M}(|\omega|) \left|\hat{f}_*(\omega)\right| = \infty. \tag{1}$$

- M specifies a certain decay rate:
  - For logarithmic decay choose e.g. $\mathrm{M} = \log(\cdot + 1)$
  - For exponential decay choose e.g. $\mathrm{M} = e^{(\cdot)}$
- Thm. 7 says roughly that the Fourier transform of an integrable signal might decay asymptotically arbitrarily slowly
- Notice that the statement in Thm. 7 is in some sense weak, since it is given only by means of limes superior.
- Thm. 7 can easily be modified as to give a statement concerning to the IFT of $L^1(\mathbb{R})$ functions:
  "For a fixed asymptotic decay rate M, there exists a signal f, which is in turn the IFT of some $L^1(\mathbb{R})$-signals, failing to possess M as an comparable asymptotic decay rate. Furthermore, there is such a signal, which is in addition <u>almost</u> band-limited to $[-\pi, \pi]$."

- Problem reformulation:
  Find $f_* \in L^1([-\pi,\pi])$ s.t.:

$$\limsup_{|\omega|\to\infty} |\Psi_{\omega,\mathrm{M}} f_*| = \limsup_{|\omega|\to\infty} \mathrm{M}(|\omega|) \left|\hat{f}_*(\omega)\right| = \infty,$$

  where $\Psi_{\omega,\mathrm{M}} : L^1([-\pi,\pi]) \to \mathbb{C}$, $f \mapsto \mathrm{M}(|\omega|)\Psi_\omega(f) = \mathrm{M}(|\omega|)\widehat{f}(\omega)$, a bounded operator, $\forall \omega \in \mathbb{R}$, and $L^1([-\pi,\pi])$ and $\mathbb{C}$ Banach spaces.

- Let $\omega \in \mathbb{R}$ be arbitrary but fixed. Write $\Psi_{\omega,\mathrm{M}} = \Psi_\omega \mathrm{M}(|\omega|)$, where $\Psi_\omega : L^1([-\pi,\pi]) \to \mathbb{C}$, $f \to \int_{-\pi}^{\pi} f(t)e^{-i\omega t}\mathrm{d}t$. Subsequently, recognize to following norm behaviour of Ψ_ω:

$$\|\Psi_\omega\| = \sup_{\|f\|_{L^1([-\pi,\pi])}\leqslant 1} |\Psi_\omega f| = 1$$

  - $\|\Psi_\omega\| \leqslant 1$ is easy to see.
  - $\|\Psi_\omega\| \geqslant 1$ can be shown by means of the following sequence of functions of unit norm in $L^1(\mathbb{R})$:

$$f_n(t) := \begin{cases} n & t \in [-\frac{1}{2n}, \frac{1}{2n}] \\ 0 & \text{else}, \end{cases} \qquad n \in \mathbb{N}$$

- $\|\Psi_\omega\| = 1$, $\forall \omega \in \mathbb{R}$ asserts that for all $\omega \in \mathbb{R}$, the norm of $\Psi_{\omega,\mathrm{M}}$ is determined by M:
$$\|\Psi_{\omega,\mathrm{M}}\| = \mathrm{M}(|\omega|).$$
- $\sup_{\omega\in\mathbb{R}_0^+} \mathrm{M}(\omega) = \lim_{\omega\to\infty} \mathrm{M}(\omega) = \infty \quad \Longrightarrow \quad \sup_{\omega\in\mathbb{R}} \|\Psi_{\omega,\mathrm{M}}\| = \infty$
- By Banach-Steinhaus Thm. (Cor. 5): there exists a function $f_* \in L^1([-\pi,\pi])$ s.t. $\limsup_{|\omega|\to\infty} |\Psi_{\omega,\mathrm{M}} f_*| = \infty$, and correspondingly the desired statement.

Corollary 8

Let M *be an arbitrary function as described in Thm. 7. The set of all* $f \in L^1([-\pi, \pi])$*, for which:*

$$\limsup_{|\omega| \to \infty} \mathrm{M}(|\omega|) \left| \widehat{f}(\omega) \right| = \infty \qquad (2)$$

holds, is a residual set in $L^1([-\pi, \pi])$*, .*

Proof.

An immediate consequence of the principle of condensation of singularities. □

- Thm. 7 and Cor. 8 also holds in case $L^1([-\pi,\pi])$ is replaced by $L^1(\mathbb{R})$: Let M be an appropriate function specifying a certain decay rate. Then typically the FT of signal f in $L^1(\mathbb{R})$ fails to possess M as a decay rate.
- Thm. 7 and Cor. 8 also holds in case $L^1([-\pi,\pi])$ is replaced by $L^p([-\pi,\pi])$, where $p \in [1,\infty]$.
- Since FT and IFT is almost identical and by previous remark, Thm. 7 and Cor. 8 can be modified as to give statements concerning to the decay behaviour of the IFT integrable signals:
  Band-limited signals[4] $\mathcal{PW}_\pi^p$, $p \in [1,\infty]$ arbitrary, decays typically arbitrarily slowly.
- One may also to modify the statements in Thm. 7 and Cor. 8 as to give a description to the decay behaviour of the Fourier coefficients:
  Typically, the Fourier coefficients of a signal $f \in L^1(\mathbb{T})$ decays arbitrarily slowly toward 0.

[4] $\mathcal{PW}_{\omega_g}^p := \{f : f(t) = \int_{-\omega_g}^{\omega_g} \hat{f}(\omega)e^{it\omega}\mathrm{d}\omega,\ \hat{f} \in L^1([-\omega_g,\omega_g])\}$

- Recall Thm. 7:
  Let M be a function, which specifies a certain decay rate. Then one can find a function $f \in L^1(\mathbb{R})$, which is essentially defined on $[-\pi, \pi]$, for which it holds:
  $$\limsup_{|\omega|\to\infty} \mathrm{M}(|\omega|) \left|\widehat{f}(\omega)\right| = \infty,$$
  i.e. f decays slower than the rate specified by M.
- The statement given in Thm. 7 is rather weak, since it is given by means of limes superior, which ensures in this context only a sequence $\{\omega_n\} \subset \mathbb{R}$, for which $\lim_{n\to\infty} \mathrm{M}(|\omega_n|) \left|\widehat{f}(\omega_n)\right| = \infty$
- Does there exists a stronger statement, in the sense:
  For a given function M specifying a certain decay rate, can one find a function $f \in L^1(\mathbb{R})$, s.t.:
  $$\lim_{|\omega|\to\infty} \mathrm{M}(|\omega|) \left|\widehat{f}(\omega)\right| = \infty?$$

Theorem 9

Let $\mathrm{M} : \mathbb{R}_0^+ \to \mathbb{R}^+$ *be an arbitrary monotonically increasing continuous function, with* $\mathrm{M}(0) > 0$ *and* $\lim_{|\omega|\to\infty} \mathrm{M}(\omega) = +\infty$*. Then there exists a function* $f_* \in L^1(\mathbb{R})$*, for which it holds:*

$$\lim_{|\omega|\to\infty} \left|\widehat{f_*}(\omega)\right| \mathrm{M}(|\omega|) = +\infty.$$

Further, f_* *can be chosen, s.t.* $\widehat{f_*}$ *is real and non-negative.*

To proof Thm. 9, we first need the following construction:

Theorem 10

Let $\mathrm{G} : \mathbb{R}_0^+ \to \mathbb{R}^+$ *be an arbitrary a monotonically decreasing continuous function, for which* $\lim_{\omega\to\infty} \mathrm{G}(\omega) = 0$ *holds. Then there exists a function* $f \in L^1(\mathbb{R})$ *such that* $|\widehat{f}(\omega)| \geqslant \mathrm{G}(|\omega|)$*, for all* $\omega \in \mathbb{R}$*. Furthermore, the function* f *can be found s.t.* $\widehat{f}$ *is real and non-negative.*

- For $n \in \mathbb{N}$, consider the Fejèr kernel g_n, whose FT is given by:

$$\widehat{g_n}(\omega) = \begin{cases} 1 - \frac{|\omega|}{2^n} & |\omega| \leqslant 2^n \\ 0 & |\omega| > 2^n. \end{cases}$$

It is well-known that $\int_{-\infty}^{\infty} g_n(t)\mathrm{d}t = 1$, and therefore: $\|g_n\|_{L^1(\mathbb{R})} = 1$

- For each $n \in \mathbb{N}$, define the function (de la Valée Poussin Kernel) g_n^* by $g_n^* := 2g_{n+1} - g_n$. FT of g_n^*:

$$\widehat{g_n^*}(\omega) = \begin{cases} 1 & |\omega| \leqslant 2^n \\ 1 - \frac{|\omega - 2^n|}{2^{n+1} - 2^n} & 2^n < |\omega| \leqslant 2^{n+1} \\ 0 & |\omega| > 2^{n+1}. \end{cases} \quad (3)$$

By the triangle inequality, and $\|g_n\|_{L^1(\mathbb{R})} = 1$:

$$\|g_n^*\|_{L^1(\mathbb{R})} = \|2g_{n+1} - g_n\|_{L^1(\mathbb{R})} \leqslant 3, \quad \forall n \in \mathbb{N}.$$

- Take an arbitrary function G, which fulfills the requirement given in the Theorem. For each steps $k \in \mathbb{N}$, do the following:
  - choose $n_k \in \mathbb{N}$ sufficiently large enough, s.t.:

    $$n_k > n_{k-1} \quad \text{and} \quad \mathrm{G}(2^{n_k}) \leqslant \frac{1}{2}\mathrm{G}(2^{n_{k-1}}),$$

    where $n_0 = 0$. By induction,and by involving the fact that G vanishes at infinity, those choices is always possible.
  - Define the function:

    $$f_k := f_{k-1} + \mathrm{G}(2^{n_{k-1}})g^*_{n_k},$$

    where $f_0 := 0$. By induction:
    - $\left\{\hat{f}_n\right\}$ pointwise monotonically increasing sequence of real non-negative functions:

      $$\forall \omega \in \mathbb{R} : \quad \hat{f}_k(\omega) \geqslant \hat{f}_l(\omega), \quad \text{for } k \geqslant l.$$

    - $\left\{\hat{f}_n\right\}$ dominate G in some "increasing" intervals:

      $$\forall k \in \mathbb{N} : \quad \hat{f}_k(\omega) \geqslant \mathrm{G}(|\omega|), \quad \forall\, |\omega| \leqslant 2^{n_k} \tag{4}$$

- Notice that $\forall k \in \mathbb{N}$, f_k can be written as $f_k = \sum_{l=1}^{k} \mathrm{G}(2^{n_l}) g_{n_l}^{*}$. Thus for $k, k^{'} \in \mathbb{N}$, $k \geqslant k^{'}$:

$$\left\| f_k - f_{k'} \right\|_{L^1(\mathbb{R})} \leqslant \sum_{l=k^{'}+1}^{k} \mathrm{G}(2^{n_l}) \left\| g_{n_l}^{*} \right\|_{L^1(\mathbb{R})} \leqslant 3\mathrm{G}(0) \sum_{l=k^{'}+1}^{k} \tfrac{1}{2^l},$$

- From above computations, and From the fact that the series $\sum_{k=1}^{\infty} 1/2^k$ converges, one can easily conclude, that $\{f_k\}$ is a Cauchy sequence in $L^1(\mathbb{R})$, and accordingly, completeness of $L^1(\mathbb{R})$ asserts the existence of $f \in L^1(\mathbb{R})$, for which $\lim_{n\to\infty} f_n = f$ w.r.t. the norm of $L^1(\mathbb{R})$.
- Continuity of the FT seen as an element of operator between $L^1(\mathbb{R})$ and $C_0(\mathbb{R})$[5], it follows that $\widehat{f_k}$ converges uniformly to $\widehat{f}$, and clearly also pointwise.
- Let $\omega \in \mathbb{R}$ be arbitrary but fixed. There exists of course an $k_0 \in \mathbb{N}$, s.t. $|\omega| \leqslant 2^{n_{k_0}}$. From the fact that $\{\widehat{f_n}(\omega)\}_n$ is monotonically increasing, $\widehat{f_n}(\omega)$ converges to $\widehat{f}(\omega)$, and (4), we have as desired $\widehat{f}(\omega) \geqslant \widehat{f_{k_0}}(\omega) \geqslant \mathrm{G}(|\omega|)$.

[5] $C_0(\mathbb{R})$ denotes the space of continuous functions f vanishing at infinity, i.e. $\lim_{x\to\infty} f(x) = 0$

- Choose a function M, which fulfills the requirements given in Thm. 9, i.e. $\mathrm{M} : \mathbb{R}_0^+ \to \mathbb{R}^+$ monotonically increasing continuous function, with $\mathrm{M}(0) > 0$ and $\lim_{|\omega| \to \infty} \mathrm{M}(\omega) = +\infty$.
- Define another function $\mathrm{G} := 1/\sqrt{\mathrm{M}}$. Notice that G fulfills the requirements given in Thm. 10, i.e. $\mathrm{G} : \mathbb{R}_0^+ \to \mathbb{R}^+$ is monotonically decreasing, continuous, and fulfills $\lim_{\omega \to \infty} \mathrm{G}(\omega) = 0$.
- By Thm. 10, there exists a function f_*, for which :

$$\left|\widehat{f_*}(\omega)\right| \geqslant \mathrm{G}(|\omega|) = \frac{1}{\sqrt{\mathrm{M}(|\omega|)}}, \quad \forall \omega \in \mathbb{R},$$

and accordingly:

$$\left|\widehat{f_*}(\omega)\right| \mathrm{M}(|\omega|) \geqslant \sqrt{\mathrm{M}(|\omega|)}, \quad \forall \omega \in \mathbb{R},$$

which gives the desired statement $\lim_{|\omega| \to \infty} \left|\widehat{f_*}(\omega)\right| \mathrm{M}(|\omega|) = \infty$.

- Since FT and IFT are almost identical, we modify Thm. 10 and Thm. 9 as follows:
  - Let $\mathrm{G} : \mathbb{R}_0^+ \to \mathbb{R}^+$ be a monotonically decreasing continuous function, for which $\lim_{\omega\to\infty} \mathrm{G}(\omega) = 0$. Then there exists an $f \in L^1(\mathbb{R})$, for which it holds:
  $$\left|\check{f}(t)\right| \geqslant \mathrm{G}(|t|), \quad \forall t \in \mathbb{R}.$$
  Further, f can be chosen s.t. $\check{f}$ is real and non-negative.
  - Let $\mathrm{M} : \mathbb{R}_0^+ \to \mathbb{R}^+$ be an arbitrary monotonically increasing continuous function, with $\mathrm{M}(0) > 0$ and $\lim_{|\omega|\to\infty} \mathrm{M}(\omega) = +\infty$. Then there exists a function $f \in L^1(\mathbb{R})$, for which it holds:
  $$\lim_{|\omega|\to\infty} \left|\check{f}(\omega)\right| \mathrm{M}(|\omega|) = +\infty.$$
  Further, f can be chosen, s.t. $\check{f}$ is real and non-negative.
- In contrast to Corollary 8, for a fixed choice of M, the set of functions $f \in L^1(\mathbb{R})$ for which $\lim_{|\omega|\to\infty} \left|\check{f}(\omega)\right| \mathrm{M}(|\omega|) = +\infty$ holds, is not a residual set in $L^1(\mathbb{R})$. Rather, it is only a small set, i.e. of first category, in $L^1(\mathbb{R})$, as asserted in[6].

[6] H. Boche and U. J. Mönich. "A General Approach for Convergence Analysis of Adaptive Sampling-Based Signal Processing". In: *Sampling Theory and Applications (SampTA), 2015 International Conference on*. 2015, pp. 211–215.

- Strong divergence of the Shannon sampling series (SSS)[7]:

Theorem 11 (Boche,Farrell)

For each $\omega_g > 0$, there exists an $f_{\omega_g} \in \mathcal{PW}^1_{\omega_g}$ possessing a strong divergence SSS, i.e.:

$$\lim_{N\to\infty} \sup_{t\in\mathbb{R}} \left| \frac{1}{\omega_g} \sum_{k=-N}^{N} f_{\omega_g}\left(\frac{k\pi}{\omega_g}\right) \frac{\sin(\omega_g(t-\frac{k\pi}{\omega_g}))}{t-\frac{k\pi}{\omega_g}} \right| = \infty. \quad (5)$$

- Paley-Wiener space $\mathcal{PW}^p_{\omega_g}$, $p \in [1,\infty]$, $\omega_g > 0$ - "Banach space of signals band-limited to ω_g":

$$\mathcal{PW}^p_{\omega_g} := \left\{ f: \ f(t) = \int_{-\omega_g}^{\omega_g} \widehat{f}(\omega) e^{it\omega} \mathrm{d}\omega, \ \widehat{f} \in L^p([-\omega_g,\omega_g]) \right\}$$

[7] H. Boche and B. Farrell. "Strong divergence of reconstruction procedures for the Paley-Wiener space $\mathcal{PW}^1_\pi$ and the Hardy space $\mathcal{H}^1$". In: *Journal of Approximation Theory* 183 (2014), pp. 98–117.

- Convergence[8] of the SSS of functions in $\mathcal{PW}^1_{\omega_g}$ on compact subsets: For any $\omega_g > 0$, it holds for every $f \in \mathcal{PW}^1_{\omega_g}$:

$$\lim_{N\to\infty} \sup_{t\in K} \left| f(t) - \frac{1}{\omega_g} \sum_{k=-N}^{N} f_{\omega_g}\left(\frac{k\pi}{\omega_g}\right) \frac{\sin(\omega_g(t-\frac{k\pi}{\omega_g}))}{t-\frac{k\pi}{\omega_g}} \right| = 0,$$

where $K \subset \mathbb{R}$ is a compact subset.

- Weak divergence[9] of the SSS of functions in $\mathcal{PW}^1_{\omega_g}$: For any $\omega_g > 0$, it holds that the set of f for which:

$$\limsup_{N\to\infty} \sup_{t\in\mathbb{R}} \left| \frac{1}{\omega_g} \sum_{k=-N}^{N} f_{\omega_g}\left(\frac{k\pi}{\omega_g}\right) \frac{\sin(\omega_g(t-\frac{k\pi}{\omega_g}))}{t-\frac{k\pi}{\omega_g}} \right| = \infty,$$

holds is a residual set in $\mathcal{PW}^1_{\omega_g}$

[8] J. L. Brown. "On the error in reconstructing a nonbandlimited function by means of the bandpass sampling theorem". In: *Journal of Mathematical Analysis and Applications* 18 (1967), pp. 75–84.

[9] H. Boche and U. J. Mönich. "There exists no globally uniformly convergent reconstruction for the Paley-Wiener space $\mathcal{PW}^1_\pi$ of bandlimited functions sampled at Nyquist rate". In: *IEEE Trans. Signal Process.* 56.7 (2008), 31703179.

- Strong divergence of SSS (5) for $f_{\omega_g} \in \mathcal{PW}^1_{\omega_g}$ is stronger than the previous statement, since it abnegates the existence of a subsequence $\{N_k\}_k \subset \mathbb{N}$, for which the following holds:

$$\limsup_{k\to\infty} \sup_{t\in\mathbb{R}} \left| \frac{1}{\omega_g} \sum_{k=-N_k}^{N_k} f_{\omega_g}\left(\frac{k\pi}{\omega_g}\right) \frac{\sin(\omega_g(t-\frac{k\pi}{\omega_g}))}{t-\frac{k\pi}{\omega_g}} \right| < \infty,$$

and hence the possibility that above expression convergence to f_{ω_g}.

- Thm. 10 allows us to give an alternative stronger proof of the strong divergence of SSS given in[10]:
  There exists a "universal" function $f \in L^1(\mathbb{R})$, s.t. for every $\omega_g > 0$, one can construct by means of f another function f_* (depends on ω_g!), whose band-limited interpolation[11] $\check{f}_{*,\omega_g} \in \mathcal{PW}^1_{\omega_g}$ of its IFT $\check{f}_*$ has a strong divergence SSS, i.e.:

$$\lim_{N\to\infty} \sup_{t\in\mathbb{R}} \left| \frac{1}{\omega_g} \sum_{k=-N}^{N} \check{f}_{*,\omega_g}\left(\frac{k\pi}{\omega_g}\right) \frac{\sin\left(\omega_g\left(t-\frac{k\pi}{\omega_g}\right)\right)}{t-\frac{k\pi}{\omega_g}} \right| = \infty. \tag{6}$$

[10] H. Boche and B. Farrell. "Strong divergence of reconstruction procedures for the Paley-Wiener space $\mathcal{PW}^1_\pi$ and the Hardy space $\mathcal{H}^1$". In: *Journal of Approximation Theory* 183 (2014), pp. 98–117.

[11] In this context, the band-limited interpolation of a function f, with band-limit $\omega_g > 0$, is the function (in case it is well-defined!) $f_{\omega_g} \in \mathcal{PW}^1_{\omega_g}$, for which $f\left(\frac{k\pi}{\omega_g}\right) = f_{\omega_g}\left(\frac{k\pi}{\omega_g}\right)$, $k \in \mathbb{Z}$.

An Alternative Proof of the Strong Divergence of the Shannon Sampling Series for Band-Limited Functions (Sketch)

- First suppose that we have a function $\mathrm{G} : \mathbb{R}_0^+ \to \mathbb{R}^+$ fulfilling the conditions given in Thm. 10 (we shall soon discuss about the choice of such function), for which the following holds:

$$\lim_{N\to\infty} \sum_{k=0}^{N} \mathrm{G}\left(\frac{k\pi}{\omega_g}\right) \frac{1}{N+\frac{1}{2}-k} = \infty, \quad \forall \omega_g \in \mathbb{R}^+. \tag{7}$$

- Thm. 10, and the fact that FT and IFT is almost identical, assert that we can find a function $f \in L^1(\mathbb{R})$, whose IFT is real and non-negative, and fulfills $\check{f}(t) \geqslant \mathrm{G}(|t|)$, for all $t \in \mathbb{R}$. In particular, we have for every $\omega_g \in \mathbb{R}^+$:

$$\check{f}\left(\frac{k\pi}{\omega_g}\right) \geqslant \mathrm{G}\left(\left|\frac{k\pi}{\omega_g}\right|\right), \quad \forall k \in \mathbb{Z}. \tag{8}$$

- Now, let $\omega_g \in \mathbb{R}^+$ be fixed. Define another function f_* by:

$$f_*(\omega) := f(\omega + \omega_g), \quad \forall \omega \in \mathbb{R}.$$

It is not hard to see that $f_* \in L^1(\mathbb{R})$, and that the following holds:

$$\check{f}_*\left(\frac{k\pi}{\omega_g}\right) = (-1)^k \check{f}\left(\frac{k\pi}{\omega_g}\right), \quad k \in \mathbb{Z}. \tag{9}$$

- Of course we can give the band-limited interpolation of $\check{f}_*$, i.e. the function $\check{f}_{*,\omega_g} \in \mathcal{PW}^1_{\omega_g}$, for which it holds:

$$\check{f}_{*,\omega_g}\left(\frac{k\pi}{\omega_g}\right) = \check{f}_*\left(\frac{k\pi}{\omega_g}\right), \tag{10}$$

  by setting: $f_{*,\omega_g}(\omega) := \sum_{k=-\infty}^{+\infty} f_*(\omega + 2\omega_g k)$, $\forall\, |\omega| \leqslant \omega_g$, and 0 else (see e.g.[12]).
- By some efforts involving (9), one can give explicitly the behaviour of the SSS of $\check{f}_*$, and resp. $\check{f}_{*,\omega_g}$ by (10), at the time instances $\tilde{t}_N := t_N(\pi/\omega_g)$, where $t_N := (N + (1/2))$, $N \in \mathbb{N}$:

$$\left| \frac{1}{\omega_g} \sum_{k=-N}^{N} \check{f}_{*,\omega_g}\left(\frac{k\pi}{\omega_g}\right) \frac{\sin(\omega_g(\tilde{t}_N - \frac{k\pi}{\omega_g}))}{\tilde{t}_N - \frac{k\pi}{\omega_g}} \right| \geqslant \frac{1}{\pi} \sum_{k=0}^{N} \frac{\mathrm{G}(\frac{k\pi}{\omega_g})}{N + \frac{1}{2} - k},$$

- Collecting all the previous observations, and by assumption (7), it is not hard to see that (6) holds, as desired.

[12] Holger Boche and Ezra Tampubolon. "On the Existence of the Band-Limited Interpolation of a Non-Band-Limited Signals". In: *In Preparation* (2015).

- Now, it remains to construct the function G, for which (7) holds. Notice that it is sufficient to require that:

$$\forall \omega_g \in \mathbb{R}^+ : \quad \lim_{N \to \infty} \mathrm{G}\left(\frac{N\pi}{\omega_g}\right) \log(N+2) = \infty.$$

For instance, the function G given by:

$$\mathrm{G}(t) = \begin{cases} 1 & 0 \leqslant t \leqslant 10 \\ \frac{\log(\log(10))}{\log(\log(t))} & t > 10, \end{cases}$$

fulfills above condition and hence (7).

- The 2. Statement of the Riemann-Lebesgue's Lemma:
  The Fourier transform of an integrable function is continuous
- It is possible to specify the continuity/smoothness behaviour of the FT?
- (Local) modulus of continuity:
  Let $f : \mathbb{R} \to \mathbb{C}$ be continuous. A continuous monotonically increasing function $\gamma_{g,\omega} : \mathbb{R}^+ \to \mathbb{R}^+$ vanishing at 0 is said to be a *(local) modulus of continuity (MOC)* of g at ω, if it holds:

$$\forall h > 0 : \quad |g(\omega + h) - g(\omega)| \leqslant \gamma_{g,\omega}(|h|).$$

- In some sense, MOC specifies the continuity behaviour of the function g at ω, and gives a (microscopic) measure on the smoothness of g at the point ω

The 2. statement of the Riemann-Lebesgue Lemma can be specified as follows:

Theorem 12

Let $\mu : \mathbb{R}^+ \to \mathbb{R}^+$ an arbitrary monotonically increasing continuous function, with $\mu(0) := \lim_{h\to 0+} \mu(h) = 0$. Given an arbitrary point $\omega_* \in \mathbb{R}$. Then the set of all $f \in L^1(\mathbb{R})$, for which:

$$\limsup_{h\to 0} \frac{\left|\hat{f}(\omega_* + h) - \hat{f}(\omega_*)\right|}{\mu(h)} = +\infty,$$

holds, is a residual set.

In other words:

Given a frequency $\omega_* \in \mathbb{R}$, and a function μ satisfying the conditions in Thm. 12. Then every functions in $L^1(\mathbb{R})$ have generically a FT, which does not admit μ as the modulus of continuity at ω_*

- For fixed $\omega \in \mathbb{R}$, we have:

$$\hat{f}(\omega_* + h) - \hat{f}(\omega_*) := \int_{-\infty}^{+\infty} f(t) e^{-i\omega_* t}(e^{iht} - 1)\mathrm{d}t, \quad h > 0$$

- We aim to analyze for $\omega_* \in \mathbb{R}$ and $h > 0$, the behaviour of the functional $\Psi_{\omega_*,h} : L^1(\mathbb{R}) \to \mathbb{C}$, given by:

$$\Psi_{\omega_*,h} f := \int_{-\infty}^{+\infty} f(t) e^{-i\omega_* t}(e^{iht} - 1)\mathrm{d}t.$$

  - Now, for $c \in \mathbb{R}^+$, define the function f_c, by $f_c(t) := ce^{i\omega_* t}$, for $|t| \leqslant 1/2c$, and $f_c(t) := 0$ else.
  - By simple computations, one obtains:

$$\Psi_{\omega_*,h} f_c = \left[\frac{\sin\left(\frac{h}{2c}\right)}{\frac{h}{2c}} - 1 \right].$$

  - For a fixed choice of $h > 0$, set $c_* = h/2\pi$, which yields the estimation $\left|\Psi_{\omega_*,h} f_{c_*}\right| = 1$, implying:

$$\left\|\Psi_{\omega_*,h}\right\| \geqslant 1, \quad \forall h \in \mathbb{R}^+ \tag{11}$$

- Now let μ be an arbitrary function fulfilling the requirements given in Thm. 12. Define by this choice the functional $\Psi_{\omega_*,h,\mu}$ on $L^1(\mathbb{R})$ by:

$$\Psi_{\omega_*,h,\mu} f := \frac{\hat{f}(\omega_* + h) - \hat{f}(\omega_*)}{\mu(h)} = \frac{\Psi_{\omega_*,h} f}{\mu(h)}.$$

- From (11), we have $\left\|\Psi_{\omega_*,h,\mu}\right\| \geqslant 1/\mu(h)$, and correspondingly:

$$\lim_{h\to 0} \left\|\Psi_{\omega_*,h,\mu}\right\| \geqslant \lim_{h\to 0} \frac{1}{\mu(h)} = +\infty.$$

Thus $\sup_{h>0} \left\|\Psi_{\omega_*,h,\mu}\right\| = +\infty$, and correspondingly by cor. 8, we obtain the desired result.

Theorem 13

Let μ be a function fulfilling the requirements given in Theorem 12. The set $\mathcal{D}_\mu$ of all $f \in L^1(\mathbb{R})$, such that the set:

$$\mathcal{P}_{Div}^{(\mu)}(f) := \left\{ \omega \in \mathbb{R} : \quad \limsup_{h \to 0} \frac{\left| \widehat{f}(\omega + h) - \widehat{f}(\omega) \right|}{\mu(h)} = +\infty \right\}$$

is a residual set in $\mathbb{R}$, forms a residual set in $L^1(\mathbb{R})$.

In other words:

"Given an appropriate function μ. Typically, signals in $L^1(\mathbb{R})$ fails to possess μ as a modulus of continuity at typical points on the real line."

Conclusions - Tightening of Riemann-Lebesgue lemma

- The Fourier transform (resp. the inverse Fourier transform) of an integrable function (also $L^p([-\pi, \pi])$, $p \in [1, \infty]$ arbitrary) can decay arbitrarily slowly.
- Tightening: The Fourier transform (resp. the inverse Fourier transform) of an integrable function (also $L^p([-\pi, \pi])$, $p \in [1, \infty]$ arbitrary) typically decay arbitrarily slowly –
- Important conclusion: Band-limited signals $\mathcal{PW}_\pi^p$, $p \in [1, \infty]$, typically decay arbitrarily slowly.

$\lightning$ The statements is only given weakly by means of the limes superior.

- Tightening: It is possible to construct a function in $L^1(\mathbb{R})$, whose FT decays strongly slower than a certain decay rate.
- But: The set of signals $f \in L^1(\mathbb{R})$ whose FT decaying strongly slower than a certain decay rate might be a negligible set in $L^1(\mathbb{R})$.
- The corresponding construction gives a stronger proof of the strong divergence of the Shannon's sampling series for band-limited signals $\mathcal{PW}^1_{\omega_g}$, where $\omega_g > 0$.

- The Fourier transform (resp. the inverse Fourier transform) of an integrable might possesses arbitrarily weak continuous behaviour.

↯ The statement is only given weakly by means of the limes superior.

- Tightening: The Fourier transform (resp. the inverse Fourier transform) of an integrable function typically possesses arbitrarily weak continuous behaviour.
- Further tightening: The Fourier transform (resp. the inverse Fourier transform) of an integrable typically possesses arbitrarily weak continuous behaviour on typical points on the real line.

Thank you!