Peak Value Blowup of Approximations of the Hilbert Transform of Signals with Finite Energy

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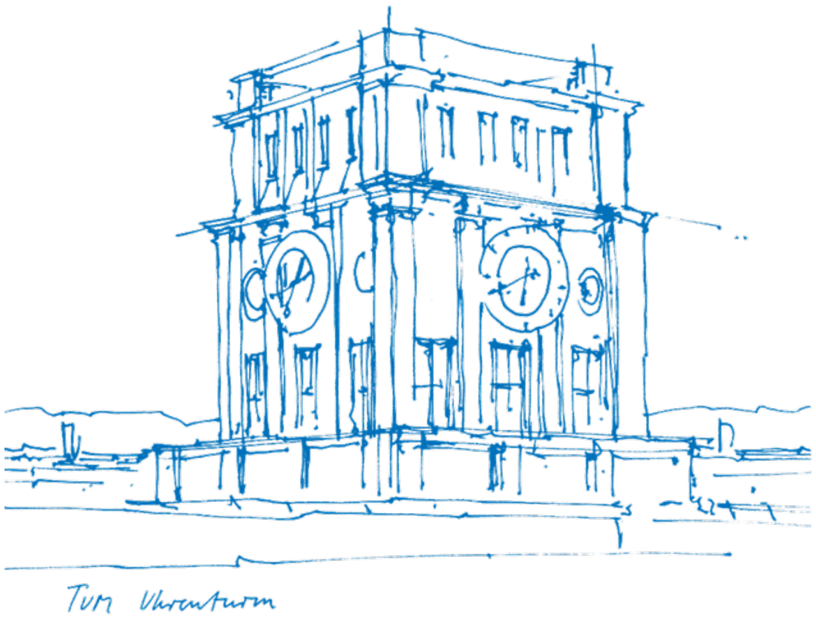

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Introduction

We consider the problem of calculating numerically the (finite) Hilbert transform

$$\widetilde{f}(\mathrm{e}^{\mathrm{i}\theta}) = (\mathrm{H}f)(\mathrm{e}^{\mathrm{i}\theta}) = \lim_{\varepsilon\to 0} \frac{1}{2\pi} \int_{\varepsilon\leq|\theta-\tau|\leq\pi} \frac{f(\mathrm{e}^{\mathrm{i}\tau})}{\tan([\theta-\tau]/2)}\,\mathrm{d}\tau\,, \qquad \theta\in[-\pi,\pi)\,. \tag{HT}$$

- This transformation plays an important role in science and engineering.
- H is also known as Kramers-Kronig relation.
- It is related to causality:
  - The real and imaginary part of a causal signal is related by the Hilbert transform.
  - The phase of a causal signal is determined by its amplitude.
  - Prediction and estimation of stationary time series – spectral factorization.

Challenges

- Singular integral kernel $\Rightarrow$ principal value integral in (**??**)
- Calculation on digital computers $\Rightarrow$ calculation of (**??**) has to be based on finitely many samples $\{f(\mathrm{e}^{\mathrm{i}\theta_n})\}_{n=1}^{N}$ of the function f

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Hilbert Transform Approximations

Hilbert Transform: $$\widetilde{f}(e^{i\theta}) = (Hf)(e^{i\theta}) = \lim_{\varepsilon\to 0} \frac{1}{2\pi} \int_{\varepsilon \le |\theta-\tau| \le \pi} \frac{f(e^{i\tau})}{\tan([\theta-\tau]/2)}\, d\tau\,, \qquad \theta \in [-\pi,\pi)\,. \quad \text{(HT)}$$

- Given a sequence of discrete sampling sets:

$$Z_N = \{\zeta_1, \zeta_2, \dots, \zeta_N\} \subset \mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}\,, \qquad N \in \mathbb{N}\,.$$

- Design a sequence $\{H_N\}_{N=1}^{\infty}$ of bounded linear operators H_N (each H_N is concentrated on Z_N) such that

$$\lim_{N\to\infty} \left\| H_N f - Hf \right\|_{\infty} = \lim_{N\to\infty} \max_{\theta\in[-\pi,\pi)} \left| (H_N f)(e^{i\theta}) - (Hf)(e^{i\theta}) \right|_{\infty} = 0 \qquad \text{for all } f \in \mathscr{B}\,,$$

wherein $\mathscr{B}$ is our signal space (which has to be specified).

Question

For which signal spaces $\mathscr{B}$ we can always find such approximation sequences $\{H_N\}_{N\in\mathbb{N}}$?

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The Hilbert Transform on $L^2(\mathbb{T})$

- Let $f \in L^2(\mathbb{T})$ be a square integrable function on the *unit circle* $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$.
- f can be represented by its Fourier series

$$f(\mathrm{e}^{\mathrm{i}\theta}) = \sum_{n=-\infty}^{\infty} c_n(f)\,\mathrm{e}^{\mathrm{i}n\theta} \qquad \text{with Fourier coefficients} \qquad c_n(f) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\mathrm{e}^{\mathrm{i}\tau})\,\mathrm{e}^{-\mathrm{i}n\tau}\,\mathrm{d}\tau$$

- Its harmonic *conjugate* $\widetilde{f}$ is given by the Hilbert transform of f

$$\widetilde{f}(\mathrm{e}^{\mathrm{i}\theta}) = (\mathrm{H}f)(\mathrm{e}^{\mathrm{i}\theta}) = -\mathrm{i}\sum_{n=-\infty}^{\infty} \mathrm{sgn}(n)\,c_n(f)\,\mathrm{e}^{\mathrm{i}n\theta} \qquad \text{with} \qquad \mathrm{sgn}(n) = \begin{cases} -1 & : \; n < 0 \\ 0 & : \; n = 0 \\ 1 & : \; n > 0 \end{cases}$$

Properties

- The Hilbert transform is a bounded mapping $\mathrm{H} : L^p(\mathbb{T}) \to L^p(\mathbb{T})$, $1 < p < \infty$.
- The Hilbert transform is a bounded mapping $\mathrm{H} : L^\infty \to BMO$.
- For $f \in \mathscr{C}(\mathbb{T})$, we have $\widetilde{f} = \mathrm{H}f \in L^p(\mathbb{T})$ for every $1 \leq p < \infty$ but $\widetilde{f} = \mathrm{H}f \notin \mathscr{C}(\mathbb{T})$, in general.

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Example of a Hilbert Transform Approximation

- For every $N \in \mathbb{N}$, we consider the equidistant sampling set

$$Z_N = \left\{ \zeta_{k,N} = \mathrm{e}^{\mathrm{i}\pi k/N} \,:\, k = 0,1,\dots,2N-1 \right\}$$

- First, we approximate $f \in L^2(\mathbb{T})$ by its partial Fourier series

$$\big(\mathrm{D}_N f\big)(\mathrm{e}^{\mathrm{i}\theta}) = \sum_{n=-N+1}^{N-1} c_{n,N}(f)\,\mathrm{e}^{\mathrm{i}n\theta}$$

but where we exchanged the exact Fourier coefficients $c_n(f)$ for approximations $c_{n,N}(f)$.

- The approximations $c_{n,N}(f)$ obtained by replacing the integral in the formula for the Fourier coefficients with the left Riemann sum with nodes Z_N.

$$c_n(f) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\mathrm{e}^{\mathrm{i}\tau})\,\mathrm{e}^{-\mathrm{i}n\tau}\,\mathrm{d}\tau \qquad \mapsto \qquad c_{n,N}(f) = \tfrac{1}{2N}\sum_{k=0}^{2N-1} f\big(\zeta_{k,N}\big)\,\mathrm{e}^{-\mathrm{i}\pi nk/N}$$

- To get an approximation of $\widetilde{f} = \mathrm{H}f$, we apply H to the trigonometric polynomial $\mathrm{D}_N f$

$$\big(\widetilde{\mathrm{D}}_N f\big)(\mathrm{e}^{\mathrm{i}\theta}) := \big(\mathrm{H}\mathrm{D}_N f\big)(\mathrm{e}^{\mathrm{i}\theta}) = -\mathrm{i}\sum_{n=-(N-1)}^{N-1} \operatorname{sgn}(n)\,c_{n,N}(f)\,\mathrm{e}^{\mathrm{i}n\theta} = \sum_{k=0}^{2N-1} f\big(\zeta_{k,N}\big)\,\widetilde{\mathscr{D}}_N\Big(\theta - k\frac{\pi}{N}\Big)\,.$$

with the kernel $\widetilde{\mathscr{D}}_N(\theta) = \frac{1}{N}\sum_{n=1}^{N-1}\sin(n\theta)$.

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Problem Statement

The sequence $\{\widetilde{\mathrm{D}}_N\}_{N\in\mathbb{N}}$ satisfies

$$\lim_{N\to\infty} \left\| \widetilde{\mathrm{D}}_N f - \mathrm{H} f \right\|_{L^2(\mathbb{T})} = 0 \qquad \text{for all} \qquad f \in L^2(\mathbb{T})\,.$$

Questions

- For which subset $\mathscr{B} \in L^2(\mathbb{T})$ do we even have

$$\lim_{N\to\infty} \left\| \widetilde{\mathrm{D}}_N f - \mathrm{H} f \right\|_{\infty} = 0 \qquad \text{for all} \qquad f \in \mathscr{B}\,.$$

- More general: For which spaces $\mathscr{B} \subset L^2(\mathbb{T})$ is it possible to find sequences of bounded linear operators $\{\mathrm{H}_N\}_{N\in\mathbb{N}}$ such that

$$\lim_{N\to\infty} \left\| \mathrm{H}_N f - \mathrm{H} f \right\|_{\infty} = 0 \qquad \text{for all} \qquad f \in \mathscr{B}\,.$$

- Which properties of $\{\mathrm{H}_N\}_{N\in\mathbb{N}}$ are necessary/sufficient for convergence on $\mathscr{B}$?

Uniform Norm

- to control peak value of the approximation: hardware requirements (dynamic range)
- H^∞-control, stability $L^2(\mathbb{T}) \to L^2(\mathbb{T})$

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Outline of the Paper

1. We introduce a scale of Banach space $\{\mathscr{B}^s\}_{s\geq 0}$ of continuous functions of finite energy.
   - These are „good“ for the Hilbert transform.
   - The parameter $s \geq 0$ characterizes the energy concentration of the signals.
2. We introduce a class of sampling based Hilbert transform approximations $\{\mathrm{H}_N\}_{N\in\mathbb{N}}$.
   - This class is characterizes by two simple axioms.
   - This class contains basically all practically relevant approximation methods.
3. Divergence results for the spaces $\mathscr{B}^s$ with $s \leq 1/2$.
   - For these spaces, there exists no Hilbert transform approximation in our class.
4. Convergence results for spaces $\mathscr{B}^s$ with $s > 1/2$.
   - For these spaces, there always exist a Hilbert transform approximation in our class.
   - Simple examples of convergent methods can be found.

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Signal Spaces of Finite Energy

Space of all continuous functions $f \in \mathscr{C}(\mathbb{T})$ with a continuous conjugate $\widetilde{f}$

$$\mathscr{B} = \left\{ f \in \mathscr{C}(\mathbb{T}) : \widetilde{f} = \mathrm{H}f \in \mathscr{C}(\mathbb{T}) \right\} \qquad \text{with norm} \qquad \|f\|_{\mathscr{B}} = \max\left(\|f\|_\infty, \|\mathrm{H}f\|_\infty\right)$$

$L^2(\mathbb{T})$ subpaces with energy concentration – Sobolev spaces

For $s \geq 0$, we define

$$W^{s,2} = \left\{ f \in L^2(\mathbb{T}) \ : \ \sum_{n \in \mathbb{Z}} |n|^{2s} |c_n(f)|^2 < \infty \right\} \quad \text{with} \quad \|f\|_{s,2} = \left(|c_0(f)|^2 + \sum_{n=-\infty}^{\infty} |n|^{2s} |c_n(f)|^2 \right)^{1/2} .$$

- $s \geq 0$ characterizes the smoothness of the functions $f \in W^{s,2}$: As larger s as smoother f.
- For $s > 1/2$, one has $W^{s,2} \subset \mathscr{C}(\mathbb{T})$.
- $s \geq 0$ characterizes the energy concentration. As larger s as more energy is concentrated in the low frequency components.

Our signal spaces

$$\mathscr{B}^s = W^{s,2} \cap \mathscr{B} \qquad \text{with} \qquad \|f\|_{\mathscr{B}^s} = \max\left\{ \|f\|_{s,2} , \|f\|_{\mathscr{B}} \right\} , \qquad s \geq 0 .$$

Relation to the Dirichlet Problem

Dirichlet Problem on the Circle

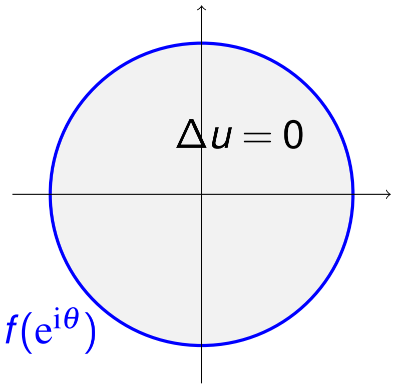
Let f be a given function on the unit circle $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$. We look for an u inside the unit circle $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ such that

1. $\frac{\partial^2 u}{\partial x^2}(z) + \frac{\partial^2 u}{\partial y^2}(z) = (\Delta u)(z) = 0$ for all $z = x + \mathrm{i}y \in \mathbb{D}$
2. $u(\mathrm{e}^{\mathrm{i}\theta}) = f(\mathrm{e}^{\mathrm{i}\theta})$ for all $\mathrm{e}^{\mathrm{i}\theta} \in \mathbb{T}$

Dirichlet's Principle

The solution of the Dirichlet problem can be obtained by minimizing the Dirichlet energy

$$D(u) = \frac{1}{2\pi} \iint_{\mathbb{D}} \left\|(\operatorname{grad} u)(z)\right\|_{\mathbb{R}^2}^2 \mathrm{d}z = \sum_{n=-\infty}^{\infty} |n| \, |c_n(f)|^2 = \|f\|_{1/2,2} \, .$$



- The boundary function of solutions of the Dirichlet problem belongs to $W^{1/2,2}$.
- If f is additionally in $\mathscr{C}(\mathbb{T})$ then $f \in \mathscr{B}^{1/2}$.

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A Class of Hilbert Transform Approximations

We consider sequences $\{\mathrm{H}_N\}_{N\in\mathbb{N}}$ of bounded linear operators $\mathrm{H}_N : \mathscr{B} \to \mathscr{B}$ which satisfy the following two axioms:

(A) Concentration on a sampling set:
To every $N \in \mathbb{N}$ there exists a finite set $Z_N = \{\zeta_{n,N} : n = 1, \dots, M_N\} \subset \mathbb{T}$ such that for all $f_1, f_2 \in \mathscr{B}$

$$f_1(\zeta_{n,N}) = f_2(\zeta_{n,N}) \qquad \text{for all } \zeta_{n,N} \in Z_N$$
$$\text{implies } (\mathrm{H}_N f_1)(\zeta) = (\mathrm{H}_N f_2)(\zeta) \qquad \text{for all } \zeta \in \mathbb{T}\,.$$

(B) Weak convergence on $\mathscr{B}$:
For every $f \in \mathscr{B}$, the sequence $\{\mathrm{H}_N f\}_{N\in\mathbb{N}}$ converges weakly to $\mathrm{H}f$, i.e.

$$\lim_{N\to\infty} \langle \mathrm{H}_N f, \varphi \rangle_2 = \langle \mathrm{H}f, \varphi \rangle_2 \qquad \text{for all } \varphi \in \mathscr{C}^\infty(\mathbb{T})\,.$$

Remark:
If $\{\mathrm{H}_N\}_{N\in\mathbb{N}}$ satisfies Axiom (A) then each H_N has the form

$$(\mathrm{H}_N f)(\mathrm{e}^{\mathrm{i}\theta}) = \sum_{n=1}^{M_N} f(\zeta_{n,N})\, h_{n,N}(\mathrm{e}^{\mathrm{i}\theta}) \qquad \text{with} \qquad \{h_{1,N}, h_{2,N}, \dots, h_{M_N,N}\} \subset \mathscr{B}\,.$$

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A Strong Divergence Result

A Technical Axiom

We say that $\{\mathrm{H}_N\}_{N\in\mathbb{N}}$ of bounded linear operator $\mathrm{H}_N : \mathscr{B} \to \mathscr{B}$ satisfies Axiom (C) if the following holds: Let $f \in \mathscr{B}$ be such that there is a closed arch $\mathbb{J} \subset \mathbb{T}$ such that $\widetilde{f} = \mathrm{H}f \in \mathscr{C}^\infty(\mathbb{J})$. Then on every closed sub-arch $\mathbb{I} \subset \mathbb{J}$ one has

$$\lim_{N\to\infty} \max_{\zeta\in\mathbb{I}} \left|\widetilde{f}(\zeta) - (\mathrm{H}_N f)(\zeta)\right| = 0\,.$$

Theorem (Strong Peak Value Blowup)

Let $\{\mathrm{H}_N\}_{N\in\mathbb{N}}$ be a sequence satisfying Axioms (A), (B), and (C). Then there exists $f_* \in \mathscr{B}$ such that

$$\lim_{N\to\infty} \left\|\mathrm{H}_N f_*\right\|_\infty = +\infty\,.$$

Remarks

- Strong assumptions: Axioms (A), (B), (C) and we consider the largest space $\mathscr{B}$.
- Strong divergence: There exists no convergent subsequence.

Questions

- Can we get convergence if we restrict our signal space: $\mathscr{B}^s \subset \mathscr{B}$? – yes!
- Can we avoid Axiom (C)? – yes, but we get (at the moment) slightly weaker divergence results.

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Divergence on Spaces $\mathscr{B}^s$ with $s \in [0, 1/2]$

Theorem (Weak Peak Value Blowup)

Let $\{\mathrm{H}_N\}_{N\in\mathbb{N}}$ be an arbitrary sequence of bounded linear operators $\mathrm{H}_N : \mathscr{B} \to \mathscr{B}$ which satisfies Axioms (A) and (B). Then for every $0 \leq s \leq 1/2$ there exists a residual set $\mathscr{E}_s \subset \mathscr{B}^s$ such that for every

$$\limsup_{N\to\infty} \left\|\mathrm{H}_N f\right\|_\infty = \infty \qquad \text{for all } f \in \mathscr{E}_s \,.$$

Remarks

- This result implies in particular

$$\limsup_{N\to\infty} \left\|\mathrm{H}_N f - \mathrm{H} f\right\|_\infty = \infty \qquad \text{for all } f \in \mathscr{E}_s \,.$$

- There is no sampling based Hilbert transform approximation on the spaces $\mathscr{B}^s$ with $0 \leq s \leq 1/2$.
- In particular, not on the set of all solutions of the Dirichlet problem (finite Dirichlet energy).
- We only require Axioms (A) and (B).
- We only have weak divergence,
  i.e. to every $f \in \mathscr{B}^s$ there may exist a subsequence $\{N_k = N_k(f)\}_{k\in\mathbb{N}}$ such that

$$\lim_{k\to\infty} \left\|\mathrm{H}_{N_k} f\right\|_\infty < \infty \qquad \text{or even} \qquad \lim_{k\to\infty} \left\|\mathrm{H}_{N_k} f - \mathrm{H} f\right\|_\infty = 0 \,.$$

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Weak Divergence versus Strong Divergence

- Given an approximation sequence $\{\mathrm{H}_N\}_{N\in\mathbb{N}}$ which diverges weakly

$$\limsup_{N\to\infty} \left\|\mathrm{H}_N f - \mathrm{H} f\right\|_\infty = \infty \qquad \text{for all } f \in \mathscr{E}_s\,.$$

To every $f \in \mathscr{E}_s$ there may exist a subsequence $\{N_k = N_k(f)\}_{k\in\mathbb{N}}$ such that

$$\lim_{k\to\infty} \left\|\mathrm{H}_{N_k} f - \mathrm{H} f\right\|_\infty = 0\,.$$

Then $\{\mathrm{H}_{N_k(f)}\}_{k\in\mathbb{N}}$ is a convergent approximation method adapted to f.

- Assume $\{\mathrm{H}_N\}_{N\in\mathbb{N}}$ diverges strongly

$$\lim_{N\to\infty} \left\|\mathrm{H}_N f - \mathrm{H} f\right\|_\infty = \infty \qquad \text{for all } f \in \mathscr{E}_s\,.$$

Then no convergent subsequence exists $\Longrightarrow$ adaption does not help.

- ... every sequence $\{\mathrm{H}_N\}_{N\in\mathbb{N}}$ diverges weakly on $\mathscr{B}^s$ $\Rightarrow$ there exists no non-adaptive approximation methods on $\mathscr{B}^s$
- ... every sequence $\{\mathrm{H}_N\}_{N\in\mathbb{N}}$ diverges strongly on $\mathscr{B}^s$ $\Rightarrow$ there exists no adaptive (and non-adaptive) approximation methods on $\mathscr{B}^s$

Spaces with Convergent Approximation Methods

Theorem

For any $s > 1/2$ there exit sequences $\{\mathrm{H}_N\}_{N\in\mathbb{N}}$ of bounded linear operators $\mathrm{H}_N : \mathscr{B} \to \mathscr{B}$ which satisfy Axioms (A) and (B) such that

$$\lim_{N\to\infty} \left\| \mathrm{H}_N f - \mathrm{H} f \right\|_\infty = 0 \qquad \text{for all } f \in \mathscr{B}^s .$$

- If the energy of the signals is sufficiently good concentrated then there always exist sampling based approximation methods which converge for all signals in the space $\mathscr{B}^s$.
- Theorem can be proved by constructing particular methods.

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Characterization of Convergent Method

Theorem

Let $\{\mathrm{H}_N\}_{N\in\mathbb{N}}$ be a sequence of bounded linear operators $\mathrm{H}_N : \mathscr{B} \to \mathscr{B}$ such that

1. *For every $n \in \mathbb{N}$ holds*

$$\lim_{N\to\infty} \left\|\mathrm{H}_N[\cos(n\cdot)] - \sin(n\cdot)\right\|_\infty = 0 \qquad \text{and} \qquad \lim_{N\to\infty} \left\|\mathrm{H}_N[\sin(n\cdot)] + \cos(n\cdot)\right\|_\infty = 0\,.$$

2. *There exists a constant C such that*

$$\max\left(\left\|\mathrm{H}_N[\cos(n\cdot)]\right\|_\infty, \left\|\mathrm{H}_N[\sin(n\cdot)]\right\|_\infty\right) \leq C \qquad \text{for all } N \in \mathbb{N}\,.$$

Then for every $s > 1/2$ one has

$$\lim_{N\to\infty} \left\|\mathrm{H}_N f - \mathrm{H} f\right\|_\infty = 0 \qquad \text{for all } f \in \mathscr{B}^s\,.$$

Thus, if an approximation method $\{\mathrm{H}_N\}_{N\in\mathbb{N}}$

- converges for the sine- and cosine functions (i.e. for the pure frequencies), and
- if the approximations of the pure frequencies are uniformly bounded

then the method $\{\mathrm{H}_N\}_{N\in\mathbb{N}}$ converges for all $f \in \mathscr{B}^s$ with $s > 1/2$.

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An Example of a Convergent Hilbert Transform

We consider again the sequence $\{\widetilde{\mathrm{D}}_N\}_{N\in\mathbb{N}}$ of the sampled conjugate Fourier series

$$(\widetilde{\mathrm{D}}_N f)(\mathrm{e}^{\mathrm{i}\theta}) := (\mathrm{H}\mathrm{D}_N f)(\mathrm{e}^{\mathrm{i}\theta}) = -\mathrm{i}\sum_{n=-(N-1)}^{N-1} \mathrm{sgn}(n)\, c_{n,N}(f)\, \mathrm{e}^{\mathrm{i}n\theta} = \sum_{k=0}^{2N-1} f(\zeta_{k,N})\, \widetilde{\mathscr{D}}_N\Big(\theta - k\frac{\pi}{N}\Big)$$

with the conjugate Dirichlet kernel $\widetilde{\mathscr{D}}_N(\theta) = \frac{1}{N}\sum_{n=1}^{N-1}\sin(n\theta)$ and which are concentrated on the equidistant sampling sets

$$Z_N = \big\{\zeta_{k,N} = \mathrm{e}^{\mathrm{i}\pi k/N} : k = 0,1,\dots,2N-1\big\}\,.$$

It is fairly easy to show that this sequence $\{\widetilde{\mathrm{D}}_N\}_{N\in\mathbb{N}}$

- satisfies Axioms (A) and (B).
- has the two properties of the previous theorem which characterized all convergent methods.

So we have

$$\lim_{N\to\infty}\big\|\widetilde{\mathrm{D}}_N f - \mathrm{H}f\big\|_\infty = 0 \qquad \text{for all } f\in\mathscr{B}^s \quad \text{with} \quad s > 1/2\,.$$

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Conclusions

- We introduced a scale of Banach spaces $\mathscr{B}^s$, $s \geq 0$ of functions
  - which are continuous with a continuous Hilbert transform
  - of finite energy
  - with different energy concentration, characterized by s
- In the scale $\{\mathscr{B}^s\}_{s \geq 0}$, we characterized precisely those spaces on which
  - there do not exists any sampling based linear Hilbert transform approximations: $s \in [0, 1/2]$
  - there do exists sampling based Hilbert transform approximations: $s > 1/2$
- For $s > 1/2$ even very simple approximations methods (sampled conjugate Fourier series) work

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Thank You! – Questions?