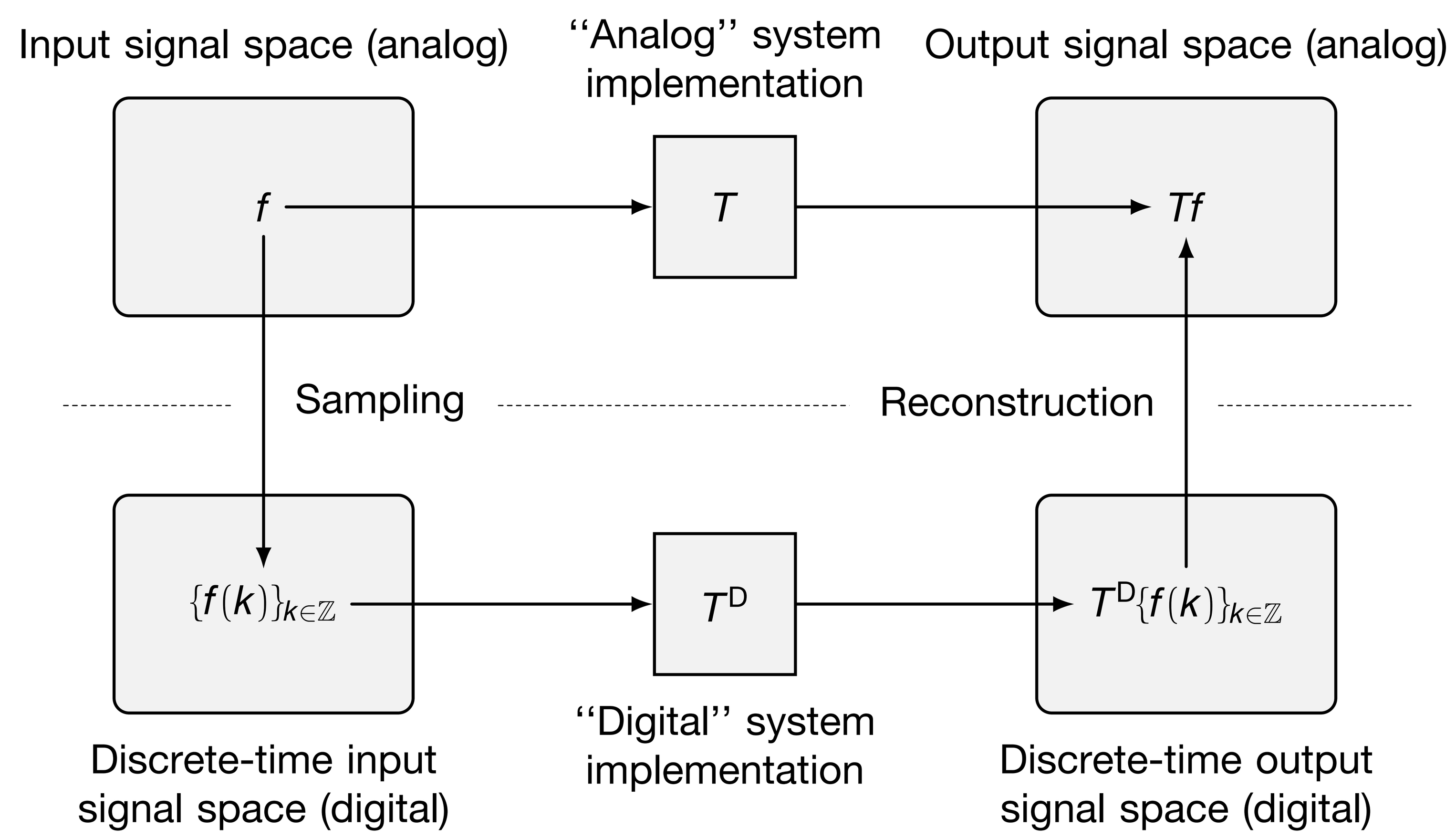
# ENERGY BLOWUP FOR TRUNCATED STABLE LTI SYSTEMS

Holger Boche and Ullrich J. Mönich
Technische Universität München
Lehrstuhl für Theoretische Informationstechnik

## Introduction

### Analog vs. digital implementation of a stable LTI system $T$




**Signal reconstruction (classical sampling):** Reconstruct a bandlimited signal $f$ from its samples $\{f(k)\}_{k\in\mathbb{Z}}$.

Sampling series: $$f(t) = \sum_{k=-\infty}^{\infty} f(k) \frac{\sin(\pi(t-k))}{\pi(t-k)}$$

**System approximation:** Approximate the output $Tf$ of a stable LTI system $T$ from the samples $\{f(k)\}_{k\in\mathbb{Z}}$ of the input signal $f$.

Approximation process 1: $$(Tf)(t) = \sum_{k=-\infty}^{\infty} f(k) h_T(t-k)$$

Approximation process 2: $$(Tf)(t) = \sum_{k=-\infty}^{\infty} f(t-k) h_T(k)$$

The interpolation kernel is $h_T = T\,\mathrm{sinc}$.

## Notation

**Paley–Wiener space $\mathcal{PW}^2_\sigma$:** Space of signals $f$ with a representation $f(z) = 1/(2\pi)\int_{-\sigma}^{\sigma} \hat{f}(\omega)\, \mathrm{e}^{iz\omega}\, \mathrm{d}\omega$, $z \in \mathbb{C}$, for some $\hat{f} \in L^2[-\sigma,\sigma]$. Norm: $\|f\|_{\mathcal{PW}^2_\sigma} = (1/(2\pi)\int_{-\sigma}^{\sigma} |g(\omega)|^2\, \mathrm{d}\omega)^{1/2}$.

**Stable LTI systems:** A linear system $T : \mathcal{PW}^2_\pi \to \mathcal{PW}^2_\pi$ is called stable linear time invariant (LTI) system if:

- $T$ is bounded, i.e., $\|T\| = \sup_{\|f\|_{\mathcal{PW}^2_\pi}\le 1} \|Tf\|_{\mathcal{PW}^2_\pi} < \infty$ and
- $T$ is time invariant, i.e., $\big(Tf(\cdot - a)\big)(t) = (Tf)(t-a)$ for all $f \in \mathcal{PW}^2_\pi$ and $t, a \in \mathbb{R}$.

Representation: For every stable LTI system $T : \mathcal{PW}^2_\pi \to \mathcal{PW}^2_\pi$ there is exactly one function $\hat{h}_T \in L^\infty[-\pi,\pi]$ such that $(Tf)(t) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \hat{h}_T(\omega)\hat{f}(\omega)\, \mathrm{e}^{i\omega t}\, \mathrm{d}\omega$ for all $f \in \mathcal{PW}^2_\pi$. We have $h_T = T\,\mathrm{sinc}$.

## Signal Reconstruction

**Shannon sampling series:**

$$(S_N f)(t) = \sum_{k=-N}^{N} f(k) \frac{\sin(\pi(t-k))}{\pi(t-k)}, \quad t \in \mathbb{R}.$$

**Convergence in the $\mathcal{PW}^2_\pi$-norm:**

$$\lim_{N\to\infty} \int_{-\infty}^{\infty} |f(t) - (S_N f)(t)|^2\, \mathrm{d}t = 0$$

## Basics of System Approximation

**System approximation process 1:**
The time variable $t \in \mathbb{R}$ is in the argument of $h_T$.

$$(T_N^{(1)} f)(t) = \sum_{k=-N}^{N} f(k) h_T(t-k), \quad t \in \mathbb{R}$$

For all $f \in \mathcal{PW}^2_\pi$ and all stable LTI systems $T\colon \mathcal{PW}^2_\pi \to \mathcal{PW}^2_\pi$ we have:

norm convergence: $$\lim_{N\to\infty} \int_{-\infty}^{\infty} \left|(Tf)(t) - (T_N^{(1)} f)(t)\right|^2\, \mathrm{d}t = 0$$

and

uniform convergence: $$\lim_{N\to\infty} \max_{t\in\mathbb{R}} \left|(Tf)(t) - (T_N^{(1)} f)(t)\right| = 0.$$

→ The system approximation process $T_N^{(1)} f$ converges in the $\mathcal{PW}^2_\pi$-norm and uniformly on the real axis.

**System approximation process 2:**
The time variable $t \in \mathbb{R}$ is in the argument of $f$.

$$(T_N^{(2)} f)(t) = \sum_{k=-N}^{N} f(t-k) h_T(k), \quad t \in \mathbb{R} \qquad (*)$$

For all $f \in \mathcal{PW}^2_\pi$ and all stable LTI systems $T\colon \mathcal{PW}^2_\pi \to \mathcal{PW}^2_\pi$ we have

uniform convergence: $$\lim_{N\to\infty} \max_{t\in\mathbb{R}} \left|(Tf)(t) - (T_N^{(2)} f)(t)\right| = 0.$$

→ $T_N^{(1)} f$ and $T_N^{(2)} f$ have the same global convergence behavior.

## Question

**Observation:**

$$\sum_{k=-\infty}^{\infty} f(k) h_T(t-k) \quad \text{and} \quad \sum_{k=-\infty}^{\infty} f(t-k) h_T(k)$$

are equiconvergent with respect to the maximum norm.

**Question:**
Do we have equiconvergence also with respect to the $\mathcal{PW}^2_\pi$-norm?

## Linear Structures / Spaceability

Linearity is an important property of signal spaces.
Lineability and spaceability are two mathematical concepts to study the existence of linear structures in general sets.

**Definition:** A subset $S$ of a Banach space $X$ is said to be lineable if $S \cup \{0\}$ contains an infinite dimensional subspace.
A subset $S$ of a Banach space $X$ is said to be spaceable if $S \cup \{0\}$ contains a closed infinite dimensional subspace.

Easy to see linear structure for convergence:

- $f_1, f_2$ such that (*) converges ⇒ convergence for $f_1 + f_2$

Difficult to show a linear structure for divergence:

- $f_1, f_2$ such that (*) diverges ⇒ not necessarily divergence for $f_1 + f_2$

**Example:** $f_1 = u_c + u_d$ and $f_2 = u_c - u_d$, where $u_c$ is any signal with convergent and $u_d$ any signal with divergent approximation process.
→ For $f_1 + f_2 = 2u_c$ we do not have divergence.

→ The sum of two signals, each of which leads to divergence, does not necessarily lead to divergence.

## Energy Blowup

Let $\mathcal{T}$ denote the set of all stable LTI systems $T\colon \mathcal{PW}^2_\pi \to \mathcal{PW}^2_\pi$.

**Theorem:** *There exist an infinite dimensional closed subspace $D_{sig} \subset \mathcal{PW}^2_\pi$ and an infinite dimensional closed subspace $D_{sys} \subset \mathcal{T}$ such that for all $f \in D_{sig}$, $f \not\equiv 0$, and all $T \in D_{sys}$, $T \not\equiv 0$, we have*

$$\limsup_{N\to\infty} \int_{-\infty}^{\infty} \left| \sum_{k=-N}^{N} f(t-k) h_T(k) \right|^2 \, dt = \infty.$$

→ The system approximation process $T_N^{(2)} f$ can be divergent with respect to the $L^2$-norm.
→ The sets of functions in $\mathcal{PW}^2_\pi$ and energetically stable LTI systems having this property are jointly spaceable.

## Discussion

For $\mathcal{PW}^1_\pi$ and LTI systems $T\colon \mathcal{PW}^1_\pi \to \mathcal{PW}^1_\pi$ joint spaceability can be shown even for pointwise divergence. In [BM16] it was sown that for arbitrary $t \in \mathbb{R}$ the sets of functions $f \in \mathcal{PW}^1_\pi$ and stable LTI systems $T\colon \mathcal{PW}^1_\pi \to \mathcal{PW}^1_\pi$ that satisfy

$$\limsup_{N\to\infty} \left| \sum_{k=-N}^{N} f(t-k) h_T(k) \right| = \infty$$

are jointly spaceable.

[BM16] H. Boche and U. J. Mönich, “Signal and system spaces with non-convergent sampling representation,” in *Proceedings of European Signal Processing Conference (EUSIPCO)*, Aug. 2016, pp. 2131–2135