STRUCTURE OF THE SET OF SIGNALS WITH STRONG DIVERGENCE OF THE SHANNON SAMPLING SERIES

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Shannon sampling series

Shannon sampling series:

$$(S_N f)(t) = \sum_{k=-N}^{N} f(k) \frac{\sin(\pi(t-k))}{\pi(t-k)}, \quad t \in \mathbb{R}.$$

Local uniform convergence (Brown): For all $f \in \mathcal{PW}_\pi^1$ and $\tau > 0$ fixed we have

$$\lim_{N\to\infty} \max_{t\in[-\tau,\tau]} |f(t) - (S_N f)(t)| = 0.$$

Global Behavior

Peak value of the reconstruction error:

$$P_N f = \max_{t\in\mathbb{R}} |f(t) - (S_N f)(t)|$$

Divergence of the peak value $P_N f$:

There exists a signal $f \in \mathcal{PW}_\pi^1$ such that

$$\limsup_{N\to\infty} \max_{t\in\mathbb{R}} |f(t) - (S_N f)(t)| = \infty,$$

or equivalently,

$$\limsup_{N\to\infty} \max_{t\in\mathbb{R}} |(S_N f)(t)| = \infty.$$

- The divergence is only in terms of the lim sup.
- Weak notion of divergence: existence of a subsequence $\{N_n\}_{n\in\mathbb{N}}$ of the natural numbers such that $\lim_{n\to\infty} P_{N_n} f = \infty$.
- Leaves the possibility that there is a different subsequence $\{N_n^*\}_{n\in\mathbb{N}}$ such that $\lim_{n\to\infty} P_{N_n^*} f = 0$.

Idea of adaptive signal processing: With an adaptive choice of the subsequence $\{N_n^*\}_{n\in\mathbb{N}}$ (in general $\{N_n^*\}_{n\in\mathbb{N}}$ will depend on the signal f) we can create convergence.

Weak and Strong Divergence

For a sequence $\{a_n\}_{n\in\mathbb{N}}$ we distinguish two modes of divergence:

Weak divergence if $\limsup_{n\to\infty} |a_n| = \infty$.
(existence of a subsequence $\{N_n\}_{n\in\mathbb{N}}$ such that $\lim_{n\to\infty} |a_{N_n}| = \infty$)
$\to$ adaptivity can help

Strong divergence if $\lim_{n\to\infty} |a_n| = \infty$.
($\lim_{n\to\infty} |a_{N_n}| = \infty$ for all subsequences $\{N_n\}_{n\in\mathbb{N}}$)
$\to$ adaptivity does not help

Notation

Paley–Wiener Space $\mathcal{PW}_\sigma^p$: Space of signals f with a representation $f(z) = 1/(2\pi) \int_{-\sigma}^{\sigma} g(\omega) e^{iz\omega} \, d\omega$, $z \in \mathbb{C}$, for some $g \in L^p[-\sigma, \sigma]$, $1 \leqslant p \leqslant \infty$. Norm: $\|f\|_{\mathcal{PW}_\sigma^p} = (1/(2\pi) \int_{-\sigma}^{\sigma} |g(\omega)|^p \, d\omega)^{1/p}$.

Strong Divergence

Strong divergence of the peak value:

There exists a signal $f \in \mathcal{PW}_\pi^1$ such that peak value of $S_N f$ diverges strongly, i.e., that

$$\lim_{N\to\infty} \max_{t\in\mathbb{R}} \left| \sum_{k=-N}^{N} f(k) \frac{\sin(\pi(t-k))}{\pi(t-k)} \right| = \infty.$$

$\to$ Adaptivity cannot be used to control the peak value of the Shannon sampling series.

H. Boche and B. Farrell, "Strong divergence of reconstruction procedures for the Paley-Wiener space PW_π^1 and the Hardy space H^1," Journal of Approximation Theory, Elsevier, 2014, 183, 98–117

Questions

What is the structure / size of the set of signals for which we have strong divergence?

Does this set contain a subset with linear structure?

Linear Structure / Spaceability

Linearity is an important property of signal spaces.

Lineability and spaceability are two mathematical concepts to study the existence of linear structures in general sets.

Definition:

A subset S of a Banach space X is said to be lineable if $S \cup \{0\}$ contains an infinite dimensional subspace.

A subset S of a Banach space X is said to be spaceable if $S \cup \{0\}$ contains a closed infinite dimensional subspace.

Easy to see linear structure for convergence:

- f_1, f_2 such that $P_N f$ converges $\Rightarrow$ convergence for $f_1 + f_2$

Difficult to show a linear structure for divergence:

- f_1, f_2 such that $P_N f$ diverges $\Rightarrow$ not necessarily divergence for $f_1 + f_2$

Example:
$f_1 = u_c + u_d$ and $f_2 = u_c - u_d$, where u_c is any signal with convergent and u_d any signal with divergent approximation process.
$\to$ For f_1 and f_2 we have divergence.
$\to$ For $f_1 + f_2 = 2u_c$ we do not have divergence.

$\to$ The sum of two signals, each of which leads to divergence, does not necessarily lead to divergence.

Spaceability and Strong Divergence

The set of signals with strong divergence of the peak value of the Shannon sampling series is spaceable.

Theorem: *The set of signals $f \in \mathcal{PW}_\pi^1$ for which the peak value of $S_N f$ diverges strongly, i.e., for which*

$$\lim_{N\to\infty} \max_{t\in\mathbb{R}} \left| \sum_{k=-N}^{N} f(k) \frac{\sin(\pi(t-k))}{\pi(t-k)} \right| = \infty \qquad (*)$$

is spaceable. That is, there exists an infinite dimensional closed subspace $\mathcal{D}_{\text{Shannon}} \subset \mathcal{PW}_\pi^1$ such that $(*)$ holds for all $f \in \mathcal{D}_{\text{Shannon}}$, $f \not\equiv 0$.

- Strong divergence of the Shannon sampling series is a frequent event.
- We have strong divergence for infinitely many signals that form an infinitely dimensional vector space.
- Any linear combination of signals from this vector space, that is not the zero signal, is again a signal that creates divergence.

Discussion

The subspace $\mathcal{D}_{\text{Shannon}}$ from the proof has interesting properties.

- $\mathcal{D}_{\text{Shannon}}$ has an unconditional basis , i.e., there exists a sequence of functions $\{\zeta_n\}_{n\in\mathbb{N}} \subset \mathcal{D}_{\text{Shannon}}$ such that for all $f \in \mathcal{D}_{\text{Shannon}}$ there exists a unique sequence of coefficients $\{a_n(f)\}_{n\in\mathbb{N}}$ such that $\lim_{N\to\infty} \|f - \sum_{n=1}^{N} a_n(f)\zeta_n\|_{\mathcal{PW}_\pi^1} = 0$.
- There exist two constants $C_1, C_2 > 0$ such that for all $f \in \mathcal{D}_{\text{Shannon}}$
$$C_1 \left(\sum_{n=1}^{\infty} |a_n(f)|^2 \right)^{\frac{1}{2}} \leqslant \|f\|_{\mathcal{PW}_\pi^1} \leqslant C_2 \left(\sum_{n=1}^{\infty} |a_n(f)|^2 \right)^{\frac{1}{2}}.$$
- $\mathcal{D}_{\text{Shannon}}$ is isomorphic to the Hilbert spaces l^2 and $\mathcal{PW}_\pi^2$.
- If we equip the space $\mathcal{D}_{\text{Shannon}}$ with the norm $\|f\|_{\mathcal{D}_{\text{Shannon}}} = \left(\sum_{n=1}^{\infty} |a_n(f)|^2\right)^{1/2}$ then it becomes a Hilbert space.

Conjecture

Non-equidistant sampling:

$$\sum_{k=-\infty}^{\infty} f(t_k)\phi_k(t), \quad t \in \mathbb{R} \qquad (**)$$

$\{t_k\}_{k\in\mathbb{Z}}$ is the sequence of sampling points, ϕ_k reconstruction functions

Theorem: For a large subclass of the set of sine type functions, if $\{t_k\}_{k\in\mathbb{Z}}$ is the zero set of a function in this class, then there exists a signal $f \in \mathcal{PW}_\pi^1$ such that the peak value of $(**)$ is weakly divergent, i.e., such that $\limsup_{N\to\infty} \max_{t\in\mathbb{R}} \left|\sum_{k=-\infty}^{\infty} f(t_k)\phi_k(t)\right| = \infty$.

Conjecture: We have strong divergence for a set that is spaceable.