The Arbitrarily Varying Wiretap Channel – Secret Randomness, Stability and Super-Activation

Janis Nötzel,
Technische Universität München

Joint work with Moritz Wiese and Holger Boche

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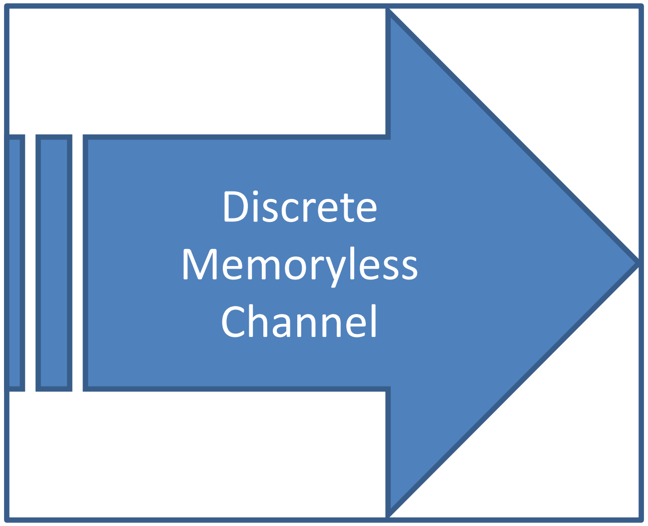
(insights from a multi-letter formula)

arXiv : 1501.07439

The Model



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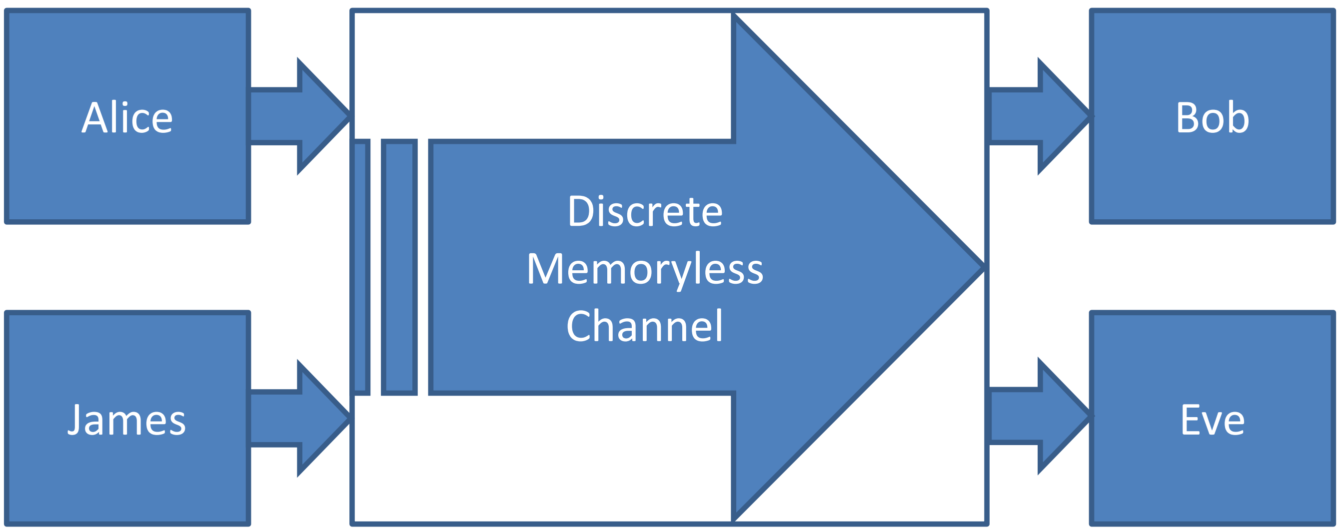

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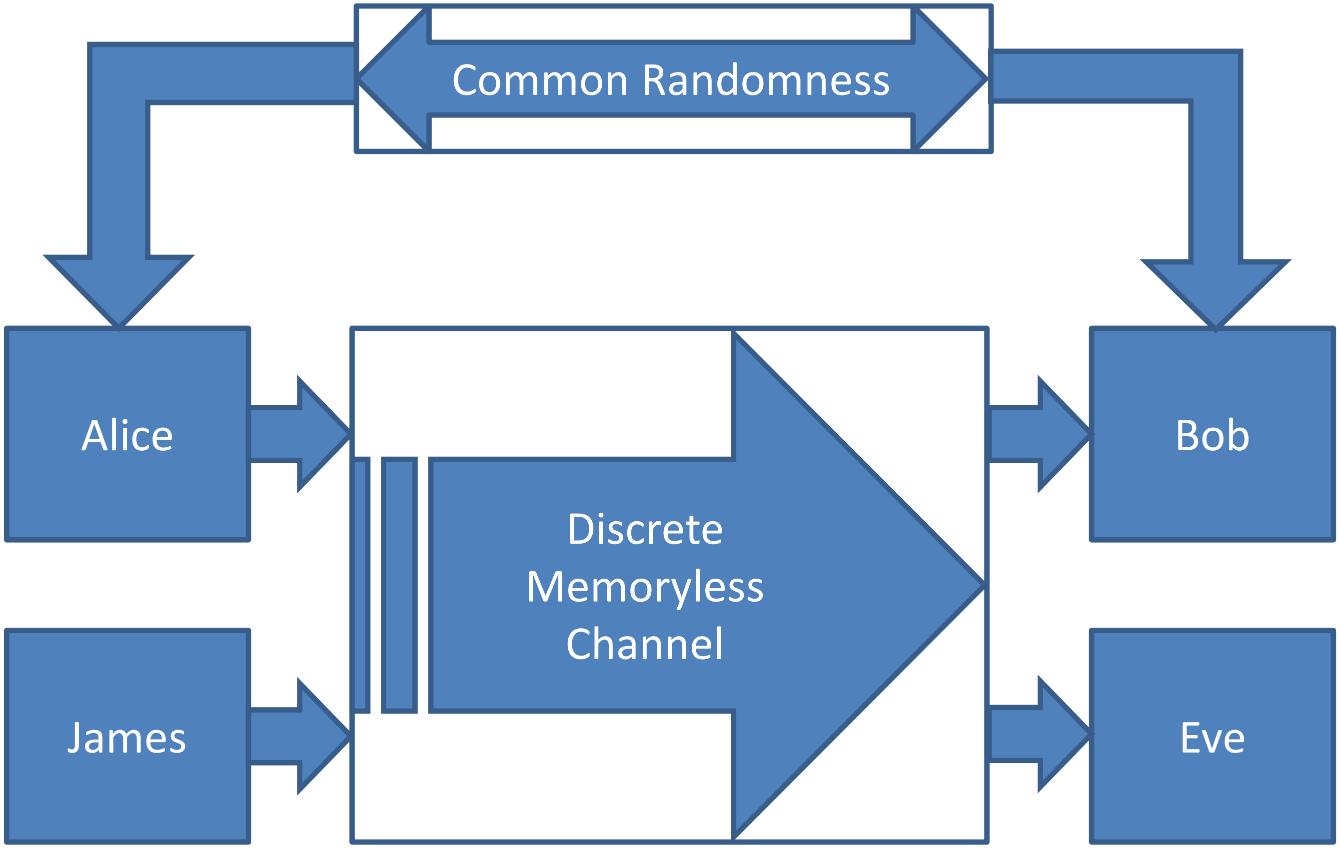
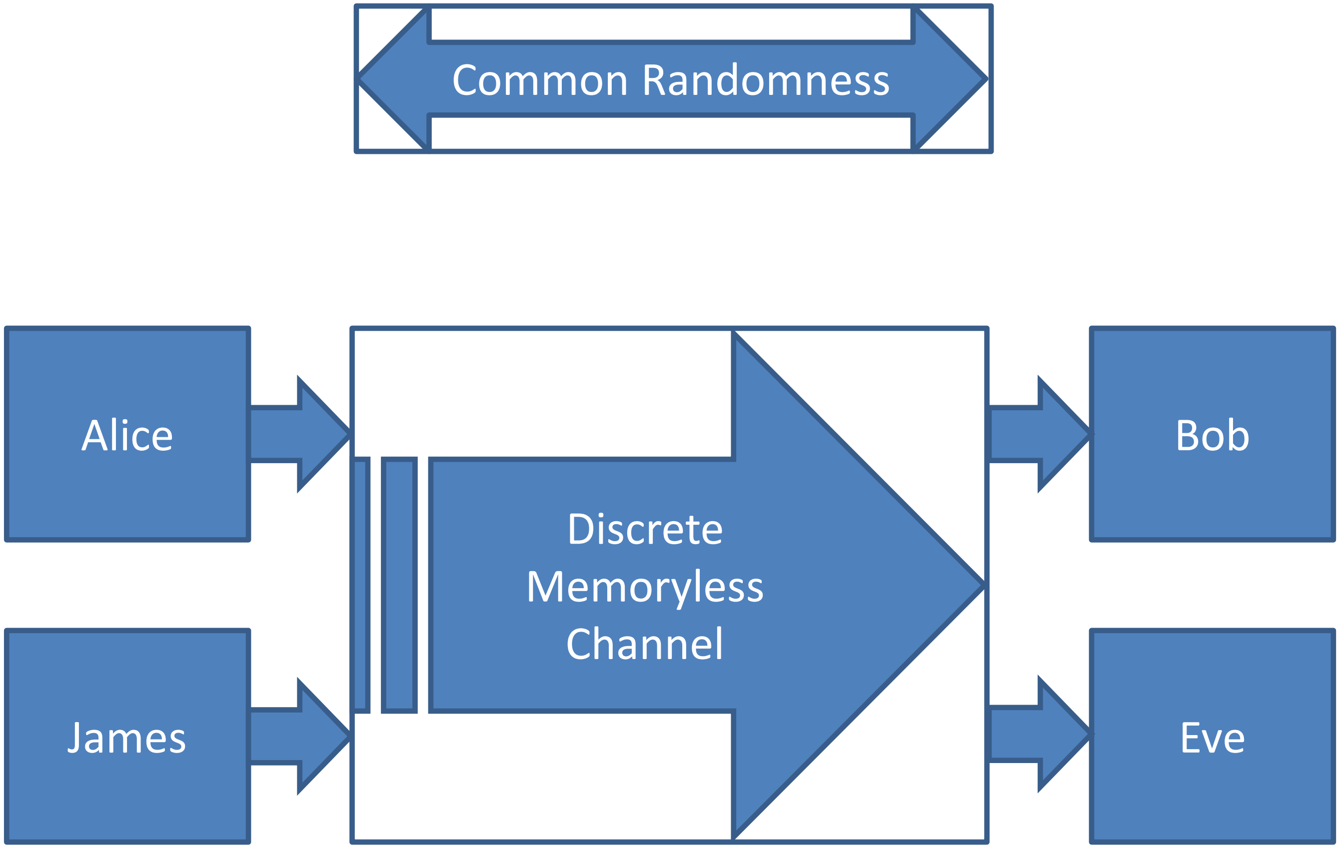

Common Randomness
Alice
James
Discrete
Memoryless
Channel
Bob
Eve

The Model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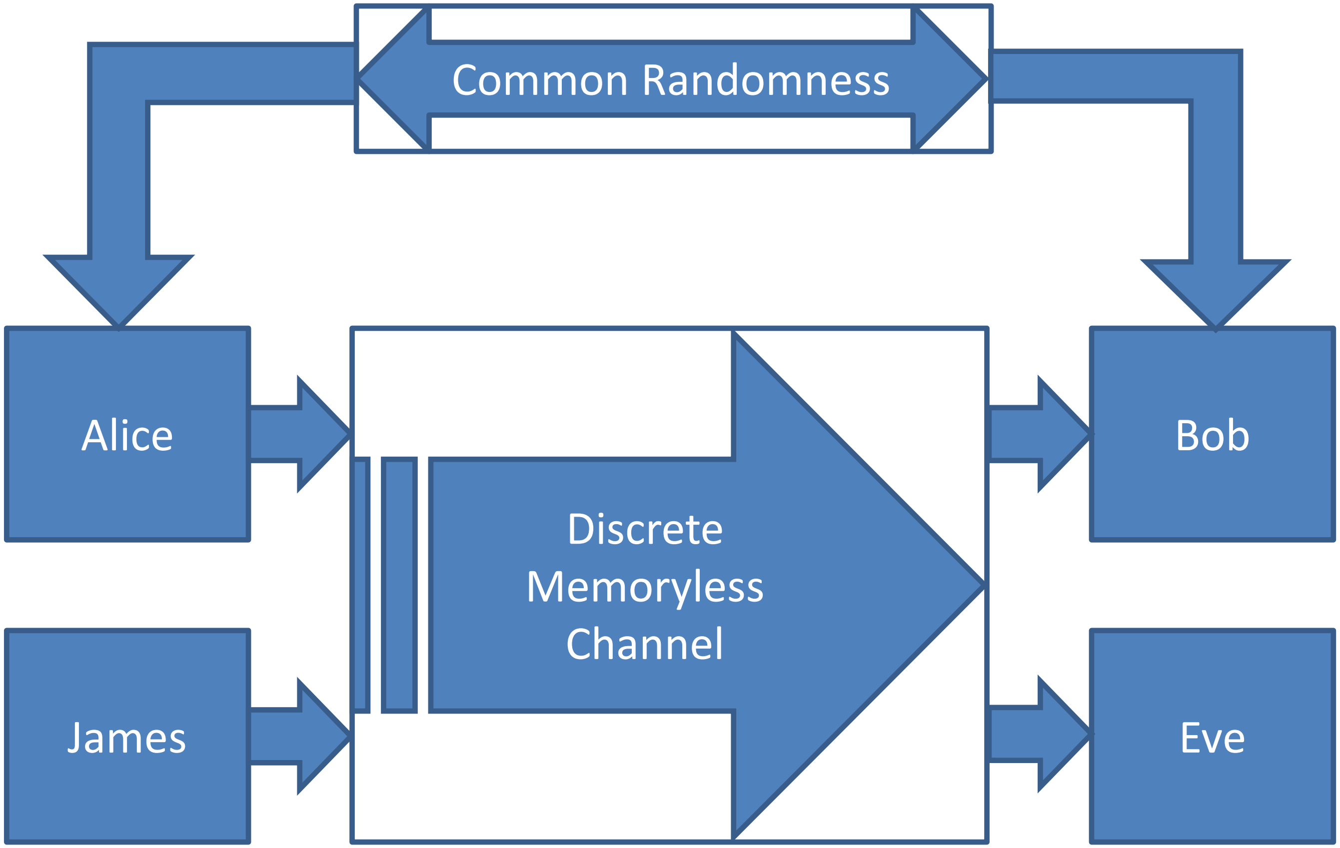

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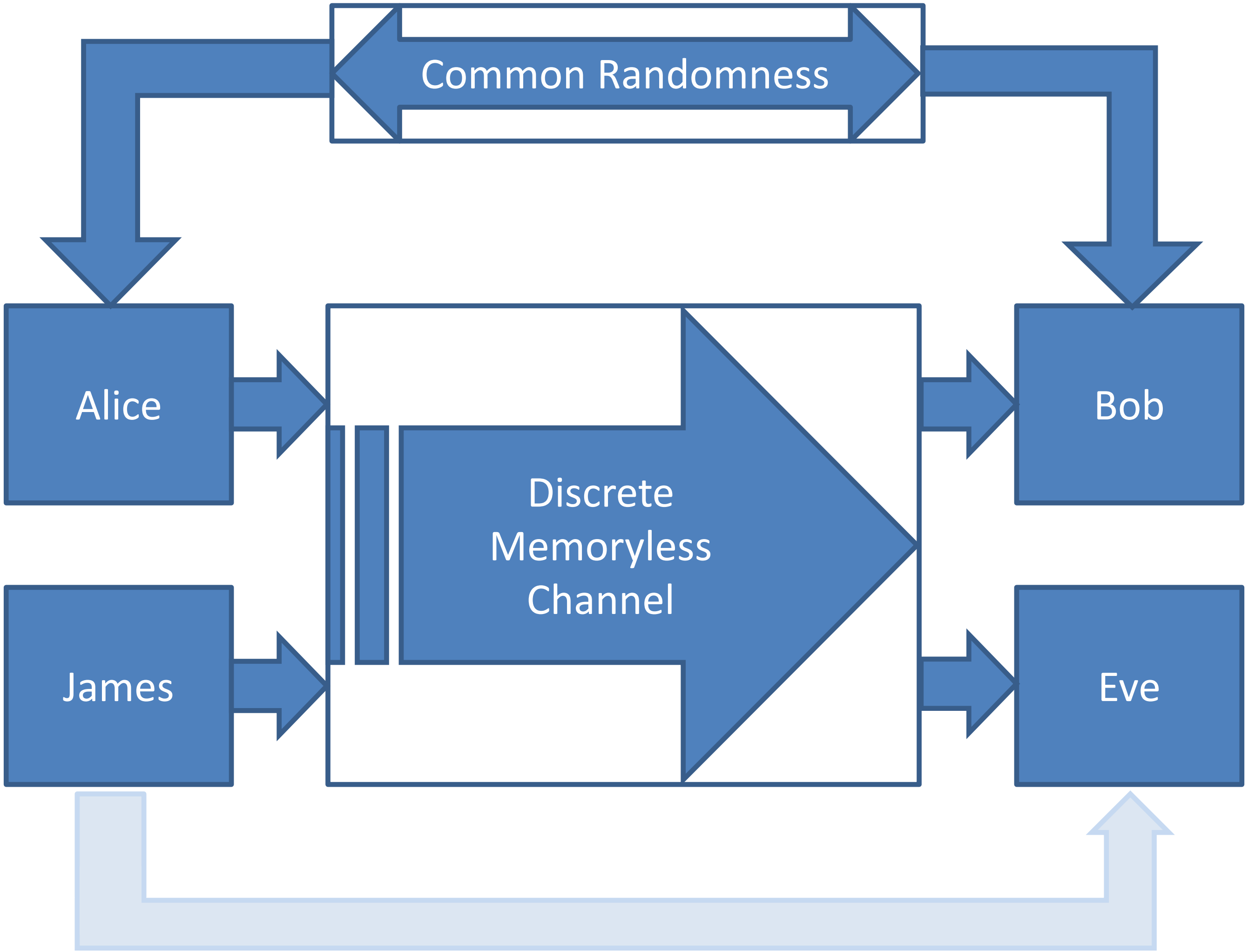
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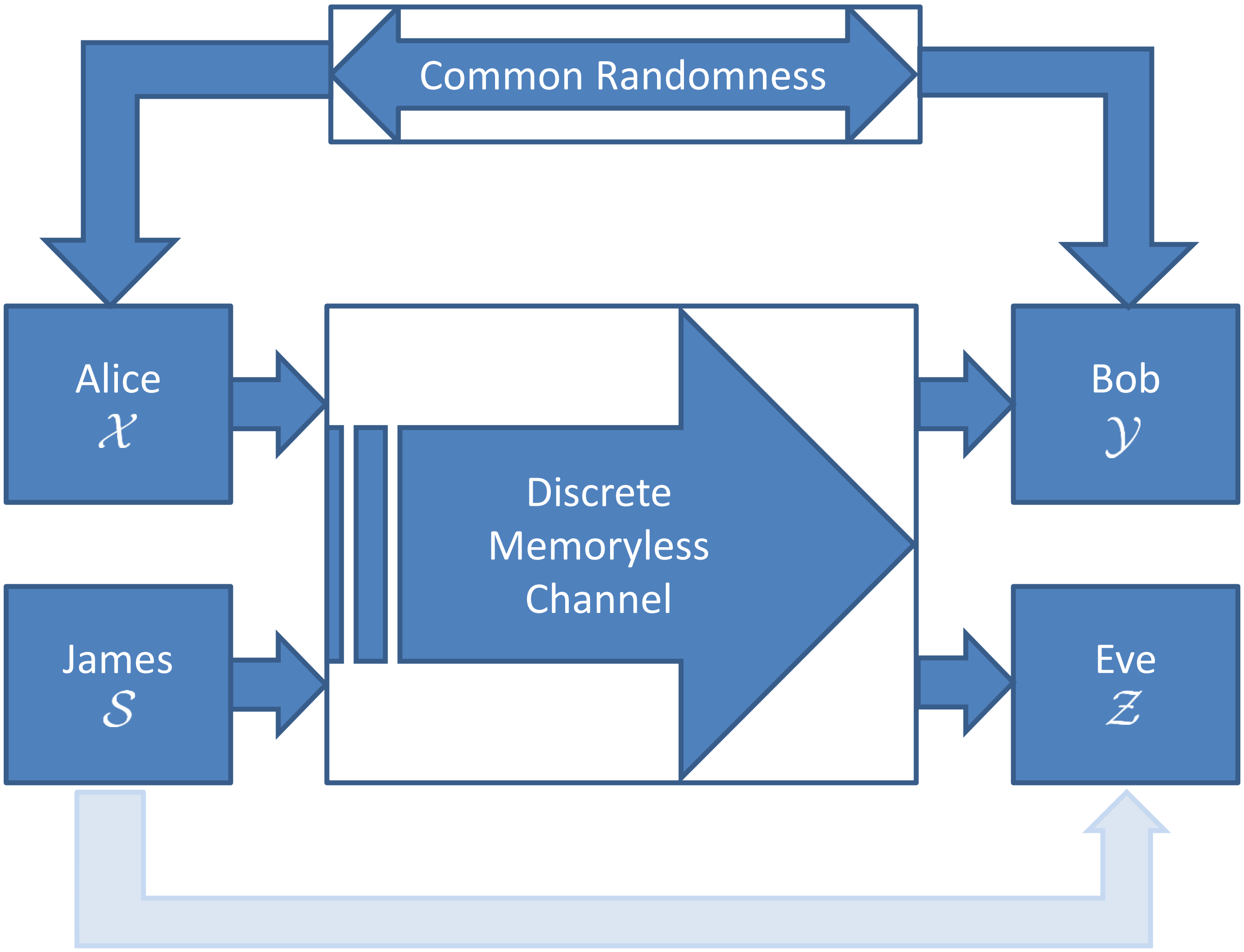
Common Randomness
Alice
$\mathcal{X}$
Discrete
Memoryless
Channel
Bob
$\mathcal{Y}$
James
$\mathcal{S}$
Eve
$\mathcal{Z}$

Topics of the talk

- We consider message transmission under two uncoordinated attacks: jamming & eavesdropping.
- This model is called the arbitrarily varying wiretap channel (AVWC).
- We consider the impact of *secret* common randomness (secret CR) as compared to common randomness (CR) which is available at Eve's site as well.
  - ➡ We provide a capacity formula for the scenario with secret CR.
- The capacity of the arbitrarily varying channel has been proven to have discontinuity points.
  - ➡ We prove that this carries over to AVWCs.
  - ➡ We also prove a very positive result: the capacity depends continuously on the channel to Eve!
- The AVWC exhibits super-activation when no CR is used.
  - ➡ We give a precise characterization of the phenomenon.
  - ➡ We provide a link to the (conjectured) super-activation of the CR assisted capacity.

Notation

- $\mathcal{P}(\mathcal{A})$ - the probability distributions on $\mathcal{A}$.
- $\mathcal{A}^n := \{(a_1, \ldots, a_n) : a_i \in \mathcal{A}\ \forall i \in \{1, \ldots, n\}\ \}$.
- The set of channels from $\mathcal{A}$ to $\mathcal{B}$ is $C(\mathcal{A}, \mathcal{B})$. $w \in C(\mathcal{A}, \mathcal{B})$ is identified with the transition probabilities $(w(b|a))_{a\in\mathcal{A}, b\in\mathcal{B}}$.
- $w \in C(\mathcal{A}, \mathcal{B})$, $w' \in C(\mathcal{A}', \mathcal{B}') \Rightarrow w \otimes w' \in C(\mathcal{A} \times \mathcal{A}', \mathcal{B} \times \mathcal{B}')$ via $(w \otimes w')((b, b')|(a, a')) := w(b|a)w'(b'|a')$ $(\forall a, b, a', b')$.
- The mutual information of a bipartite random variable (X, Y) is denoted $I(X; Y)$.
- For the remainder of the talk, fix $\mathcal{S}$, $\mathcal{X}$, $\mathcal{Y}$ and $\mathcal{Z}$. The channel to Bob is $w \in C(\mathcal{S} \times \mathcal{X}, \mathcal{Y})$. The channel to Eve is $v \in C(\mathcal{S} \times \mathcal{X}, \mathcal{Z})$.
- w and v incorporate all the necessary details for this model.
- Equivalent representation: $\mathfrak{W} := (w(\cdot|s, \cdot))_{s\in\mathcal{S}} \in C(\mathcal{X}, \mathcal{Y})^{|\mathcal{S}|}$ and $\mathfrak{V} := (v(\cdot|s, \cdot))_{s\in\mathcal{S}} \in C(\mathcal{X}, \mathcal{Z})^{|\mathcal{S}|}$. Denote the AVWC by $(\mathfrak{W}, \mathfrak{V})$.

Definition of codes

DEF I. Let $n \in \mathbb{N}$. A CR assisted code $\mathcal{K}_n$ for n uses of $(\mathfrak{W}, \mathfrak{V})$ consists of: $K, \Gamma \in \mathbb{N}$, a set of encoders $\{E^\gamma\}_{\gamma=1}^{\Gamma} \subset C(\{1, \dots, K\}, \mathcal{X}^n)$ and a collection $(D_k^\gamma)_{k,\gamma=1}^{K,\Gamma}$ of subsets D_k^γ of $\mathcal{Y}^n$ satisfying $D_k^\gamma \cap D_{k'}^\gamma = \emptyset$ for all $\gamma \in [\Gamma]$, whenever $k \neq k'$. Every such code defines the random variables $S_{s^n} := (\mathfrak{K}_n, \mathfrak{K}'_n, \mathfrak{d}_n, \mathfrak{X}_n, \mathfrak{Y}_{s^n}, \mathfrak{Z}_{s^n})$ $(s^n \in \mathcal{S}^n)$ via

$$\begin{aligned}\mathbb{P}(&S_{s^n} = (k, k', \gamma, x^n, y^n, z^n)) \\ &:= \frac{1}{\Gamma \cdot K} E^\gamma(x^n|k) \mathbb{1}_{D_{k'}^\gamma}(y^n) w^{\otimes n}(y^n|s^n, x^n) v^{\otimes n}(z^n|s^n, x^n).\end{aligned}$$

The average error of $\mathcal{K}_n$ is

$$e(\mathcal{K}_n) = 1 - \max_{s^n \in \mathcal{S}^n} \frac{1}{K\Gamma} \sum_{k,\gamma=1}^{K,\Gamma} \sum_{x^n} E^\gamma(x^n|k) w^{\otimes n}(D_k^\gamma|s^n, x^n).$$

Definition of coding schemes

DEF II. A CR assisted secure coding scheme for $(\mathfrak{W}, \mathfrak{V})$ operating at rate R consists of a sequence $(\mathcal{K}_n)_{n\in\mathbb{N}}$ of CR assisted codes such that

$$\lim_{n\to\infty} e(\mathcal{K}_n) = 0, \qquad \liminf_{n\to\infty} \frac{1}{n}\log(K_n) = R,$$
$$\text{and} \qquad \limsup_{n\to\infty} \max_{s^n\in\mathcal{S}^n} I(\mathfrak{K}_n; \mathfrak{Z}_{s^n}|\mathfrak{d}_n) = 0.$$

If $\Gamma_n = 1$ for all $n \in \mathbb{N}$, $(\mathcal{K}_n)_{n\in\mathbb{N}}$ is called deterministic coding scheme.

DEF III. A secure CR assisted secure coding scheme $\mathcal{K}$ for $(\mathfrak{W}, \mathfrak{V})$ operating at rate R and using an amount $G > 0$ of secret CR consists of a sequence $(\mathcal{K}_n)_{n\in\mathbb{N}}$ of CR assisted codes satisfying

$$\lim_{n\to\infty} \frac{1}{n}\log\Gamma_n = G, \qquad \liminf_{n\to\infty} \frac{1}{n}\log(K_n) = R,$$
$$\lim_{n\to\infty} e(\mathcal{K}_n) = 0, \qquad \limsup_{n\to\infty} \max_{s^n\in\mathcal{S}^n} I(\mathfrak{K}_n; \mathfrak{Z}_{s^n}) = 0.$$

Definition of capacities

DEF IV. Let $G > 0$. $C_s(\mathfrak{W}, \mathfrak{V}, G)$ is the supremum over all $R \geq 0$ such that there is a secure coding scheme $\mathcal{K}$ for $(\mathfrak{W}, \mathfrak{V})$ operating at rate R and using an amount G of secret CR.

DEF V. $C_d(\mathfrak{W}, \mathfrak{V})$ is the supremum over all $R \geq 0$ such that there is a secure deterministic coding scheme $\mathcal{K}$ at rate R.

DEF VI. $C_r(\mathfrak{W}, \mathfrak{V})$ is the supremum over all $R \geq 0$ such that there exists a secure CR assisted coding scheme $\mathcal{K}$ at rate R.

Results

RESULT I. (Capacity with secret CR) It holds

$$C_s(\mathfrak{W}, \mathfrak{V}, G) = \min\{\ C_r(\mathfrak{W}, \mathfrak{V}) + G,\ \ C_r(\mathfrak{W}, \mathfrak{T})\ \},$$

where $\mathfrak{T} = \{t\}$ and $t(z|x,s) = 1/|\mathcal{Z}|\ \forall\ s \in \mathcal{S}$, $x \in \mathcal{X}$, $z \in \mathcal{Z}$.

REMARK. Recent (*arXiv:1410.8078*, this ISIT, paper number 2395) work by Wiese, Nötzel and Boche provided a multi-letter formula for C_r.

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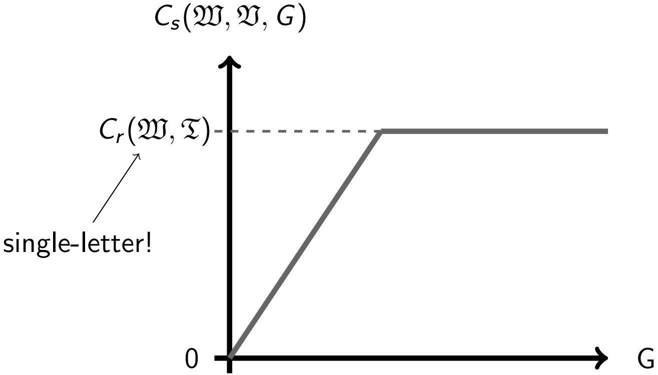

Results

RESULT II. (Symmetrizability)
1) If $\mathfrak{W}$ is symmetrizable, then $C_d(\mathfrak{W}, \mathfrak{V}) = 0$.
2) If $\mathfrak{W}$ is non-symmetrizable, then $C_d(\mathfrak{W}, \mathfrak{V}) = C_r(\mathfrak{W}, \mathfrak{V})$.

REMARK. The proof is based on [Csiszar,Narayan'78].

REMARK. An AVC $\mathfrak{W}$ is symmetrizable if there is a conditional probability distribution $(u(s|x))_{s\in\mathcal{S},x\in\mathcal{X}}$ such that

$$\forall x, \hat{x} \in \mathcal{X} : \qquad \sum_{s\in\mathcal{S}} u(s|x)w(\cdot|s,\hat{x}) = \sum_{s\in\mathcal{S}} u(s|\hat{x})w(\cdot|s,x).$$

DEFINITION. Let $M_{\mathrm{f}} := \{M \subset C(\mathcal{X}, \mathcal{Y}) : |M| < \infty\}$. Define $F : M_{\mathrm{f}} \to \mathbb{R}_+$ by

$$F(\mathfrak{W}) := \max_{x\neq x'} \min_{u} \| \sum_{s} (u(s|x)w(\cdot|s,\hat{x}) - u(s|\hat{x})w(\cdot|s,x))\|_1.$$

Then '$F(\mathfrak{W}) = 0$' is equivalent to 'the AVC $\mathfrak{W}$ is symmetrizable'.

Results

DEFINITION. As metric on the set of AVWCs (and AVCs) we use the Hausdorff-distance which is inherited from the one-norm (variational distance) on probability distributions. Let this distance be denoted by d.

RESULT III. (Discontinuity)
1) C_d is discontinuous in $(\mathfrak{W}, \mathfrak{V})$ iff: $C_r(\mathfrak{W}, \mathfrak{V}) > 0$, $F(\mathfrak{W}) = 0$ but: $\forall\ \epsilon > 0\ \exists\ \mathfrak{W}_\epsilon$ such that $d(\mathfrak{W}, \mathfrak{W}_\epsilon) < \epsilon$ and $F(\mathfrak{W}_\epsilon) > 0$.
2) If C_d is discontinuous in the point $(\mathfrak{W}, \mathfrak{V})$ then it is discontinuous for all $\hat{\mathfrak{V}}$ for which $C_r(\mathfrak{W}, \hat{\mathfrak{V}}) > 0$.

RESULT IV. (Stability) If $C_d(\mathfrak{W}, \mathfrak{V}) > 0$ then there is $\epsilon > 0$ such that $d((\mathfrak{W}, \mathfrak{V}), (\mathfrak{W}', \mathfrak{V}')) \leq \epsilon$ implies $C_d(\mathfrak{W}', \mathfrak{V}') > 0$.

Super-activation: preliminaries

For two AVWCs $(\mathfrak{W}_1, \mathfrak{V}_1)$ and $(\mathfrak{W}_2, \mathfrak{V}_2)$, we define $(\mathfrak{W}_1 \otimes \mathfrak{W}_2, \mathfrak{V}_1 \otimes \mathfrak{V}_2)$ to equal

$$((w_1(\cdot|\cdot, s) \otimes w_2(\cdot|\cdot, s'))_{s,s' \in \mathcal{S}}, (v_1(\cdot|\cdot, s) \otimes v_2(\cdot|\cdot, s'))_{s,s' \in \mathcal{S}}),$$

Since all state alphabets are assumed to be finite, there is no loss of generality in this definition. Then,

$$C_d(\mathfrak{W}_1 \otimes \mathfrak{W}_2, \mathfrak{V}_1 \otimes \mathfrak{V}_2) \geq C_d(\mathfrak{W}_1, \mathfrak{V}_1) + C_d(\mathfrak{W}_2, \mathfrak{V}_2)$$

follows trivially from the definition of C_d. In contrast, if

$$C_d(\mathfrak{W}_1 \otimes \mathfrak{W}_2, \mathfrak{V}_1 \otimes \mathfrak{V}_2) > C_d(\mathfrak{W}_1, \mathfrak{V}_1) + C_d(\mathfrak{W}_2, \mathfrak{V}_2)$$

holds, we speak of *super-additivity* and if even

$$\begin{aligned} C_d(\mathfrak{W}_1, \mathfrak{V}_1) = C_d(\mathfrak{W}_2, \mathfrak{V}_2) &= 0, \\ C_d(\mathfrak{W}_1 \otimes \mathfrak{W}_2, \mathfrak{V}_1 \otimes \mathfrak{V}_2) &> 0 \end{aligned}$$

we speak of *super-activation*.

Super-activation: results

There exist AVWCs which exhibit super-activation [Boche,Schaefer'14].
We give a precise characterization of the effect.

RESULT III. (Super-activation) Let $(\mathfrak{W}_i, \mathfrak{V}_i)_{i=1,2}$ be AVWCs.
...
3) If C_r shows super-activation for $(\mathfrak{W}_1, \mathfrak{V}_1)$ and $(\mathfrak{W}_2, \mathfrak{V}_2)$, then C_d can show super-activation for $(\mathfrak{W}_1, \mathfrak{V}_1)$ and $(\mathfrak{W}_2, \mathfrak{V}_2)$ if and only if at least one of $\mathfrak{W}_1$ or $\mathfrak{W}_2$ is non-symmetrizable.
4) If C_r shows no super-activation for $(\mathfrak{W}_1, \mathfrak{V}_1)$ and $(\mathfrak{W}_2, \mathfrak{V}_2)$ then super-activation of C_d can only happen if $\mathfrak{W}_1$ is non-symmetrizable and $\mathfrak{W}_2$ is symmetrizable and $C_r(\mathfrak{W}_1, \mathfrak{V}_1) = 0$ and $C_r(\mathfrak{W}_2, \mathfrak{V}_2) > 0$.

Conclusions

- We provided a complete characterization of the secrecy capacity of AVWCs
  - This characterization uses a multi-letter formula
  - nonetheless, we were able to make nontrivial statements:
  - Complete characterization of discontinuity points in terms of functions which are continuous themselves
  - Complete characterization of super-activation of C_d
  - Single-letterization is an open and potentially hard problem

- Compare to zero-error capacity which [Ahlswede'70] is deeply connected to AVCs

- It was conjectured [Shannon'56] that the zero-error capacity of a DMC is additive. This conjecture was proven to be wrong [Alon'98]
  - ➡ Super-additivity can occur for the zero-error capacity

THANKS FOR YOUR ATTENTION

Related work

- [Boche,Schaefer'14] Boche and Schaefer "Capacity results and super-activation for wiretap channels with active wiretappers"(2014)
- [Csiszar,Narayan'88] Csiszar and Narayan "The arbitrarily varying channel revisited: positivity, constraints" (1988)
- [Ahlswede'78] Ahlswede "Elimination of correlation for arbitrarily varying channels" (1978)
- [Shannon'56] Shannon, "The zero-error capacity of a noisy channel" (1956)
- [Ahlswede'70] Ahlswede, "A Note on the Existence of the Weak Capacity for Channels with Arbitrarily Varying Channel Probability Functions and Its Relation to Shannon's Zero Error Capacity" (1970)
- [Alon'98] Alon, "The Shannon capacity of a union" (1998)