On the Computability of the Hilbert Transform

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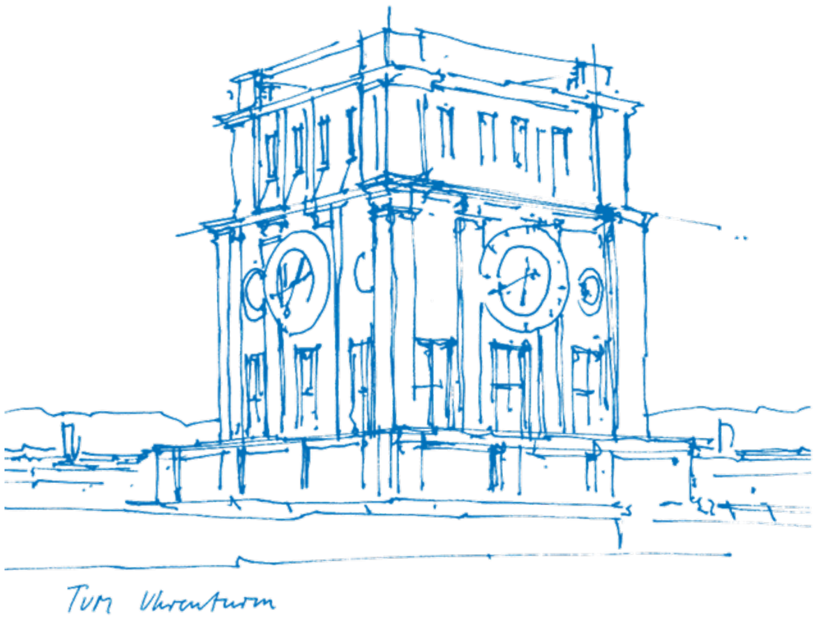


Outline

1. What is the Hilbert transform?
2. Why it is important?
3. How to calculate the Hilbert transform?
4. Our approach to investigate the computability of the Hilbert transform
   - Signal spaces of finite energy $\{\mathscr{B}_\beta\}_{\beta \geq 1}$
   - Axiomatic for approximation operators
5. Main Results: Regions of Divergence and Convergence
6. Turing computability
7. Outlook

What is the Hilbert transform?

The Hilbert Transform

- ▷ We consider continuous functions f on $\mathbb{T} = [-\pi, \pi]$ with $f(-\pi) = f(\pi)$.
- ▷ Assume f can be represented by its Fourier series

$$f(t) = \sum_{n=-\infty}^{\infty} c_n(f)\, \mathrm{e}^{\mathrm{i}nt} \qquad \text{with Fourier coefficients} \qquad c_n(f) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\tau)\, \mathrm{e}^{-\mathrm{i}n\tau}\, \mathrm{d}\tau\,.$$

- ▷ Its harmonic *conjugate $\widetilde{f}$* is given by

$$\widetilde{f}(t) = (\mathrm{H}f)(t) = -\mathrm{i} \sum_{n=-\infty}^{\infty} \mathrm{sgn}(n)\, c_n(f)\, \mathrm{e}^{\mathrm{i}nt} \qquad \text{with} \qquad \mathrm{sgn}(n) = \begin{cases} -1 & : \; n < 0 \\ 0 & : \; n = 0 \\ 1 & : \; n > 0 \end{cases}$$

such that

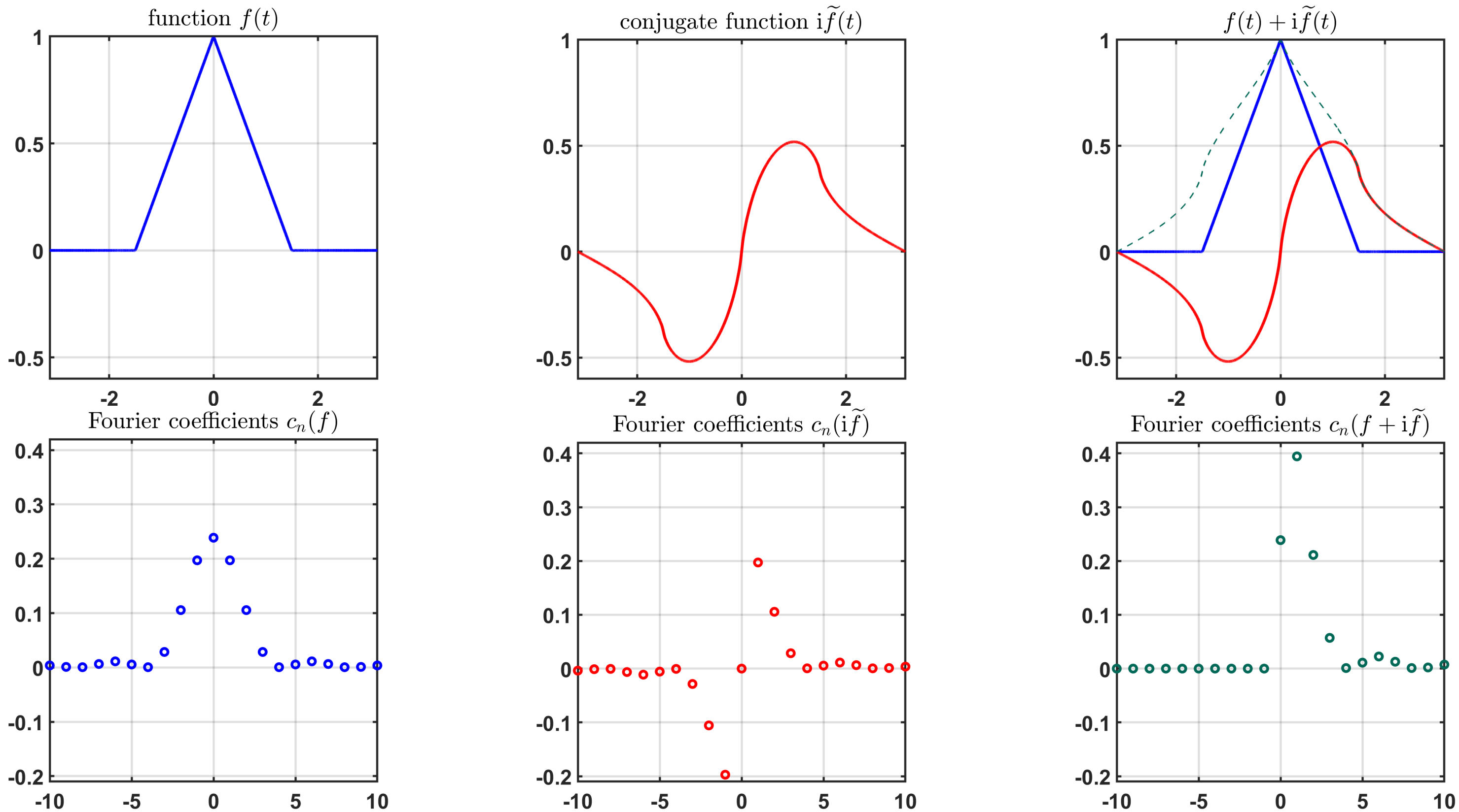
$$F(t) = f(t) + \mathrm{i}\widetilde{f}(t) = c_0(f) + 2 \sum_{n=1}^{\infty} c_n(f)\, \mathrm{e}^{\mathrm{i}nt} = \sum_{n=0}^{\infty} c_n(F)\, \mathrm{e}^{\mathrm{i}nt}\,,$$

i.e. such that

$$c_n(F) = c_n(f + \mathrm{i}\widetilde{f}) = 0 \qquad \text{for all } n < 0\,.$$

- ▷ The mapping $\mathrm{H} : f \mapsto \widetilde{f}$ is called *Hilbert transform*.

The Hilbert Transform – Illustration



$$f(t) = \sum_{n=-\infty}^{\infty} c_n(f)\, \mathrm{e}^{\mathrm{i}nt}$$

$$\mathrm{i}\widetilde{f}(t) = \sum_{n=-\infty}^{\infty} \mathrm{sgn}(n)\, c_n(f)\, \mathrm{e}^{\mathrm{i}nt}$$

$$f(t) + \mathrm{i}\widetilde{f}(t) = c_0(f) + 2\sum_{n=1}^{\infty} c_n(f)\, \mathrm{e}^{\mathrm{i}nt}$$

The Hilbert Transform – Closed Form

So the Hilbert transform is the mapping

$$\begin{aligned} \mathrm{H} \;:\; & f & \mapsto \; & \widetilde{f} \\ \mathrm{H} \;:\; & \textstyle\sum_{n\in\mathbb{Z}} c_n(f)\,\mathrm{e}^{\mathrm{i}nt} & \mapsto \; & -\mathrm{i}\textstyle\sum_{n\in\mathbb{Z}} \operatorname{sgn}(n)\, c_n(f)\,\mathrm{e}^{\mathrm{i}nt} \end{aligned} \tag{1}$$

with the Fourier coefficients

$$c_n(f) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\tau)\,\mathrm{e}^{-\mathrm{i}n\tau}\,\mathrm{d}\tau \tag{2}$$

Inserting the Fourier coefficients (2) into (1), one obtains a closed form expression for $\mathrm{H} \;:\; f \mapsto \widetilde{f}$

$$\widetilde{f}(t) = \big(\mathrm{H}f\big)(t) = \lim_{\varepsilon\to 0}\frac{1}{2\pi}\int_{\varepsilon\le|t-\tau|\le\pi}\frac{f(\tau)}{\tan([t-\tau]/2)}\,\mathrm{d}\tau\,, \qquad t\in\mathbb{T}=[-\pi,\pi]\,. \tag{HT}$$

The Hilbert Transform – Trigonometric Form

- Assume f can be represented by its Fourier series

$$f(t) = \sum_{n=-\infty}^{\infty} c_n(f)\,\mathrm{e}^{\mathrm{i}nt} \qquad \text{with Fourier coefficients} \qquad c_n(f) = \frac{1}{2\pi}\int_{\mathbb{T}} f(\tau)\,\mathrm{e}^{-\mathrm{i}n\tau}\,\mathrm{d}\tau\ .$$

- The Fourier series of f can be written equivalently as

$$f(t) = \frac{a_0(f)}{2} + \sum_{n=1}^{\infty} a_n(f)\cos(nt) + b_n(f)\sin(nt)\ .$$

with the Fourier coefficients

$$a_n(f) = \frac{1}{\pi}\int_{\mathbb{T}} f(t)\cos(nt)\,\mathrm{d}t \qquad \text{and} \qquad b_n(f) = \frac{1}{\pi}\int_{\mathbb{T}} f(t)\sin(nt)\,\mathrm{d}t\ .$$

- Then its harmonic conjugate is given by

$$\widetilde{f}(t) = (\mathrm{H}f)(t) = \sum_{n=1}^{\infty} a_n(f)\sin(nt) - b_n(f)\cos(nt)\ .$$

Why the Hilbert transform is important?

Hilbert Transform and Causality

▷ The Hilbert transform is closely related to "causality".

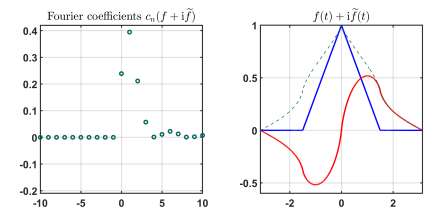
▷ Therefore, it plays an important role in science and engineering.

▷ In physics, it is also known as Kramers-Kronig relation.

Hardy spaces – Spaces of "causal functions"

- For $1 \leq p < \infty$, let $L^P(\mathbb{T})$ be the usual spaces of p-integrable functions on $\mathbb{T}$.
- For every $f \in L^p(\mathbb{T})$ its Fourier coefficients $c_n(f)$, $n = 0, \pm 1, \pm 2, \ldots$ are well defined
- Define the "causal subspaces"

$$\mathscr{H}^p = \{ f \in L^p(\mathbb{T}) \ : \ c_n(f) = 0 \text{ for all } n < 0 \} \ .$$



▷ Let $f \in \mathscr{H}^p$ be arbitrary and write $f(t) = u(t) + \mathrm{i}v(t) = |f(t)|\, \mathrm{e}^{\mathrm{i}\varphi(t)}$

▷ By the definition of the Hilbert transform, we have

$$v(t) = (\mathrm{H}u)(t) \qquad \text{Kramers-Kronig relation}$$

$$\varphi(t) = \mathrm{H}(\log|f(t)|) \qquad \text{Phase retrieval}$$

▶ V. Pohl, N. Li, H. Boche, "Phase Retrieval in Spaces of Analytic Functions on the Unit Disk," *SampTA 2017.*

Example – Causal Linear Systems

▷ Input-output relation of a linear system S

$$y_n = \sum_{k=0}^{\infty} c_k x_{n-k}\,, \qquad n \in \mathbb{Z}$$

Input sequence : $\{x_n\}_{n\in\mathbb{Z}} \subset \mathbb{C}$
Output sequence : $\{y_n\}_{n\in\mathbb{Z}} \subset \mathbb{C}$
Impulse response of S : $\{c_n\}_{n\in\mathbb{Z}} \subset \mathbb{C}$

▷ $c_k = 0$ for all $k < 0$
$\Rightarrow$ the output y_n depends only on the past input symbols $\{x_n, x_{n-1}, x_{n-2}, \dots\}$
$\Rightarrow$ S is a causal system.

▷ Take discrete-time Fourier transform (DTFT) of the input-output relation yields

$$Y(\omega) = C(\omega)X(\omega)\,, \qquad \omega \in [-\pi, \pi)$$

with the *transfer function* $C(\omega)$ of S

$$C(\omega) = \sum_{k=0}^{\infty} c_k \mathrm{e}^{\mathrm{i}k\omega}$$

▷ Because S is causal, we have $C \in \mathscr{H}^p$ and therefore

$$\Im[C(\omega)] = \mathrm{H}(\Re[C(\omega)]) \qquad \text{and} \qquad \arg[C(\omega)] = \mathrm{H}(\log|C(\omega)|)\,.$$

– So $C(\omega)$ is already uniquely determined by its real part $\Re[C(\omega)]$ or by its amplitude $|C(\omega)|$.
– The corresponding imaginary part or phase can be calculated using the Hilbert transform.

Applications and Properties

Applications

- ▷ Phase retrieval
- ▷ Prediction and estimation of stationary time series – spectral factorization.
  - Let $\boldsymbol{x} = \{x_n\}_{n=-\infty}^{\infty}$ be a w.s.s. stochastic process with spectral density $\Phi_{\boldsymbol{x}} \in L^1(\mathbb{T})$.
  - We want to have a linear estimate of future values based on past observations

$$\widehat{x}_{n+\tau} = \sum_{k=0}^{\infty} c_k x_{n-k} \ , \qquad \tau = 1, 2, \ldots$$

  - The transfer function $C(\omega)$ of the optimal (MMSE) filter is determined by the spectral factor $\Phi_{\boldsymbol{x}}^{+}$.
  - The mapping $\Phi_{\boldsymbol{x}} \to \Phi_{\boldsymbol{x}}^{+}$ involves the Hilbert transform (Kolmogorov or ceptral method).
- ▷ Solution of the Kardion, Quanteninformation theory, etc.

Properties

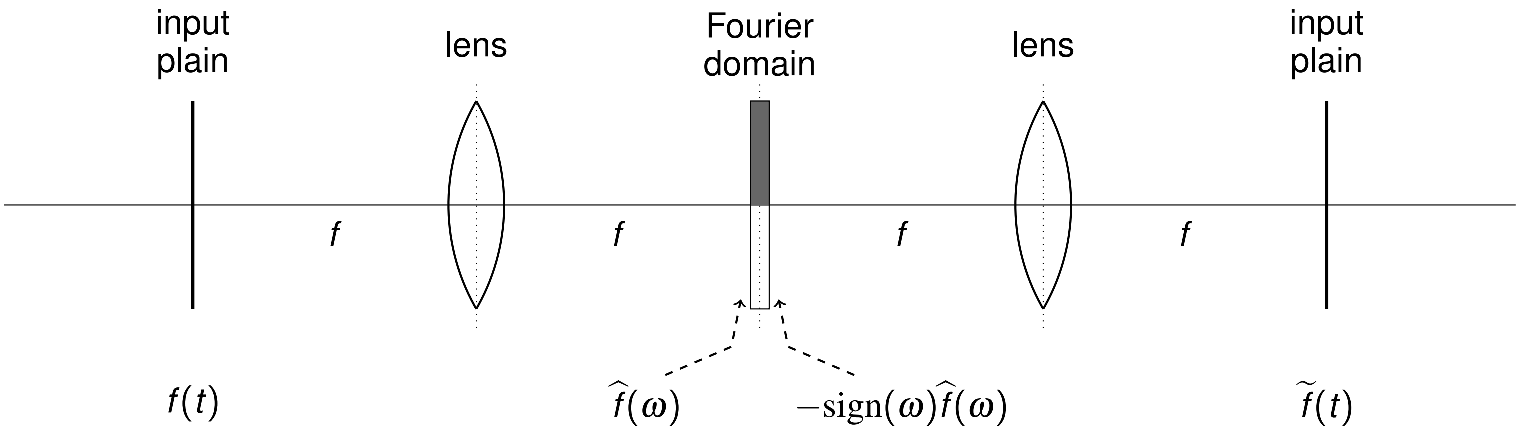
- ▷ Hilbert transform is a bounded mapping $\mathrm{H} : L^p(\mathbb{T}) \to L^p(\mathbb{T})$, $1 < p < \infty$.
- ▷ The Hilbert transform is a bounded mapping $\mathrm{H} : L^\infty(\mathbb{T}) \to BMO$.
- ▷ $\mathrm{H} : \mathscr{C}(\mathbb{T}) \to \mathscr{C}(\mathbb{T})$ is not bounded but $\mathrm{H} : \mathscr{C}^\alpha(\mathbb{T}) \to \mathscr{C}(\mathbb{T})$ is bounded.
- ▷ For $f \in \mathscr{C}(\mathbb{T})$, we have $\widetilde{f} = \mathrm{H}f \in L^p(\mathbb{T})$ for every $1 \leq p < \infty$ but $\widetilde{f} = \mathrm{H}f \notin \mathscr{C}(\mathbb{T})$, in general.

How to calculate the Hilbert transform?

Calculation on Analog Computers

There do exist analog computers for calculating the Hilbert transform.

Example (Hilbert Transformation with a $4f$ imaging system)



Calculation on Digital Computers

We want calculate numerically the Hilbert transform of a given $f \in \mathscr{B} = \left\{ f \in \mathscr{C}(\mathbb{T}) \ : \ \widetilde{f} \in \mathscr{C}(\mathbb{T}) \right\}$

$$\widetilde{f}(t) = \big(\mathrm{H}f\big)(t) = \lim_{\varepsilon \to 0} \frac{1}{2\pi} \int_{\varepsilon \leq |t-\tau| \leq \pi} \frac{f(\tau)}{\tan([t-\tau]/2)} \mathrm{d}\tau \,, \qquad t \in \mathbb{T} = [-\pi, \pi] \,. \tag{HT}$$

Challenges

- ▷ Singular integral kernel $\Rightarrow$ principal value integral in (HT)
- ▷ Calculation on digital computers

  $\Rightarrow$ calculation of (HT) has to be based on finitely many samples of the function f:

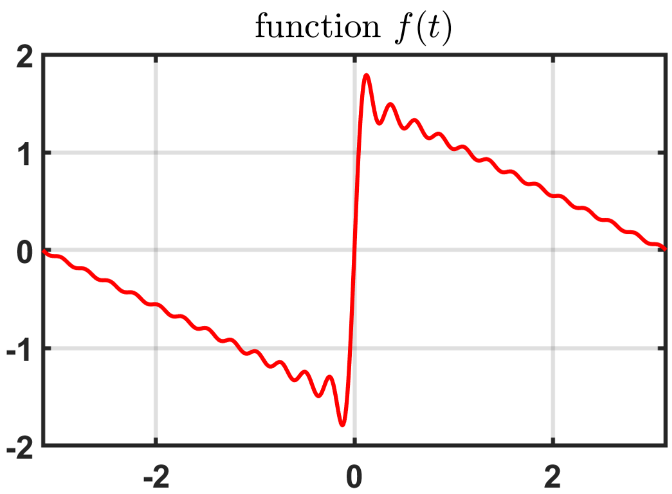
$$\left\{ f(\lambda_n) \ : \ n = 1, 2, \ldots, N \right\}$$

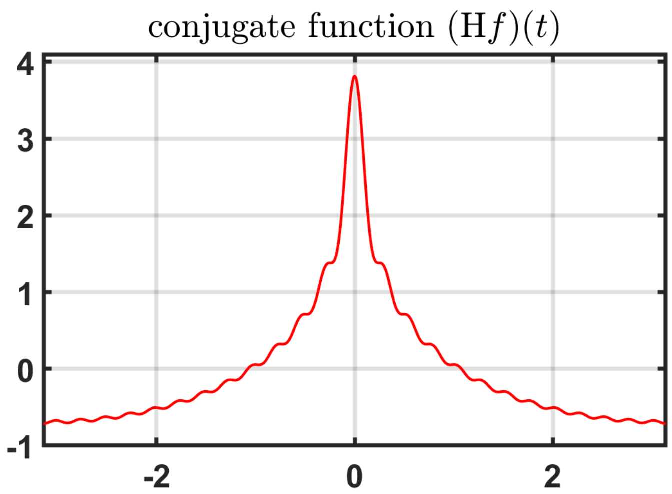
  – finite storage (memory)
  – finite computing time

  $\Rightarrow$ we neglect quantization errors
- ▷ Find an approximation $\mathrm{H}_N : \{f(\lambda_n)\}_{n=1}^N \mapsto \widetilde{f}_N$ such that

$$\lim_{N\to\infty} \left\| \widetilde{f} - \widetilde{f}_N \right\|_\infty = \lim_{N\to\infty} \left\| \widetilde{f} - \mathrm{H}_N f \right\|_\infty = \lim_{N\to\infty} \max_{t \in \mathbb{T}} \left| \widetilde{f}(t) - (\mathrm{H}_N f)(t) \right| = 0 \qquad \text{for all } f \in \mathscr{B} \,.$$

Calculation on Digital Computers – Illustration





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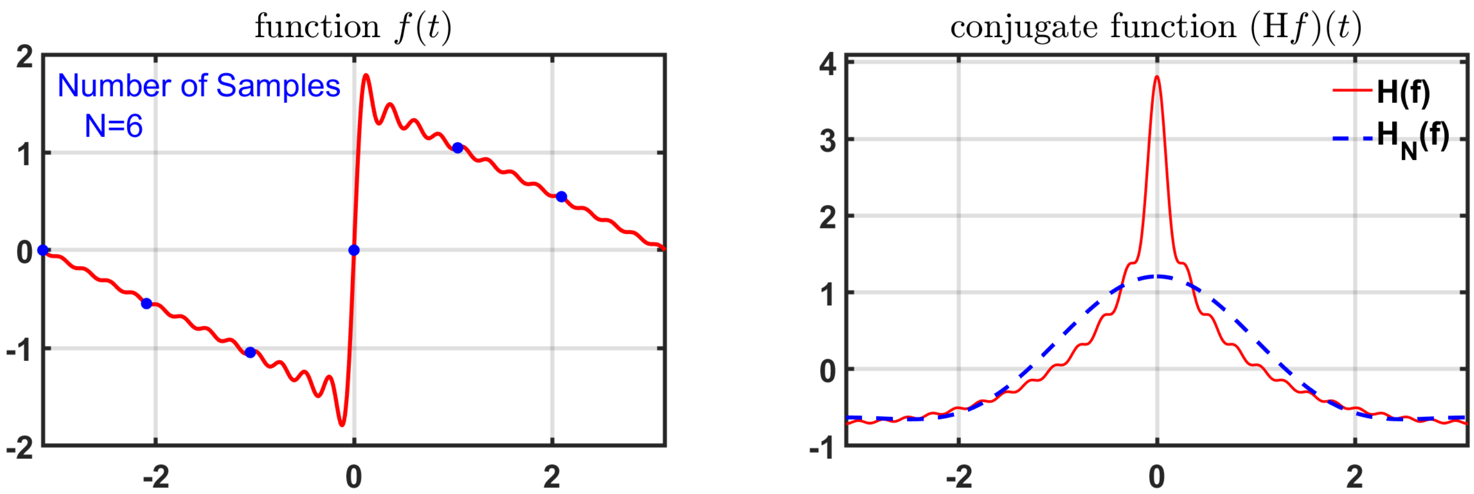

A linear approximation procedure

Sampling set: $\Lambda_N = \left\{\lambda_{N,n} = \left(\frac{2n-N}{N}\right)\pi \;:\; n = 0,1,2,\dots,N-1\right\}$

Operators: $(\mathrm{H}_N f)(t) = \sum_{n=0}^{N-1} f(\lambda_{N,n})\mathscr{D}_N(t-\lambda_{N,n})$

Interpolation kernel: $\mathscr{D}_N(t) = \frac{2}{N}\sum_{n=1}^{N/2-1}\sin(nt)$

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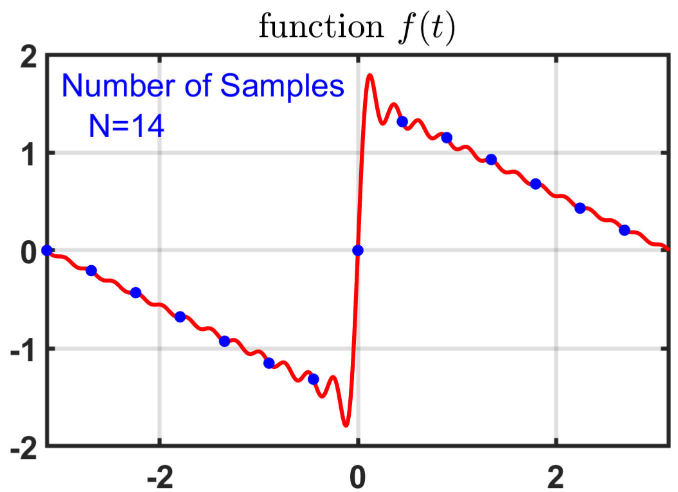

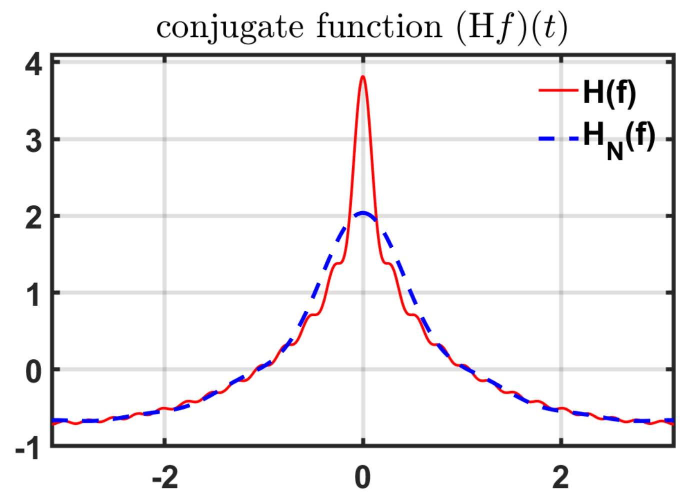

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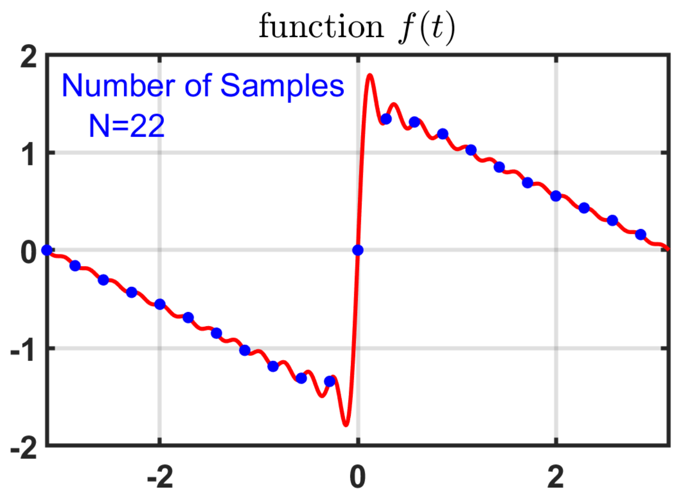

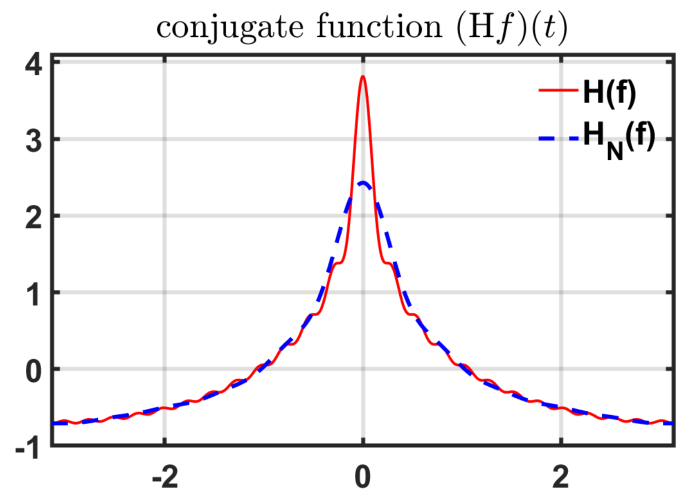

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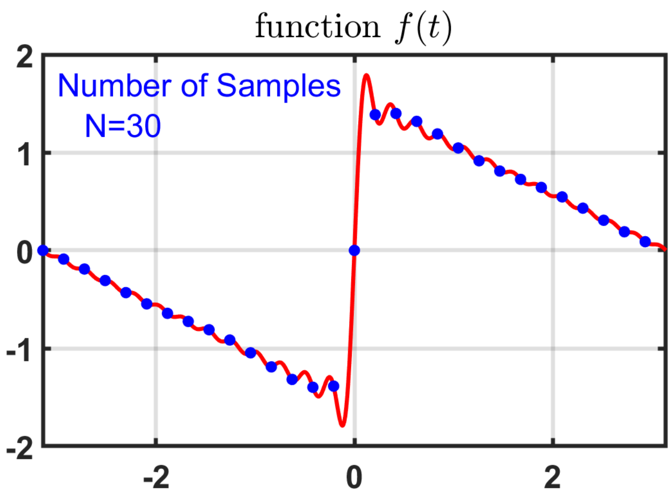

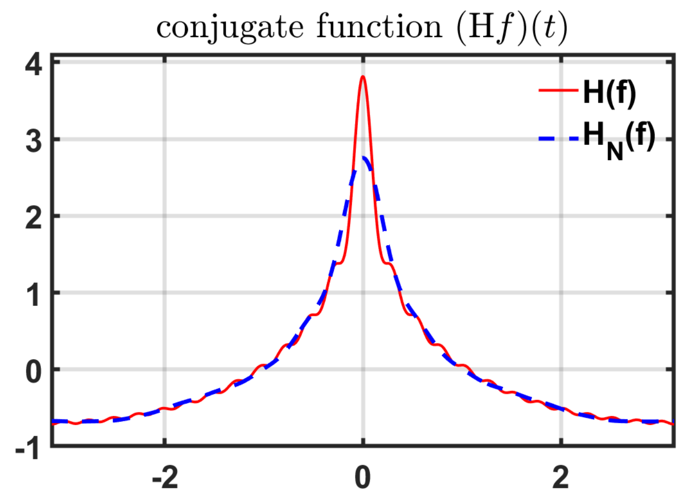

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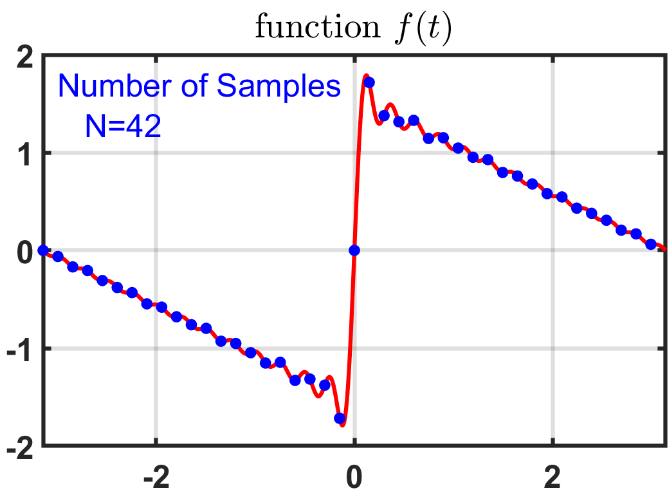

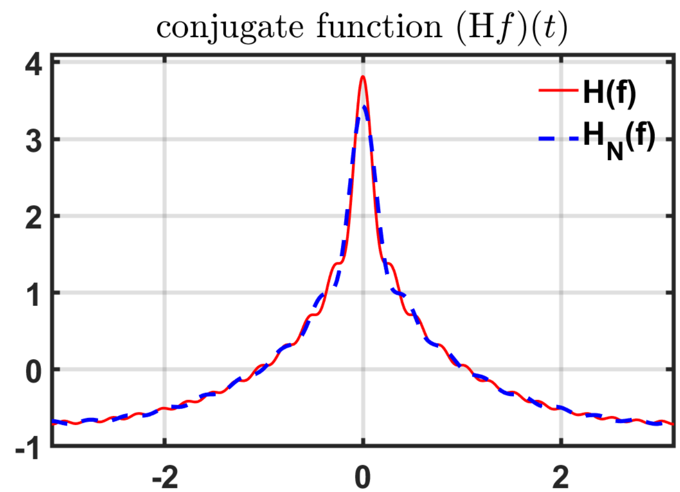

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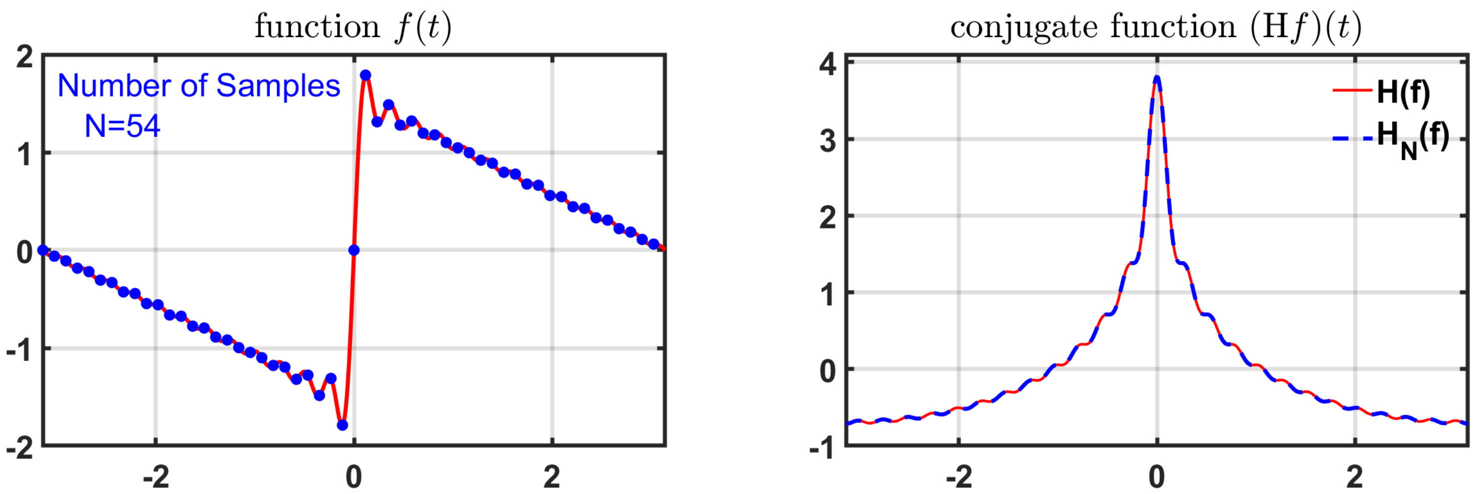

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1. Example of a Hilbert Transform Approximation

- For every $N \in \mathbb{N}$, we define the equidistant sampling set

$$\Lambda_N = \left\{ \lambda_{N,n} = \tfrac{2n-N}{N}\pi \,:\, n = 0, 1, \dots, N-1 \right\} .$$

- We know that

$$\widetilde{f}(t) = (\mathrm{H}f)(t) = -\mathrm{i} \sum_{n=-\infty}^{\infty} \mathrm{sgn}(n)\, c_n(f)\, \mathrm{e}^{\mathrm{i}nt} \qquad \text{with} \qquad c_n(f) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\tau)\, \mathrm{e}^{-\mathrm{i}n\tau}\, \mathrm{d}\tau$$

- We approximate $\widetilde{f}$ by its partial conjugate Fourier series

$$(\mathrm{H}_N f)(t) = -\mathrm{i} \sum_{n=-(N/2-1)}^{N/2-1} \mathrm{sgn}(n)\, c_{N,n}(f)\, \mathrm{e}^{\mathrm{i}nt}$$

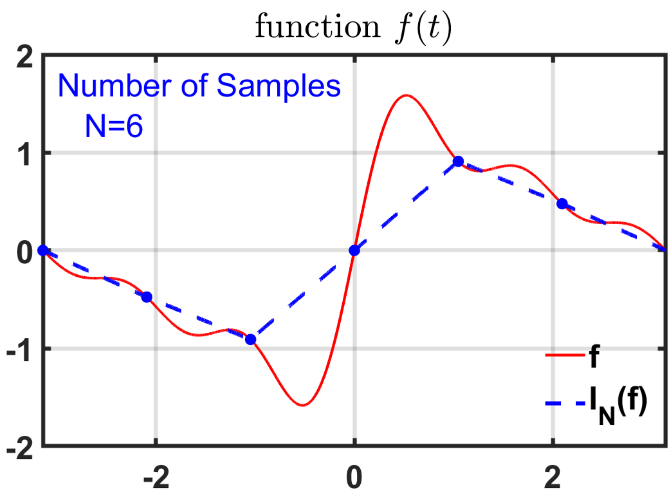
and exchange the exact Fourier coefficients $c_n(f)$ for approximations $c_{N,n}(f)$, obtained by replacing the integral in the formula for the Fourier coefficients with the left Riemann sum with nodes Λ_N.

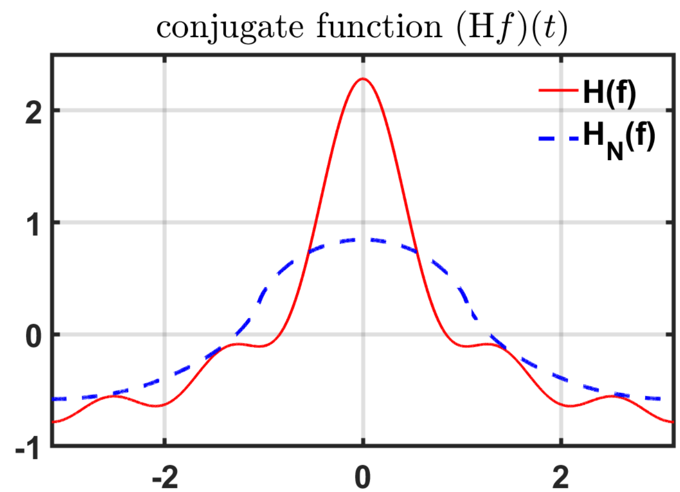
$$c_n(f) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\tau)\, \mathrm{e}^{-\mathrm{i}n\tau}\, \mathrm{d}\tau \qquad \mapsto \qquad c_{N,n}(f) = \tfrac{1}{N} \sum_{k=0}^{N-1} f(\lambda_{N,k})\, \mathrm{e}^{\mathrm{i}n\lambda_{N,k}}$$

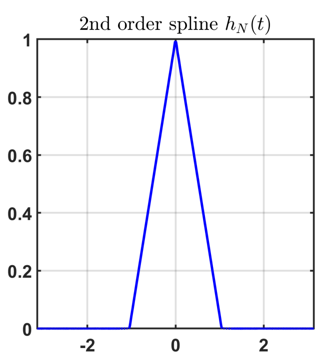
- This yields the approximation operator

$$(\mathrm{H}_N f)(t) = \sum_{n=0}^{N-1} f(\lambda_{N,n})\, \mathscr{D}_N(t - \lambda_{N,n}) \qquad \text{with kernel} \qquad \mathscr{D}_N(t) = \frac{2}{N} \sum_{n=1}^{N/2-1} \sin(nt) .$$

Example 2: Calculation via Linear Interpolation







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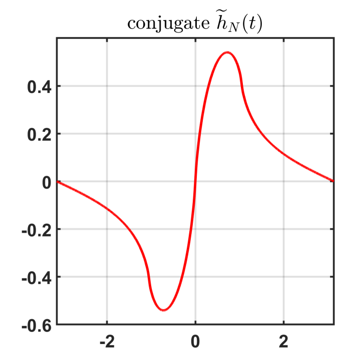

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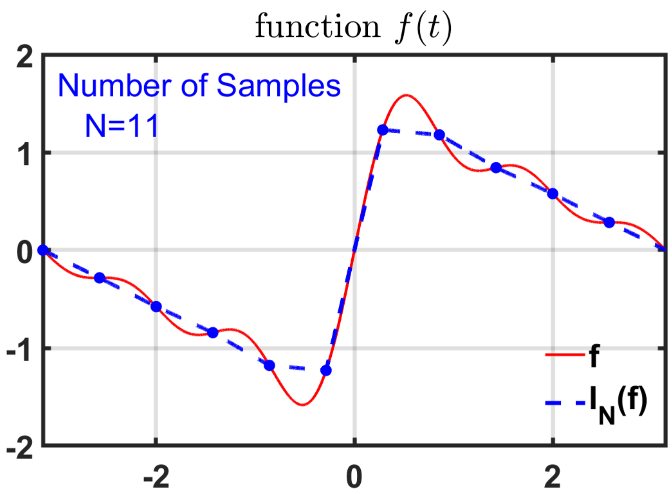

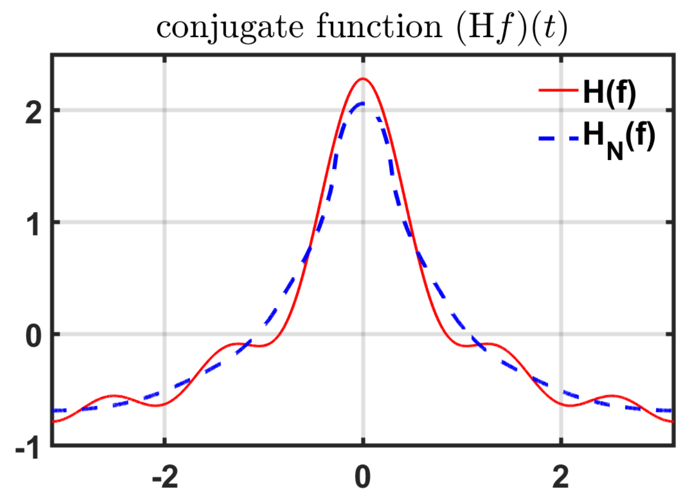

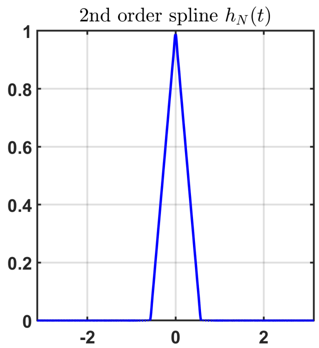

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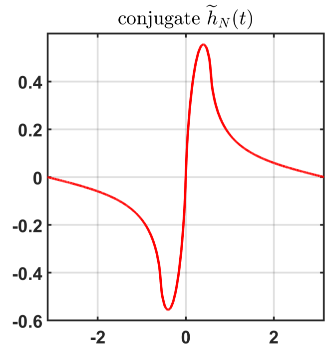

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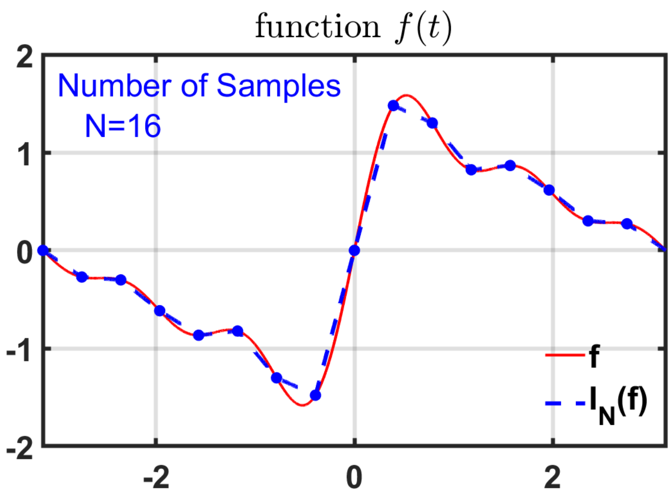

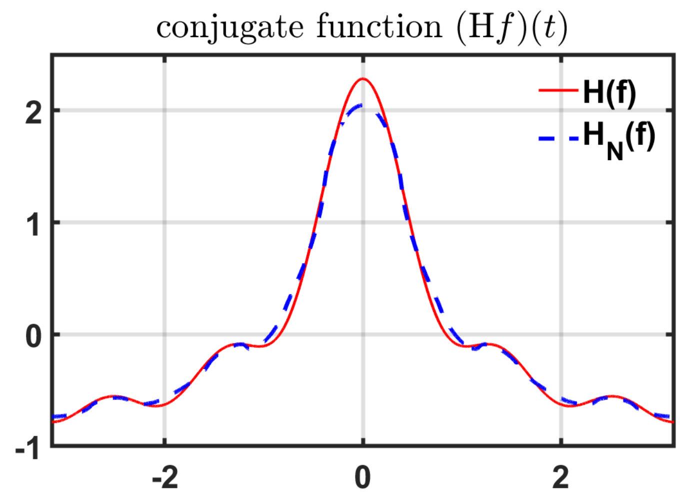

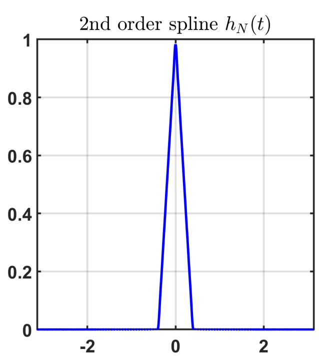

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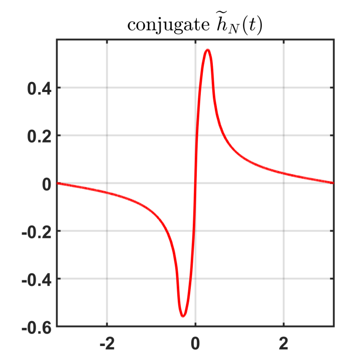
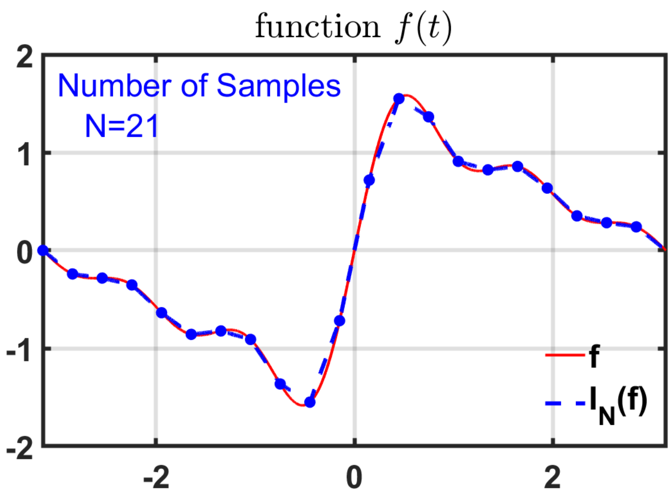
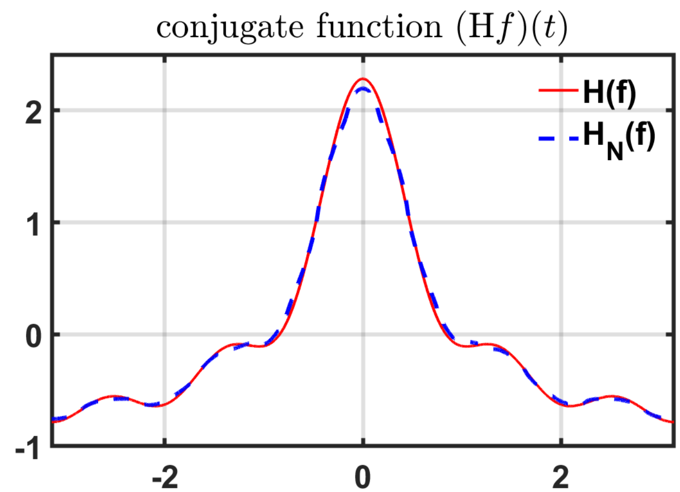
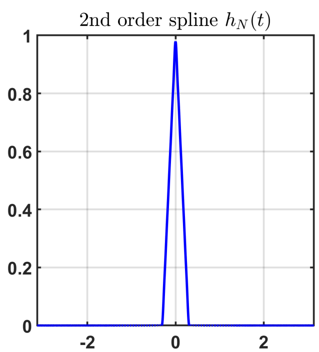


Example 2: Calculation via Linear Interpolation



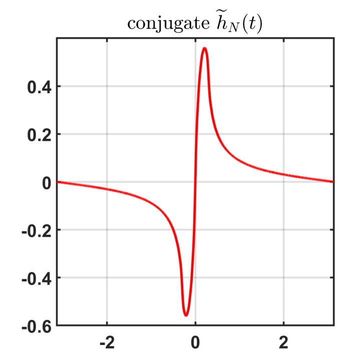




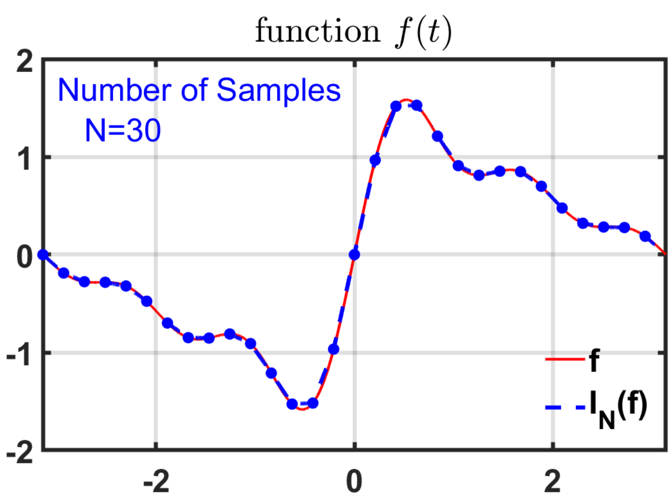
Sampling set: $\Lambda_N = \{\lambda_{N,n}\}_{n=0}^{N-1}$

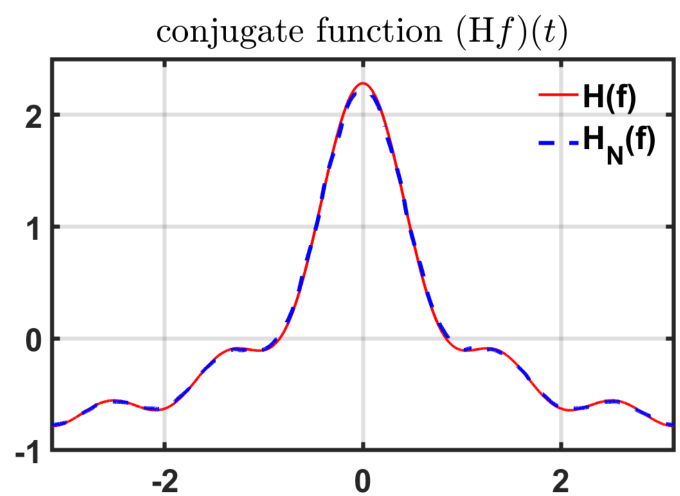
Interpolation: $(\mathrm{I}_N f)(t) = \sum_{n=0}^{N-1} f(\lambda_{N,n})\, h_N(t - \lambda_{N,n})$

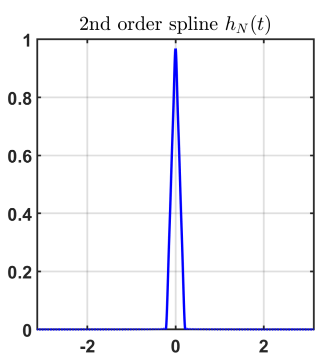
Operators: $(\mathrm{H}_N f)(t) = \sum_{n=0}^{N-1} f(\lambda_{N,n})\, \widetilde{h}_N(t - \lambda_{N,n})$



Example 2: Calculation via Linear Interpolation



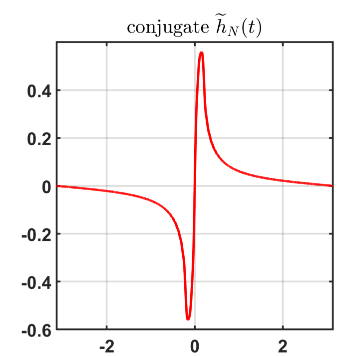




Sampling set: $\Lambda_N = \{\lambda_{N,n}\}_{n=0}^{N-1}$

Interpolation: $(\mathrm{I}_N f)(t) = \sum_{n=0}^{N-1} f(\lambda_{N,n})\, h_N(t - \lambda_{N,n})$

Operators: $(\mathrm{H}_N f)(t) = \sum_{n=0}^{N-1} f(\lambda_{N,n})\, \widetilde{h}_N(t - \lambda_{N,n})$



Variations of the Previous Examples

To obtain other approximation operators $\{\mathrm{H}_N\}_{N\in\mathbb{N}}$, the previous two examples can be varied in different ways:

1. Approximation by Fourier series

$$(\mathrm{H}_N f)(t) = -\mathrm{i} \sum_{n=-(N/2-1)}^{N/2-1} \operatorname{sgn}(n)\, c_{N,n}(f)\, \mathrm{e}^{\mathrm{i}nt} \qquad \text{with} \qquad c_{N,n}(f) = \frac{1}{N} \sum_{k=0}^{N-1} f(\lambda_{N,k})\, \mathrm{e}^{\mathrm{i}n\lambda_{N,k}}$$

- Consider non-equidistant sampling sets
- Use different numerical integration methods (trapezoidal rule, Newton–Cotes formulas, etc.)
- Consider other summation formulas (Cesaro, Fejér, etc.)

2. Linear Interpolation $(\mathrm{H}_N f)(t) = (\mathrm{H}\mathrm{I}_N t)(t)$

- Consider non-equidistant sampling sets
- Apply other interpolation methods I_N, e.g. higher order splines

---

All these linear methods will have the form

$$(\mathrm{H}_N f)(t) = \sum_{\lambda\in\Lambda_N} f(\lambda)\kappa_N(t-\lambda)$$

Hilbert Transform Approximations – Design Goal

Hilbert Transform: $$\widetilde{f}(t) = (\mathrm{H}f)(t) = \lim_{\varepsilon \to 0} \frac{1}{2\pi} \int_{\varepsilon \le |t-\tau| \le \pi} \frac{f(\tau)}{\tan([t-\tau]/2)} \mathrm{d}\tau \,, \qquad t \in [-\pi, \pi) \,.$$

▷ Design a sequence $\{\Lambda_N\}_{N \in \mathbb{N}}$ of sampling sets:

$$\Lambda_N = \{\lambda_{N,1}, \lambda_{N,2}, \dots, \lambda_{N,M}\} \subset \mathbb{T} \,, \qquad N \in \mathbb{N} \,.$$

▷ Design a sequence $\{\mathrm{H}_N\}_{N=1}^{\infty}$ of operators H_N (each H_N is concentrated on Λ_N) such that

$$\lim_{N \to \infty} \left\| \mathrm{H}_N f - \mathrm{H}f \right\|_\infty = \lim_{N \to \infty} \max_{t \in [-\pi, \pi)} \left| (\mathrm{H}_N f)(t) - (\mathrm{H}f)(t) \right| = 0 \qquad \text{for all } f \in \mathscr{B} \,,$$

wherein $\mathscr{B}$ is our signal space (which has to be specified).

Questions

- Is this always possible?
- For which signal spaces $\mathscr{B}$ this is possible?
- Does there exists signal spaces for which this is impossible?

Approach

1. We introduce a scale of Banach space $\{\mathscr{B}_\beta\}_{\beta \geq 0}$ of continuous functions of finite energy.
   - These are „good“ for the Hilbert transform.
   - The parameter $\beta \geq 0$ characterizes the energy concentration of the signals.
2. We introduce a class of sampling based Hilbert transform approximations $\{\mathrm{H}_N\}_{N \in \mathbb{N}}$.
   - This class is characterizes by two simple axioms.
   - This class contains basically all practically relevant Hilbert transform approximation methods.
   - We consider linear and non-linear methods.
3. Divergence results for the spaces $\mathscr{B}_\beta$ with $\beta \leq 1$.
   - For these spaces, there exists no Hilbert transform approximation in the class $\mathscr{B}_\beta$.
4. Convergence results for spaces $\mathscr{B}_\beta$ with $\beta > 1$.
   - For these spaces, there always exist a Hilbert transform approximation in the class.
   - Simple examples of convergent methods can be found.
5. Application: Calculating the Hilbert transform on Turing machines.

A Family of Signal Spaces with Energy Concentration

Motivation 1: Spaces of Smooth Functions

Space of all continuous functions $f \in \mathscr{C}(\mathbb{T})$ with a continuous conjugate $\widetilde{f}$

$$\mathscr{B} := \left\{ f \in \mathscr{C}(\mathbb{T}) \,:\, \widetilde{f} = \mathrm{H}f \in \mathscr{C}(\mathbb{T}) \right\} \qquad \text{with norm} \qquad \|f\|_{\mathscr{B}} = \max\left(\|f\|_\infty , \|\mathrm{H}f\|_\infty \right)$$

- Sampling operator $f \mapsto \{f(\lambda) \,:\, \Lambda \in \Lambda_N\}$ is well defined.
- The Hilbert transform $\mathrm{H} : \mathscr{B} \to \mathscr{B}$ is well defined and bounded.

There does not exist any *linear* sampling based algorithm which is able to approximate the Hilbert transform on $\mathscr{B}$!

$\Rightarrow$ Consider subspaces of $\mathscr{B}$.

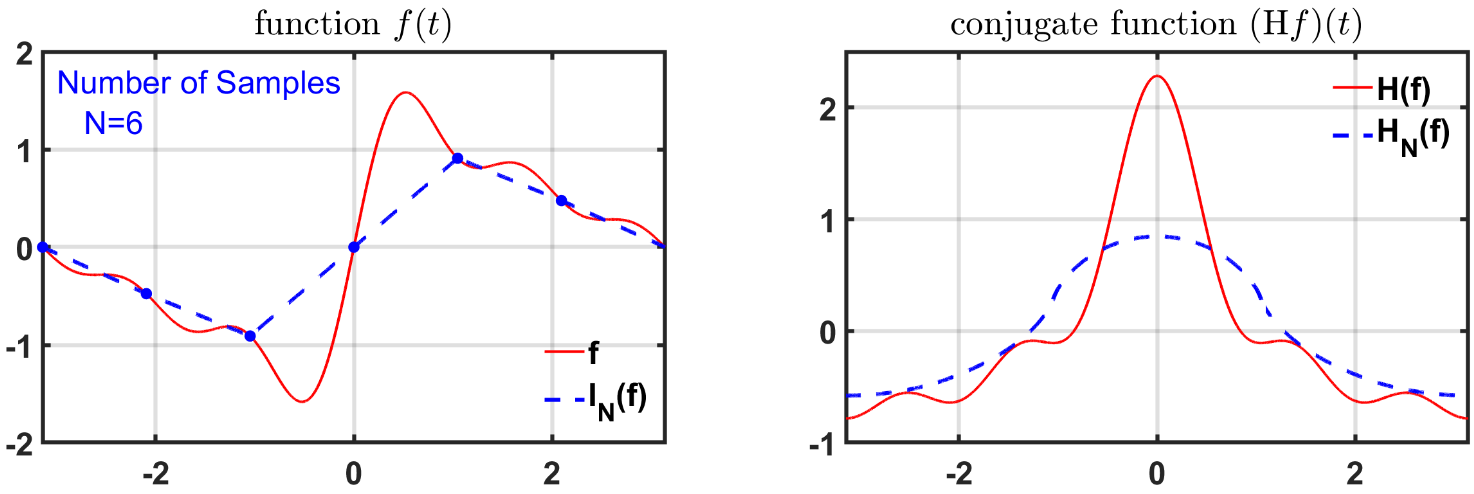
$\Rightarrow$ Consider *non linear* approximation methods.

► H. Boche, V. Pohl, "On the Calculation of the Hilbert Transform from Interpolated Data," *IEEE Trans. Inf. Theory*, vol. 54, no. 5 (May 2008).

Motivation 2: Spaces of Smooth Functions

For sufficiently smooth functions, there are standard procedures to obtain the desired sequences $\{\mathrm{H}_N\}_{N\in\mathbb{N}}$ of linear Hilbert transform approximations:

B-spline interpolation + Hilbert transform



Motivation 2: Spaces of Smooth Functions

- Assume f belongs to a Sobolev space $H^s(\mathbb{T}) = W^{s,2}(\mathbb{T})$ with $s > 1/2$.

$$\|f\|_{H^s(\mathbb{T})} = \left(\sum_{n=1}^{\infty} n^{2s}\left[|a_n(f)|^2 + |b_n|^2\right]\right)^{1/2}$$

- Sobolev embedding shows that f is *Hölder continuous* of index $0 < \alpha < s - 1/2$, i.e. $f \in \mathscr{C}^{\alpha}(\mathbb{T})$.
- Assume $\Lambda_N = \{\lambda_1, \ldots, \lambda_N\}$ is a sampling set with mesh size $r_N = \min_{n \neq m} |\lambda_n - \lambda_m|$.
- There is a unique interpolating function f_N which is continuous, piecewise linear, and which satisfies

$$f_N(\lambda_n) = (\mathrm{I}_N f)(\lambda_n) = f(\lambda_n) \qquad \text{for all } \lambda_n \in \Lambda_N\,.$$

- Since $f \in \mathscr{C}^{\alpha}(\mathbb{T})$ it follows that for all $0 < \alpha' < \alpha$

$$\|f - f_N\|_{\mathscr{C}^{\alpha'}(\mathbb{T})} \to 0 \qquad \text{as} \qquad r_N \to 0\,.$$

- Since $\mathrm{H} : \mathscr{C}^{\alpha'}(\mathbb{T}) \to \mathscr{C}(\mathbb{T})$ is known to be bounded, we set $\widetilde{f}_N = \mathrm{H} f_N$ and obtain

$$\left\|\widetilde{f}_N - \widetilde{f}\right\|_{\infty} = \|\mathrm{H}(f_N - f)\|_{\infty} \leq \|\mathrm{H}\|\ \|f - f_N\|_{\mathscr{C}^{\alpha'}(\mathbb{T})} \to 0 \qquad \text{as } r_N \to 0\,.$$

Remark

- Procedure fails for $s \leq 1/2$ because Sobolev embedding yields no longer Hölder continuity.
- Is this failure a particular property of the above procedure? $\Rightarrow$ Consider spaces close to $H^{1/2}(\mathbb{T})$.

Sobolev-like Banach Spaces

- Space of all continuous functions $f \in \mathscr{C}(\mathbb{T})$ with a continuous conjugate $\widetilde{f}$

$$\mathscr{B} := \left\{ f \in \mathscr{C}(\mathbb{T}) \ : \ \widetilde{f} = \mathrm{H}f \in \mathscr{C}(\mathbb{T}) \right\} \qquad \text{with norm} \qquad \|f\|_{\mathscr{B}} = \max\big(\|f\|_\infty, \|\mathrm{H}f\|_\infty\big)$$

- For any $f \in \mathscr{B}$, we consider the trigonometric series

$$f(t) = \frac{a_0(f)}{2} + \sum_{n=1}^{\infty} a_n(f)\cos(nt) + b_n(f)\sin(nt) \qquad \text{with} \qquad \begin{aligned} a_n(f) &= \tfrac{1}{\pi}\textstyle\int_{\mathbb{T}} f(\tau)\cos(n\tau)\,\mathrm{d}\tau \\ b_n(f) &= \tfrac{1}{\pi}\textstyle\int_{\mathbb{T}} f(\tau)\sin(n\tau)\,\mathrm{d}\tau \end{aligned}$$

- For $\beta \geq 0$, we define the seminorm

$$\|f\|_\beta = \left(\sum_{n=1}^{\infty} n(1+\log n)^\beta \left[|a_n(f)|^2 + |b_n(f)|^2\right] \right)^{1/2}$$

- For any $\beta \geq 0$, we introduce the Banach space

$$\mathscr{B}_\beta = \left\{ f \in \mathscr{B} : \|f\|_\beta < \infty \right\} \qquad \text{with norm} \qquad \|f\|_{\mathscr{B}_\beta} = \max\left(\|f\|_\infty, \|\widetilde{f}\|_\infty, \|f\|_\beta\right).$$

Our Scale of Signal Spaces – Properties

For $\beta \geq 0$ we consider the Banach spaces

$$\mathscr{B}_\beta = \left\{ f \in \mathscr{B} \,:\, \|f\|_\beta^2 = \sum_{n=1}^{\infty} n(1+\log n)^\beta \left[|a_n(f)|^2 + |b_n(f)|^2\right] < \infty \right\} .$$

- ▷ Each $\mathscr{B}_\beta$ is a Banach space
- ▷ Every $f \in \mathscr{B}_\beta$ is continuous with a continuous conjugate
- ▷ $\mathscr{B}_\beta \subset L^2(\mathbb{T})$
- ▷ Every $f \in \mathscr{B}_\beta$ has finite Dirichlet energy (cf. next slide)
- ▷ $\mathscr{B}_{\beta_2} \subset \mathscr{B}_{\beta_1} \subset \mathscr{B}_0 \subset \mathscr{B} \subset \mathscr{C}(\mathbb{T})$ for all $\beta_2 > \beta_1 > 0$.
- ▷ The parameter $\beta \geq 0$ characterizes the smoothness of the functions.
- ▷ $\beta \geq 0$ characterizes how fast the Fourier coefficients converges to zero as $n \to \infty$.
- ▷ The seminorm $\|\cdot\|_0$ corresponds to the norm in the critical Sobolev space $H^{1/2}(\mathbb{T})$, i.e.

$$\mathscr{B}_0 = \mathscr{B} \cap H^{1/2}(\mathbb{T})$$

- ▷ $H^{1/2}(\mathbb{T})$ is of fundamental importance in quantum information theory for Gaussian channels.

► A. S. Holevo, "The classical capacity of quantum Gaussian gauge-covariant channels: Beyond i.i.d.," *IEEE Inf. Theory Soc. Newsletter*, 66 (4) (2016) 3–6, Extension of the Shannon Lecture at ISIT 2016, Barcelona, Spain.

Relation to the Dirichlet Problem

Dirichlet Problem on the Unit Circle

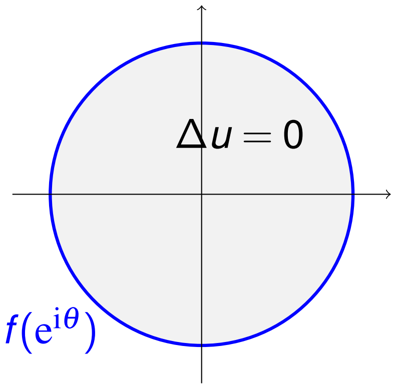
Let f be a given function on the unit circle $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$. We look for an u inside the unit circle $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ such that

1. $\dfrac{\partial^2 u}{\partial x^2}(z) + \dfrac{\partial^2 u}{\partial y^2}(z) = (\Delta u)(z) = 0$ for all $z = x + \mathrm{i}y \in \mathbb{D}$
2. $u(\mathrm{e}^{\mathrm{i}t}) = f(t)$ for all $t \in \mathbb{T} = [-\pi, \pi)$

Dirichlet's Principle

The solution of the Dirichlet problem can be obtained by minimizing the Dirichlet energy

$$D(u) = \frac{1}{2\pi} \iint_{\mathbb{D}} \|(\mathrm{grad}\, u)(z)\|^2_{\mathbb{R}^2} \,\mathrm{d}z = \sum_{n=-\infty}^{\infty} |n|\, |c_n(f)|^2 = \|f\|^2_{H^{1/2}}$$



- The boundary function of solutions of the Dirichlet problem belongs to the Sobolev space $H^{1/2}(\mathbb{T})$.
- If f is additionally in $\mathscr{B}$ then $f \in \mathscr{B}_0$.

Linear Approximation Operators

A Class of Hilbert Transform Approximations

We consider sequences $\{\mathrm{H}_N\}_{N\in\mathbb{N}}$ of bounded linear operators $\mathrm{H}_N : \mathscr{B}_\beta \to \mathscr{C}(\mathbb{T})$ which satisfy the following two axioms:

(A) Concentration on a sampling set:

To every $N \in \mathbb{N}$ there exists a finite set $\Lambda_N = \{\lambda_{N,n} : n = 1, \dots, M_N\} \subset \mathbb{T}$ such that for all $f_1, f_2 \in \mathscr{B}$

$$\begin{aligned} f_1(\lambda_{N,n}) &= f_2(\lambda_{N,n}) && \text{for all } \lambda_{N,n} \in \Lambda_N \\ \text{implies } (\mathrm{H}_N f_1)(t) &= (\mathrm{H}_N f_2)(t) && \text{for all } t \in \mathbb{T}\,. \end{aligned}$$

(B) Convergence on a dense subset of $\mathscr{B}$:

There exists a dense subset $\mathscr{M} \subset \mathscr{B}_\beta$ such that

$$\lim_{N\to\infty} \|\mathrm{H}_N f - \mathrm{H} f\|_\infty = 0 \qquad \text{for all } f \in \mathscr{M}\,.$$

Remark:

If $\{\mathrm{H}_N\}_{N\in\mathbb{N}}$ satisfies Axiom (A) then each H_N has the form

$$(\mathrm{H}_N f)(t) = \sum_{n=1}^{M_N} f(\lambda_{N,n})\, \kappa_{N,n}(t) \qquad \text{with} \qquad \{\kappa_{N,1}, \kappa_{N,2}, \dots, \kappa_{N,M_N}\} \subset \mathscr{B}_\beta\,.$$

Divergence Results for $0 \leq \beta \leq 1$

Theorem: Let $\beta \in [0,1]$ be arbitrary, and let $\boldsymbol{H} = \{\mathrm{H}_N\}_{N\in\mathbb{N}}$ be a sequence of bounded linear operators $\mathrm{H}_N : \mathscr{B}_\beta \to \mathscr{C}(\mathbb{T})$ satisfying Axioms (A), (B). Then

$$\mathscr{R}_\beta(\boldsymbol{H}) = \left\{ f \in \mathscr{B}_\beta \;:\; \limsup_{N\to\infty} \|\mathrm{H}_N f\|_\infty = +\infty \right\}$$

is a residual set (a non-meager and dense subset) in $\mathscr{B}_\beta$.

By the uniform boundedness principle for linear operators, one obtains:

Corollary: $\lim_{N\to\infty} \left\|\mathrm{H}_N\right\|_{\mathscr{B}_\beta \to \mathscr{C}(\mathbb{T})} = +\infty$.

Corollary: Let $0 \leq \beta \leq 1$ be arbitrary and let $\boldsymbol{H} = \{\mathrm{H}_N\}_{N\in\mathbb{N}}$ be a sequence of bounded linear operators $\mathrm{H}_N : \mathscr{B}_\beta \to \mathscr{C}(\mathbb{T})$ which satisfies Axiom (A). Then there exists a residual set $\mathscr{R}_\beta(\boldsymbol{H}) \subset \mathscr{B}_\beta$ such that

$$\limsup_{N\to\infty} \left\|\mathrm{H}_N f_* - \mathrm{H} f_*\right\|_\infty > 0\,, \qquad \text{for all } f \in \mathscr{R}_\beta(\boldsymbol{H})\,.$$

Corollary: There is *no* sequence $\{\mathrm{H}_N\}_{N\in\mathbb{N}}$ of bounded linear operators $\mathrm{H}_N : \mathscr{B}_\beta \to \mathscr{C}(\mathbb{T})$ which can be implemented on an idealized digital computer and which approximates the Hilbert transform $\mathrm{H}f$ for every $f \in \mathscr{B}_\beta$ with $0 \leq \beta \leq 1$.

▶ H. Boche, V. Pohl, "Calculating the Hilbert transform on spaces with energy concentration: Convergence and divergence regions" *IEEE Trans. Inf. Theory*, (2018), in Press.

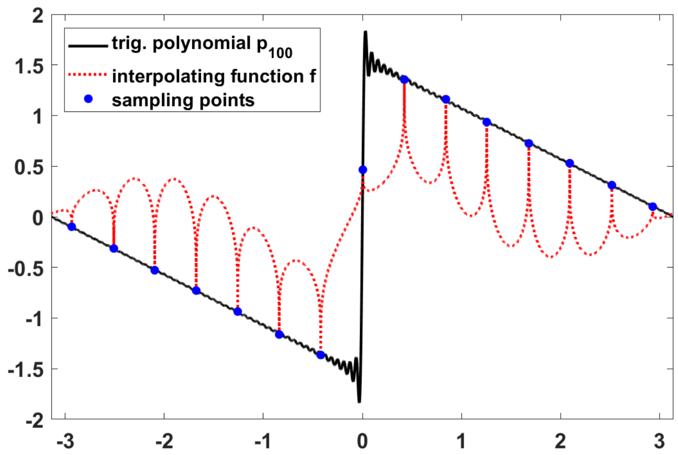
Proof – Interpolation Lemma

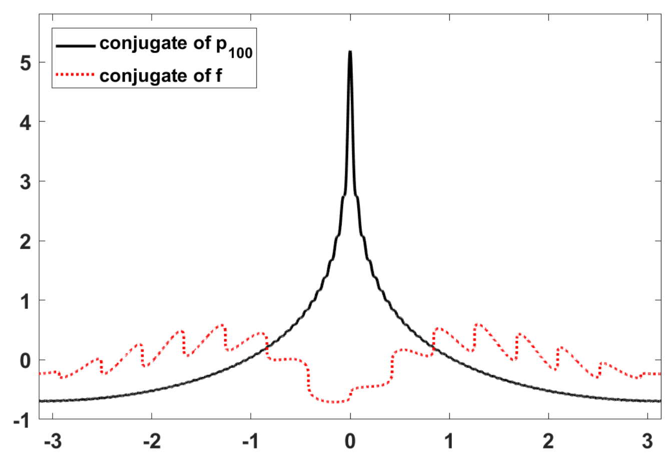
Lemma: Let $0 \le \beta \le 1$ and $0 < \varepsilon < 1$ be arbitrary and let $\Lambda = \{\lambda_1, \dots, \lambda_N\} \subset \mathbb{T}$ be a finite sampling set. To every $g \in \mathscr{C}(\mathbb{T})$ there exists a $f \in \mathscr{B}_\beta$ which solves the interpolation problem

$$f(\lambda_n) = g(\lambda_n) \qquad \text{for all } n = 1, \dots, N\,,$$

and such that $\|f\|_{\mathscr{B}_\beta} \le (1+\varepsilon)\,\|g\|_\infty$.

Remark: Note that we have to control $\|f\|_\infty$, $\|\widetilde{f}\|_\infty$, and $\|f\|_\beta$ of the interpolating function.





▷ So we can extend the operators $\mathrm{H}_N : \mathscr{B}_\beta \to \mathscr{C}(\mathbb{T})$ to operators $\mathrm{H}_N : \mathscr{C}(\mathbb{T}) \to \mathscr{C}(\mathbb{T})$.

Proof – Main Ideas

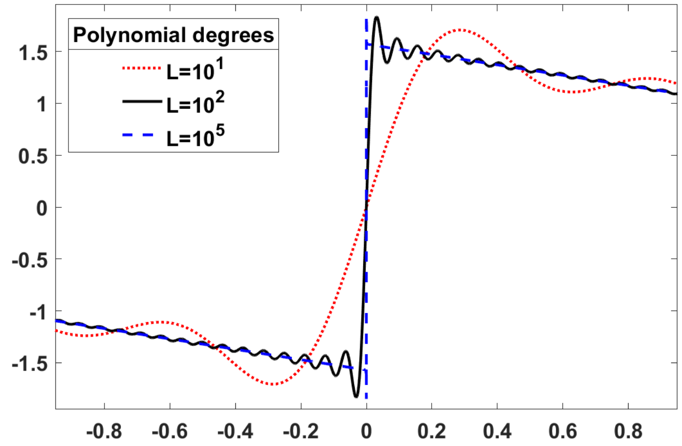
- ▷ Assume the Theorem would be wrong $\Rightarrow$ Operator norms would be uniformly bounded

$$\|\mathrm{H}_N\|_{\mathscr{B}_\beta\to\mathscr{C}(\mathbb{T})} \leq C < \infty \qquad \text{for all } N \in \mathbb{N}\,.$$

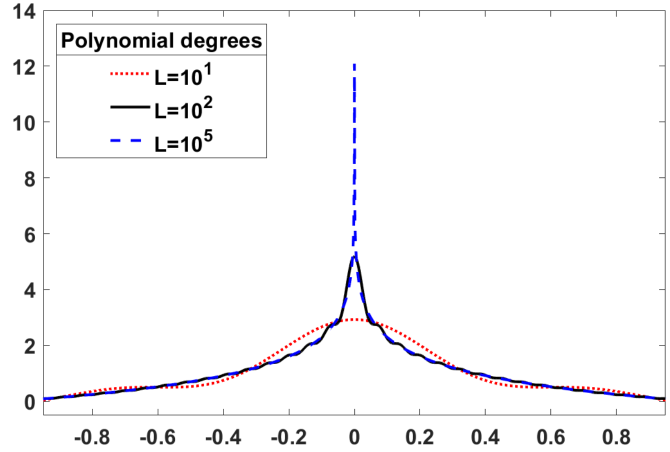
- ▷ The interpolation lemma shows that the operators $\mathrm{H}_N : \mathscr{B}_\beta \to \mathscr{C}(\mathbb{T})$ can be extendet to operators $\mathrm{H}_N : \mathscr{C}(\mathbb{T}) \to \mathscr{C}(\mathbb{T})$ and we have

$$\|\mathrm{H}_N\|_{\mathscr{C}(\mathbb{T})\to\mathscr{C}(\mathbb{T})} = \sup_{g\in\mathscr{C}(\mathbb{T})} \frac{\|\mathrm{H}_N g\|_\infty}{\|g\|_\infty} \leq (1+\varepsilon) \sup_{f\in\mathscr{B}_\beta} \frac{\|\mathrm{H}_N f\|_\infty}{\|f\|_{\mathscr{B}_\beta}} = (1+\varepsilon)\,\|\mathrm{H}_N\|_{\mathscr{B}_\beta\to\mathscr{C}(\mathbb{T})}$$

- ▷ Together with Axiom (B), it would follow that $\|\mathrm{H}\|_{\mathscr{C}(\mathbb{T})\to\mathscr{C}(\mathbb{T})} \leq (1+\varepsilon)C \Rightarrow$ Contradiction!



$p_L(t) = \sum_{k=1}^{L} \frac{1}{k}\sin(kt)$



$\widetilde{p}_L(t) = \sum_{k=1}^{L} \frac{1}{k}\cos(kt)$

Spaces with Convergent Approximation Methods

Theorem: For any $\beta > 1$ there exit sequences $\{\mathrm{H}_N\}_{N\in\mathbb{N}}$ of bounded linear operators $\mathrm{H}_N : \mathscr{B} \to \mathscr{C}(\mathbb{T})$ satisfying Axioms (A) and (B) and such that

$$\lim_{N\to\infty} \left\|\mathrm{H}_N f - \mathrm{H} f\right\|_\infty = 0 \qquad \text{for all } f \in \mathscr{B}_\beta\,.$$

- If the energy of the signals is sufficiently concentrated then there always exist sampling based approximation methods which converges for all signals in the space $\mathscr{B}_\beta$ with $\beta > 1$.
- Theorem can be proven by constructing a particular convergent approximation method (cf. motivating example).

Characterization of Convergent Method

Theorem: Let $\beta > 1$ and let $\{\mathrm{H}_N\}_{N\in\mathbb{N}}$ be a sequence of bounded linear operators $\mathrm{H}_N : \mathscr{B} \to \mathscr{C}(\mathbb{T})$ such that

1. For every $n \in \mathbb{N}$ holds
$$\lim_{N\to\infty} \left\| \mathrm{H}_N[\cos(n\cdot)] - \sin(n\cdot) \right\|_\infty = 0 \qquad \text{and} \qquad \lim_{N\to\infty} \left\| \mathrm{H}_N[\sin(n\cdot)] + \cos(n\cdot) \right\|_\infty = 0\,.$$
2. There exists a constant C such that for every $n \in \mathbb{N}$
$$\max\left(\left\| \mathrm{H}_N[\cos(n\cdot)] \right\|_\infty, \ \left\| \mathrm{H}_N[\sin(n\cdot)] \right\|_\infty \right) \le C \qquad \text{for all } N \in \mathbb{N}\,.$$

Then one has
$$\lim_{N\to\infty} \left\| \mathrm{H}_N f - \mathrm{H} f \right\|_\infty = 0 \qquad \text{for all } f \in \mathscr{B}_\beta\,.$$

Thus, if an approximation method $\{\mathrm{H}_N\}_{N\in\mathbb{N}}$

- converges for the sine- and cosine functions (i.e. for the pure frequencies), and
- if the approximations of the pure frequencies are uniformly bounded

then the method $\{\mathrm{H}_N\}_{N\in\mathbb{N}}$ converges for all $f \in \mathscr{B}_\beta$ with $\beta > 1$.

A Convergent Hilbert Transform Approximation

We consider again the sequence $\{\mathrm{D}_N\}_{N\in\mathbb{N}}$ of the sampled conjugate Fourier series from the beginning

$$(\mathrm{D}_N f)(t) = -\mathrm{i} \sum_{n=-(N-1)}^{N-1} \mathrm{sgn}(n)\, c_{N,n}(f)\, \mathrm{e}^{\mathrm{i}nt} = \sum_{k=0}^{2N-1} f(\lambda_{N,k})\, \mathscr{D}_N\Big(t - k\frac{\pi}{N}\Big) \;,$$

with the conjugate Dirichlet kernel $\mathscr{D}_N(t) = \frac{1}{N}\sum_{n=1}^{N-1} \sin(n\,t)$ and which is concentrated on the equidistant sampling sets

$$\Lambda_N = \Big\{\lambda_{N,k} = k\tfrac{\pi}{N} \;:\; k = 0, 1, \dots, 2N-1\Big\} \;.$$

It is fairly easy to show that this sequence $\{\mathrm{D}_N\}_{N\in\mathbb{N}}$

- satisfies Axioms (A) and (B)
- has the two properties of the previous theorem which characterized all convergent methods

and so, we have

$$\lim_{N\to\infty} \big\|\mathrm{D}_N f - \mathrm{H}f\big\|_\infty = 0 \qquad \text{for all } f \in \mathscr{B}_\beta \quad \text{with} \quad \beta > 1 \;.$$

Remarks

- ▷ So even on the important space $\mathscr{B}_0 = H^{1/2}(\mathbb{T}) \cap \mathscr{B}$ of functions with finite Dirichlet energy there exists no linear, sampling based approximation methods $\{\mathrm{H}_N\}_{N\in\mathbb{N}}$.
- ▷ Not only the initially given approximation procedure (based on linear interpolation and Sobolev embedding) fails, but every sampling based method fails on the spaces $\mathscr{B}_\beta$ with $0 \leq \beta \leq 1$.
- ▷ The transition between spaces on which no approximation method exists to spaces where such methods exist is slightly off the critical space $\mathscr{B}_0 = H^{1/2}(\mathbb{T}) \cap \mathscr{B}$ from the initial example.

Nonlinear Approximation Operators

Nonlinear Approximation Methods

- We replace the sequence $\{\mathrm{H}_N\}_{N\in\mathbb{N}}$ of bounded linear operators $\mathrm{H}_N : \mathscr{B}_\beta \to \mathscr{C}(\mathbb{T})$ by sequences $\{\Psi_N\}_{N\in\mathbb{N}}$ of arbitrary, not necessarily linear, operators.
- With every operator $\Psi_N : \mathscr{B}_\beta \to \mathscr{C}(\mathbb{T})$, we associate the functional

$$\Phi_N(f) = \|\Psi_N(f)\|_\infty \ .$$

- We require that these functionals are lower semicontinuous.

Definition: Let $\Phi : \mathscr{B}_\beta \to \mathbb{R}_+$ be a functional on a Banach space $\mathscr{B}_\beta$.
One says that Φ is lower semicontinuous if for every $\lambda \geq 0$

$$\{f \in \mathscr{B} \ : \ \Phi(f) \leq \lambda\}$$

is a closed set.
We write $LCS(\mathscr{B}_\beta, \mathbb{R}_+)$ for the set of all lower semicontinuous functionals $\Phi : \mathscr{B}_\beta \to \mathbb{R}_+$.

Nonlinear Approximations – Axiomatic

Let $0 \leq \beta \leq 1$ be arbitrary and let $\mathbf{\Psi} = \{\Psi_N\}_{N\in\mathbb{N}}$ be a sequence of mappings $\Psi_N : \mathscr{B}_\beta \to \mathscr{C}(\mathbb{T})$ with the associated functionals $\Phi_N(f) = \|\Psi_N(f)\|_\infty$. We say that $\mathbf{\Psi}$ satisfies Axiom

(A) Concentration on a discrete sampling set

if $\{\Phi_N\}_{N\in\mathbb{N}} \subset LSC(\mathscr{B}_\beta, \mathbb{R}_+)$ and if for every $N \in \mathbb{N}$ there exists a finite subset $\Lambda_N \subset \mathbb{T}$ such that for arbitrary $f_1, f_2 \in \mathscr{B}_\beta$

$$\begin{aligned} f_1(\lambda_n) &= f_2(\lambda_n) && \text{for all } \lambda_n \in \Lambda_N \\ \text{implies} \quad \big(\Psi_N f_1\big)(t) &= \big(\Psi_N f_2\big)(t) && \text{for all } t \in \mathbb{T}\,. \end{aligned}$$

(B) Convergence on a dense subset

if there exists a dense subset $\mathscr{M} \subset \mathscr{B}_\beta$ such that

$$\lim_{N\to\infty} \|\Psi_N(f) - \mathrm{H}f\|_\infty = 0 \qquad \text{for all } f \in \mathscr{M}\,.$$

- ▷ Axiom (A) requires basically that the approximation $\widetilde{f}_N = \Psi_N(f)$ is uniquely determined by the values of f on the finite sampling set $\Lambda_N \subset \mathbb{T}$.
- ▷ Axioms (B) describe $\{\Psi_N\}_{N\in\mathbb{N}}$ as a sequence which approximates the Hilbert transform.
- ▷ Note that the set $\mathscr{M}$ has no linear structure, in general.

Nonlinear Approximations – Divergence Result

Theorem: Let $0 \leq \beta \leq 1$ be arbitrary and let $\boldsymbol{\Psi} = \{\Psi_N\}_{N\in\mathbb{N}}$ be a sequence of mappings $\Psi_N : \mathscr{B}_\beta \to \mathscr{C}(\mathbb{T})$ which satisfies Axioms (A) and (B). Then

$$\mathscr{R}_\beta(\boldsymbol{\Psi}) = \left\{ f \in \mathscr{B}_\beta \ : \ \limsup_{N\to\infty} \left\|\Psi_N(f)\right\|_\infty = +\infty \right\}$$

is a residual set in $\mathscr{B}_\beta$.

Corollary: Let $0 \leq \beta \leq 1$ be arbitrary and let $\boldsymbol{\Psi} = \{\Psi_N\}_{N\in\mathbb{N}}$ be a sequence of operators $\Psi_N : \mathscr{B}_\beta \to \mathscr{C}(\mathbb{T})$ which satisfies Axiom (A) and let

$$\mathscr{D}_\beta(\boldsymbol{\Psi}) = \left\{ f \in \mathscr{B}_\beta \ : \ \limsup_{N\to\infty} \left\|\Psi_N(f) - \mathrm{H}f\right\|_\infty > 0 \right\} . \tag{3}$$

be the *divergence set* associated with $\boldsymbol{\Psi}$. Then for $\mathscr{D}_\beta(\boldsymbol{\Psi})$ holds at least one of the following two statements:

- $\overline{\mathscr{D}_\beta(\boldsymbol{\Psi})}^{\mathscr{B}_\beta} = \mathscr{B}_\beta$
- $\mathscr{D}_\beta(\boldsymbol{\Psi})$ contains an open ball from $\mathscr{B}_\beta$.

Proof Ingredients

1. The interpolation lemma for $\mathscr{B}_\beta$.
2. A generalization of the uniform boundedness principle.

Lemma (Generalized uniform boundedness principle):
Let $\mathscr{X}$ be a Banach space and let $\mathbf{\Phi} \subset LSC(\mathscr{X}, \mathbb{R}_+)$ be a family of functionals on $\mathscr{X}$ such that there exists a set $K \subset \mathscr{X}$ of second category so that

$$\sup_{\varphi \in \mathbf{\Phi}} \varphi(f) = M(f) < +\infty \qquad \text{for all } f \in K \,.$$

Then there exist an $M_{\mathbf{\Phi}} < \infty$, an $f_0 \in \mathscr{X}$, and a $\delta > 0$ such that for all $f \in B_\delta(f_0, \mathscr{X})$ always

$$\varphi(f) \leq M_{\mathbf{\Phi}} \qquad \text{for all } \varphi \in \mathbf{\Phi} \,,$$

where $B_\delta(f_0, B) = \{f \in \mathscr{X} : \|f - f_0\|_{\mathscr{X}} < \delta\}$.

Application: Turing Computable Approximations

Structure of non-linear Approximation Methods

Definition: We write $\mathscr{O}(M)$ for the set of all functions $F : \mathbb{R}^M \times \mathbb{T} \to \mathbb{R}$ such that for every $\mathbf{x} \in \mathbb{R}^M$

- $F(\mathbf{x}; \cdot) \in \mathscr{C}(\mathbb{T})$
- $G(\mathbf{x}) = \|F(\mathbf{x}; \cdot)\|_\infty$ is a lower semicontinuous functional.

▷ We show that each (non-linear) mapping Ψ_N in our approximation methods $\boldsymbol{\Psi} = \{\Psi_N\}_{N \in \mathbb{N}}$ is characterized by a function in the class $\mathscr{O}(M)$.

Lemma: A sequence $\boldsymbol{\Psi} = \{\Psi_N\}_{N \in \mathbb{N}}$ of mappings $\Psi_N : \mathscr{B}_\beta \to \mathscr{C}(\mathbb{T})$ satisfies Axiom (A) if and only if to every $N \in \mathbb{N}$ there exists an $M = M(N) \in \mathbb{N}$ and a function $F_N \in \mathscr{O}(M)$ such that for all $f \in \mathscr{B}_\beta$ always

$$(\Psi_N f)(t) = F_N\big(\, f(t_{N,1}), \dots, f(t_{M,N})\ ;\ t\ \big)\,, \qquad t \in \mathbb{T}\,.$$

▷ We require next that the functions F_N are Turing computable.

Computability

Definition (Computable vectors):

Let $\mathbf{x} \in \mathbb{R}^M$ be an M-dimensional real vector.

1. A sequence $\{\mathbf{x}_n\}_{n\in\mathbb{N}} \subset \mathbb{Q}^M$ of rational vectors is said to be a *rapidly converging Cauchy name* of $\mathbf{x}$, if $\mathbf{x}_n$ converges rapidly to $\mathbf{x}$ in the following sense: For all $n, m \in \mathbb{N}$ with $m > n$, we have $\|\mathbf{x}_m - \mathbf{x}_n\|_{\mathbb{R}^M} \leq 2^{-n}$.
2. The vector $\mathbf{x}$ is said to be *computable*, if there exists a rapidly converging Cauchy name of $\mathbf{x}$.
3. We write $\mathbb{R}_c^M$ for the set of all commutable vectors in $\mathbb{R}^M$.

Definition (Computable Function):

We call $F : \mathbb{R}_c^M \to \mathbb{R}_c$ a *computable function* if there is an algorithm that transforms each rapidly converging Cauchy name of an arbitrary $\mathbf{x} \in \mathbb{R}_c^M$ into a rapidly converging Cauchy name of $F(\mathbf{x})$.

Computable Approximations – Definitions

Definition (Computable approximation method):
Let $\boldsymbol{\Psi} = \{\Psi_N\}_{N\in\mathbb{N}}$ be a sequence of mappings $\mathscr{B}_\beta \to \mathscr{C}(\mathbb{T})$. We call $\boldsymbol{\Psi}$ a *computable approximation method* if to every $N \in \mathbb{N}$ there exists

- an $M = M(N) \in \mathbb{N}$,
- a finite sampling set $\Lambda_N = \left\{\lambda_{N,1}, \lambda_{N,2}, \dots, \lambda_{N,M(N)}\right\} \subset \mathbb{T}$,
- a computable function $F_N \in \mathscr{O}(M)$

such that for all $f \in \mathscr{B}_\beta$

$$\left(\Psi_N f\right)(t) = F_N\left(f(\lambda_{N,1}), f(\lambda_{N,2}), \dots, f(\lambda_{N,M(N)}); t\right) \quad \text{for all } t \in \mathbb{T}\,.$$

No-Go for Computable Approximations

Theorem: Let $0 \leq \beta \leq 1$ be arbitrary and let $\boldsymbol{\Psi} = \{\Psi_N\}_{N \in \mathbb{N}}$ be a sampling based computable approximation method such that there exists a dense subset $\mathscr{M} \subset \mathscr{B}_\beta$ so that

$$\lim_{N\to\infty} \left\| \Psi_N(f) - \mathrm{H}f \right\|_\infty = 0 \qquad \text{for all } f \in \mathscr{M}\ ,$$

then

$$\left\{ f \in \mathscr{B}_\beta \ :\ \limsup_{N\to\infty} \|\Psi_N(f)\|_\infty = +\infty \right\}$$

is a residual set in $\mathscr{B}_\beta$.

Corollary: Let $0 \leq \beta \leq 1$ be arbitrary and let $\boldsymbol{\Psi} = \{\Psi_N\}_{N \in \mathbb{N}}$ be a sampling based computable approximation method which satisfies the conditions of the previous Theorem. Then there always exists an $f_* \in \mathscr{B}_\beta$ such that

$$\lim_{N\to\infty} \left\| \Psi_N(f_*) - \mathrm{H}f_* \right\|_\infty > 0\ .$$

- H. Boche, V. Pohl, "Investigations on the approximability and computability of the Hilbert transform with applications," Appl. Comput. Harmon. Anal., (2017), submitted for publ.

Summary and Outlook

- ▷ We introduced a scale of Banach spaces $\mathscr{B}_\beta$, $\beta \geq 0$ of functions
  - which are continuous with a continuous conjugate
  - with finite (Dirichlet) energy
  - with energy concentration characterized by β
- ▷ In the scale $\{\mathscr{B}_\beta\}_{\beta \geq 0}$, we characterized precisely those spaces on which
  - there do not exist sampling based Hilbert transform approximations: $\beta \in [0, 1]$
  - there do exist sampling based Hilbert transform approximations: $\beta > 1$.
  - nonlinear approximations give no improvement (with respect to this divergence behavior) over linear approximation operators.
- ▷ For $\beta > 1$ even very simple linear approximation methods (sampled conjugate Fourier series) converge for all $f \in \mathscr{B}_\beta$.

Outlook

- ▷ Other operators: *Fourier series approximation*, *spectral factorization*, *Wiener filter*, etc.
- ▷ (Turing) Computability of these operators.

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Thank You! – Questions?