On the Strong Divergence of Hilbert Transform Approximations and a Problem of Ul'yanov

Holger Boche
(joint work with Volker Pohl)

Lehrstuhl für Theoretische Informationstechnik, Technische Universität München

Workshop – Kernel Methods in Approximation Theory and Sampling
Dortmund, March 23rd-25th, 2015

Motivation – An Approximation Problem

- Let $\mathrm{T} : \mathcal{B}_1 \to \mathcal{B}_2$ a bounded linear operator between Banach spaces.
- Approximate T by a sequence $\{\mathrm{T}_N\}_{N\in\mathbb{N}}$ of bounded linear operators with finite-dimensional rank such that

$$\lim_{N\to\infty} \|\mathrm{T}_N f - Tf\|_{\mathcal{B}_2} = 0 \qquad \text{for all} \qquad f \in \mathcal{B}_1$$

- Applications $\Rightarrow$ Restrictions on class of admissible approximation operators
  - The calculation of $\mathrm{T}_N f$ should be based on time-domain samples $\{f(\lambda_n)\}_{n=1}^{N}$.
- Minimal requirement: $\{\mathrm{T}_N f\}$ converges for all f from a dense subset of $\mathcal{B}_1$

Outline

Outline

Weakly Divergent Series

There are many important examples, where

$$\lim_{N\to\infty} \left\| \mathrm{T}_N f - \mathrm{T} f \right\|_{\mathcal{B}_2} = 0 \quad \text{for all} \quad f \in \mathcal{B}_0 \tag{1}$$

in a dense subset $\mathcal{B}_0 \subset \mathcal{B}_1$, but such that

$$\lim_{N\to\infty} \| \mathrm{T}_N f_* - \mathrm{T} f_* \|_{\mathcal{B}_2} = \infty \quad \text{for some} \quad f_* \in \mathcal{B}_1 \ . \tag{2}$$

- ϟ Usually, it is fairly easy to construct $\{\mathrm{T}_N\}_{N\in\mathbb{N}}$ such that (1) holds for some dense subset $\mathcal{B}_0$.
- ϟ It is much harder to show that (2) holds. Or alternatively that

$$\lim_{N\to\infty} \| \mathrm{T}_N f - \mathrm{T} f \|_{\mathcal{B}_2} = 0 \qquad \text{for all} \qquad f \in \mathcal{B}_1$$

- ϟ Instead of (2) one often verifies the *weaker divergence condition*

$$\limsup_{N\to\infty} \left\| \mathrm{T}_N f_* - \mathrm{T}_N f_* \right\|_{\mathcal{B}_2} = \infty \qquad \text{for some} \quad f_* \in \mathcal{B}_1 \ .$$

Weak Divergence & Banach-Steinhaus Theorem

Weak divergence results are often stated as

$$\limsup_{N\to\infty} \left\|\mathrm{T}_N f_*\right\|_{\mathcal{B}_2} = \infty \qquad \text{for some} \qquad f_* \in \mathcal{B}_1$$

and proofs are often based on the *uniform boundedness principle*.

Theorem (Banach-Steinhaus)

Let $\{\mathrm{T}_N\}_{N\in\mathbb{N}}$ *be a sequence of linear operators* $\mathrm{T}_N : \mathcal{B}_1 \to \mathcal{B}_2$ *with norm*

$$\|\mathrm{T}_N\| = \sup_{f\in\mathcal{B}_1} \frac{\|\mathrm{T}_N f\|_{\mathcal{B}_2}}{\|f\|_{\mathcal{B}_1}} .$$

If $\sup_{N\in\mathbb{N}} \|\mathrm{T}_N\| = \infty$ *then there exists an* $f_* \in \mathcal{B}_1$ *such that*

$$\sup_{N\in\mathbb{N}} \|\mathrm{T}_N f_*\|_{\mathcal{B}_2} = \infty . \qquad (\Delta)$$

In fact, the set $\mathcal{D}$ *of all* $f_* \in \mathcal{B}_1$ *which satisfy* (Δ) *is a residual set in* $\mathcal{B}_1$.

Example - Sampled Fourier Series

- Let $\mathcal{B}_1 = \mathcal{B}_2 = \mathcal{C}(\mathbb{T})$ be the set of continuous functions on $\mathbb{T} = [-\pi, \pi]$.
- Let $\mathrm{T} = \mathrm{I}_{\mathcal{C}}$ be the identity operator on $\mathcal{C}(\mathbb{T})$.

Example (Sampled trigonometric Fourier series)

$$\big(\mathrm{T}_N f\big)(t) = \sum_{k=0}^{N-1} f\big(k\,\tfrac{2\pi}{N}\big)\,\mathcal{D}_N\big(t - k\,\tfrac{2\pi}{N}\big)\ , \qquad t \in \mathbb{T}\ ,\ N \in \mathbb{N}$$

with *Dirichlet kernel*

$$\mathcal{D}_N(\tau) = \frac{\sin([N+1/2]\tau)}{\sin(\tau/2)}\ .$$

▷ **Convergence on a dense subset:**

$$\lim_{N\to\infty} \|\mathrm{T}_N p - p\|_\infty = 0 \qquad \text{for all polynomials } p \text{ on } \mathbb{T}\ .$$

▷ **Weak divergence:** There are functions $f_* \in \mathcal{C}(\mathbb{T})$ such that

$$\sup_{N\in\mathbb{N}} \|\mathrm{T}_N f_*\|_\infty = +\infty \qquad \Rightarrow \qquad \limsup_{N\to\infty} \|\mathrm{T}_N f_* - f_*\|_\infty = +\infty\ .$$

The Weakness of Weak Divergence

Definition (Weak Divergence)

A sequence $\{\mathrm{T}_N\}_{N\in\mathbb{N}}$ of bounded linear approximation operators $\mathrm{T}_N : \mathcal{B}_1 \to \mathcal{B}_2$ is said to *diverge weakly* if

$$\limsup_{N\to\infty} \left\| \mathrm{T}_N f_* - \mathrm{T} f_* \right\|_{\mathcal{B}_2} = \infty \qquad \text{for some} \qquad f_* \in \mathcal{B}_1 \ . \tag{WD}$$

- ϟ Weak divergence only implies that there exists a *„bad subsequence“* $\{N_k\}_{k\in\mathbb{N}}$ such that

$$\lim_{k\to\infty} \left\| \mathrm{T}_{N_k} f_* - \mathrm{T} f_* \right\|_{\mathcal{B}_2} = \infty \qquad \text{for some} \qquad f_* \in \mathcal{B}_1 \ .$$

- ϟ This notion of divergence does not exclude the possibility that there exist *„good subsequences“* $\{N_k = N_k(f)\}_{k\in\mathbb{N}}$ such that

$$\inf_{k\in\mathbb{N}} \left\| \mathrm{T}_{N_k} f - \mathrm{T} f \right\|_{\mathcal{B}_2} < \infty \qquad \text{or even} \qquad \lim_{k\to\infty} \left\| \mathrm{T}_{N_k} f - \mathrm{T} f \right\|_{\mathcal{B}_2} = 0$$

for all $f \in \mathcal{B}_1$.

Example – Approximation by Walsh Functions

Example (Weakly divergent with a convergent subsequence)

- Let $\{\psi_n\}_{n=0}^{\infty}$ be the orthonormal set of Walsh functions in $L^2([0,1])$.
- Let $\mathrm{P}_N : L^2([0,1]) \to \overline{\operatorname{span}}\{\psi_n : n = 0, 1, 2, \dots, N\}$ be the orthogonal projection onto the first $N+1$ Walsh functions.
- View P_N as a mapping $L^\infty([0,1]) \to L^\infty([0,1])$ with norm

$$\|\mathrm{P}_N\| = \sup\{\|\mathrm{P}_N f\|_\infty : f \in L^\infty([0,1]), \|f\|_\infty \le 1\} \, .$$

▷

$$\limsup_{N\to\infty} \|\mathrm{P}_N\| = +\infty \qquad \text{but} \qquad \|\mathrm{P}_{2^k}\| = 1 \text{ for all } k \in \mathbb{N} \, .$$

▷ Thus $\{\mathrm{P}_N\}_{N\in\mathbb{N}}$ is *weakly divergent*.

▷ There exists a (universal) subsequence $\{N_k = 2^k\}_{k=0}^{\infty}$ such that

$$\begin{aligned} \lim_{k\to\infty} \|\mathrm{P}_{N_k} f - f\|_\infty = 0 \qquad & \text{for all} \qquad f \in \mathcal{C}([0,1]) \\ \limsup_{k\to\infty} \|\mathrm{P}_{N_k} f - f\|_\infty < \infty \qquad & \text{for all} \qquad f \in L^\infty([0,1]) \end{aligned}$$

Strong Divergence

Definition (Strong Divergence)

A sequence $\{\mathrm{T}_N\}_{N\in\mathbb{N}}$ of bounded linear approximation operators $\mathrm{T}_N : \mathcal{B}_1 \to \mathcal{B}_2$ is said to *diverge strongly* if

$$\lim_{N\to\infty} \left\|\mathrm{T}_N f_* - \mathrm{T} f_*\right\|_{\mathcal{B}_2} = \infty \qquad \text{for some} \qquad f_* \in \mathcal{B}_1 \,. \tag{SD}$$

- Strong divergence excludes the possibility of the existence of *good subsequences* $\{\mathrm{T}_{N_k}(f)\}_{k\in\mathbb{N}}$ such that

$$\liminf_{k\to\infty} \left\|\mathrm{T}_{N_k(f)} f_* - \mathrm{T} f_*\right\|_{\mathcal{B}2} < \infty \,.$$

- We are going to investigate whether weakly divergent sequences $\{\mathrm{T}_N\}_{N\in\mathbb{N}}$ are even strongly divergent.

Example – Pointwise Convergent Fourier Series

Example (Divergent operator norms – Not strongly divergent)

- For any $f \in \mathcal{C}([-\pi,\pi])$, let $(\mathrm{U}_N f)(t)$ be the partial sum of the Fourier series:

$$(\mathrm{U}_N f)(t) = \sum_{k=-N}^{N} \widehat{f}_n \, \mathrm{e}^{int} \quad \text{with} \quad \widehat{f}_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\tau)\, \mathrm{e}^{-in\tau} \, \mathrm{d}\tau \ .$$

- Fix $\lambda \in [-\pi,\pi]$ arbitrary and define the functionals $\mathrm{U}_{N,\lambda} : \mathcal{C}([-\pi,\pi]) \to \mathbb{C}$ by

$$\mathrm{U}_{N,\lambda} f := (\mathrm{U}_N f)(\lambda) \ , \qquad N \in \mathbb{N} \ .$$

▷ It is easy to see that $\|\mathrm{U}_{N,\lambda}\| = \|\mathrm{U}_N\|_{\mathcal{C}\to\mathcal{C}}$. Therefore

$$\lim_{N\to\infty} \left\|\mathrm{U}_{N,\lambda}\right\| = \lim_{N\to\infty} \left\|\mathrm{U}_N\right\|_{\mathcal{C}\to\mathcal{C}} = \infty \ .$$

▷ *Fejér*: To each $f \in \mathcal{C}([-\pi,\pi])$ there is a subsequence $\{N_k = N_k(f,\lambda)\}_{k\in\mathbb{N}}$ such that

$$\lim_{k\to\infty} \mathrm{U}_{N_k,\lambda} f = \lim_{k\to\infty} (\mathrm{U}_{N_k} f)(\lambda) = f(\lambda) \ .$$

⇒ **No strong divergence of $\{\mathrm{U}_{N,\lambda}\}_{N\in\mathbb{N}}$.**

Strong Divergence and Adaptive Methods

Assume that a sequence $\{\mathrm{T}_N\}_{N\in\mathbb{N}}$ diverges weakly but not strongly.

$\Rightarrow$ To every $f \in \mathcal{B}_1$ there exists a subsequence $\{N_k(f)\}_{k\in\mathbb{N}}$ such that

$$\sup_{k\in\mathbb{N}} \left\| \mathrm{T}_{N_k(f)} f - \mathrm{T} f \right\|_{\mathcal{B}_2} < \infty \,.$$

! The convergent subsequence $\{N_k(f)\}_{k\in\mathbb{N}}$ depends always on f

$\Rightarrow$ $\{\mathrm{T}_{N_k(f)}\}_{k\in\mathbb{N}}$ is a method *adapted* to the particular function $f \in \mathcal{B}_1$.

$\Rightarrow$ $\{\mathrm{T}_{N_k(f)} f\}_{k\in\mathbb{N}}$ is a *non-linear* approximation method

weak divergence	related to	existence of **non-adaptive methods**
strong divergence	related to	existence of **adaptive methods**

- *Banach-Steinhaus Theorem* is the perfect tool for *non-adaptive methods*.
- New techniques needed to investigate adaptive approximation methods.

Historical Remark

- Paul Erdős investigated *strong divergence* of Lagrange interpolation of continuous functions on Chebyshev notes in 1941.
- But he found himself that his proof was erroneous.
- His question is still open until now.

P. Erdős, **On divergence properties of the Lagrange interpolation parabolas**
Ann. of Math. vol. 42, no. 1 (1941), pp. 309–315.

P. Erdős, **Corrections to two of my papers**
Ann. of Math. vol. 44, no. 4 (1943), pp. 647–651.

Outline

The Hilbert Transform

Definition

For any $f \in L^1(\mathbb{T})$, its **conjugate function** $\widetilde{f}$ is given by the **Hilbert transform** $\mathrm{H}f$ of f. Thus

$$\widetilde{f}(t) = \big(\mathrm{H}f\big)(t) = \lim_{\epsilon \to 0} \frac{1}{2\pi} \int_{\epsilon \leq |\tau| \leq \pi} \frac{f(\tau + t)}{\tan(\tau/2)} \, \mathrm{d}\tau$$

where the integral on the right hand side exists for almost all $t \in \mathbb{T}$.

This transformation plays a very important role in different areas of science and engineering.

- System theory: The real- and imaginary part the transfer function of a causal system are related by the Hilbert transform.
- Physics: *Kramers-Kronig-Relation*
- Control theory

Illustration – Hilbert Transform for Polynomials

Let $p \in \mathcal{P}$ be a *trigonometric polynomial*
and $p_+ \in \mathcal{P}_+$ be a *causal trigonometric polynomial*

$$p(t) = \sum_{n=-N}^{N} c_n \, \mathrm{e}^{\mathrm{i}nt} \qquad \text{and} \qquad p_+(t) = \sum_{n=0}^{N} c_n \, \mathrm{e}^{\mathrm{i}nt}$$

Definition

The polynomial $\widetilde{p} \in \mathcal{P}$ is said to be the *conjugate* of $p \in \mathcal{P}$ if

$$p + \mathrm{i}\, \widetilde{p} \in \mathcal{P}_+ \qquad \text{and} \qquad \int_{-\pi}^{\pi} \widetilde{p}(t) \, \mathrm{d}t = 0 \,.$$

Example (even trigonometric polynomials)

$$p(t) = c_0 + 2 \sum_{n=1}^{N} c_n \, \cos(nt) \qquad \Rightarrow \qquad \widetilde{p}(t) = 2 \sum_{n=1}^{N} c_n \, \sin(nt)$$

Hilbert Transform – Basic Properties

- L^p-Theory
  - ▷ $\mathrm{H} : L^1(\mathbb{T}) \to$ weak $L^1(\mathbb{T})$ (*Kolmogoroff*)
  - ▷ $\mathrm{H} : L^p(\mathbb{T}) \to L^p(\mathbb{T})$, $1 < p < \infty$
  - ▷ $\mathrm{H} : L^\infty(\mathbb{T}) \to BMO$ (*Ch. Fefferman* and *E. M. Stein*)
  - ▷ $\mathrm{H} : H^1 \to H^1$ (*L. Carleson* and *E. M. Stein*)
  - ▷ H^1–BMO Duality (*Ch. Fefferman*)

- Hilbert transform on $\mathcal{C}(\mathbb{T})$
  - ▷ $\mathrm{H} : \mathcal{C}(\mathbb{T}) \to L^p(\mathbb{T}), \; 1 \leq p < \infty$
  - ▷ $\mathrm{H} : \mathcal{C}(\mathbb{T}) \nrightarrow \mathcal{C}(\mathbb{T})$
  - ▷ $\mathrm{H} : \mathcal{C}(\mathbb{T}) \to VMO$ (*Ch. Fefferman*)

J.B. Garnett **Bounded analytic functions** Academic Press, New York, 1981.

Signal Space for Hilbert Transform Approximations

We consider the Hilbert transform on the Banach space $\mathcal{B}$ of all *continuous functions* on $\mathbb{T} = [-\pi, \pi]$ *with continuous conjugate*

$$\mathcal{B} := \left\{ f \in \mathcal{C}(\mathbb{T}) : \widetilde{f} = \mathrm{H}f \in \mathcal{C}(\mathbb{T}) \right\}$$

equipped with the norm

$$\left\| f \right\|_{\mathcal{B}} := \max \left\{ \| f \|_\infty, \| \mathrm{H}f \|_\infty \right\} \qquad \text{with} \qquad \left\| f \right\|_\infty = \max_{t \in \mathbb{T}} \left| f(t) \right| .$$

Goal

Find a (practically relevant) sequence $\{\mathrm{H}_N\}_{N \in \mathbb{N}}$ of linear operators $\mathrm{H}_N : \mathcal{B} \to \mathcal{B}$ such that

$$\lim_{N\to\infty} \left\| \mathrm{H}_N f - \widetilde{f} \right\|_{\mathcal{B}} = \lim_{N\to\infty} \left\| \mathrm{H}_N f - \mathrm{H}f \right\|_{\mathcal{B}} = 0 \qquad \text{for all} \qquad f \in \mathcal{B} .$$

Approximation from Frequency Samples

Given $f \in \mathcal{B}$ arbitrary, and let $\{\widehat{f}_n\}_{n\in\mathbb{Z}}$ be its *Fourier coefficients*

$$\widehat{f}_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\,\mathrm{e}^{\mathrm{i}nt}\,\mathrm{d}t\,, \qquad n \in \mathbb{Z}\,.$$

Consider the Nth-order *Fejér mean*

$$\big(\mathrm{F}_N f\big)(t) = \sum_{n=-N}^{N}\Big(1 - \frac{|n|}{N}\Big)\,\widehat{f}_n\,\mathrm{e}^{\mathrm{i}nt} = \frac{N}{2\pi}\int_{-\pi}^{\pi} f(\theta)\,\mathcal{F}_N(t-\theta)\,\mathrm{d}\theta$$

and define $\widetilde{\mathrm{F}}_N := \mathrm{H}\mathrm{F}_N$.

Theorem

$$\lim_{N\to\infty}\big\|\widetilde{\mathrm{F}}_N f - \widetilde{f}\big\|_\infty = 0 \qquad \textit{for all} \qquad f \in \mathcal{B}\,.$$

Proof:

$$\big\|\widetilde{\mathrm{F}}_N f - \widetilde{f}\big\|_\infty = \big\|\mathrm{H}\mathrm{F}_N f - \widetilde{f}\big\|_\infty = \big\|\widetilde{\mathrm{F}_N f} - \widetilde{f}\big\|_\infty = \big\|\mathrm{F}_N \widetilde{f} - \widetilde{f}\big\|_\infty\,.$$

Example - A Pointwise Convergent Process

Example (Divergent operator norms – Not strongly divergent)

- For any $f \in \mathcal{B}$, let $(\mathrm{U}_N f)(t)$ be the partial sum of the Fourier series.
- Define $\widetilde{\mathrm{U}}_N := \mathrm{H}\mathrm{U}_N = \mathrm{U}_N\mathrm{H}$
- Fix $\lambda \in [-\pi, \pi]$ arbitrary and define the functionals $\widetilde{\mathrm{U}}_{N,\lambda} : \mathcal{B} \to \mathbb{C}$ by

$$\widetilde{\mathrm{U}}_{N,\lambda} f := \big(\widetilde{\mathrm{U}}_N f\big)(\lambda) = \big(\mathrm{H}\mathrm{U}_N f\big)(\lambda) = \big(\mathrm{U}_N \mathrm{H} f\big)(\lambda) = \big(\mathrm{U}_N \widetilde{f}\big)(\lambda)$$

▷ It is easy to see that $\|\widetilde{\mathrm{U}}_{N,\lambda}\| = \|\widetilde{\mathrm{U}}_N\|_{\mathcal{B}\to\mathcal{B}}$. Therefore

$$\lim_{N\to\infty} \big\|\widetilde{\mathrm{U}}_{N,\lambda}\big\| = \lim_{N\to\infty} \big\|\widetilde{\mathrm{U}}_N\big\|_{\mathcal{C}\to\mathcal{C}} = \infty \,.$$

▷ Using *Fejér*: To each $f \in \mathcal{B}$ there is a subsequence $\{N_k(f,\lambda)\}_{k\in\mathbb{N}}$ such that

$$\lim_{k\to\infty} \widetilde{\mathrm{U}}_{N_k,\lambda} f = \lim_{k\to\infty} \big(\mathrm{U}_{N_k} \widetilde{f}\big)(\lambda) = \widetilde{f}(\lambda) \,.$$

⇒ **No strong divergence of $\{\widetilde{\mathrm{U}}_{N,\lambda}\}_{N\in\mathbb{N}}$.**

Practical Constraints on Approximation Sequences

- ▷ Previous operators $\{\widetilde{\mathrm{F}}_N\}_{N\in\mathbb{N}}$ and $\{\widetilde{\mathrm{U}}_N\}_{N\in\mathbb{N}}$ are based on the exact knowledge of the Fourier coefficients $\{\widehat{f_n}\}_{n=-N}^{N}$.
- ▷ Equivalently, these operators are based on the knowledge of $f \in \mathcal{B}$ on the whole interval $\mathbb{T}$.

⇒ **Analog computers/devices are needed for implementation.**

- ϟ Practical applications ⇒ **digital signal processing**.
- ϟ Signals f are only known on finite number of sampling points $\{f(\lambda_m)\}_{m=1}^{M}$.
- ϟ Previous approximation sequence $\{\widetilde{\mathrm{F}}_N\}_{N\in\mathbb{N}}$ can not be implemented.

⇒ **Consider approximation sequences $\{\mathrm{H}_N\}_{N\in\mathbb{N}}$ which are based on sampled data.**

Properties of our Approximation Sequences

(A) Concentration on a finite sampling set: For every $N \in \mathbb{N}$ there exists a finite sampling set $\Lambda_N = \{\lambda_n : n = 1, \ldots, M_N\}$ with $\lambda_n \in \mathbb{T}$ such that

$$f(\lambda) = g(\lambda) \qquad \text{for all } \lambda \in \Lambda_N$$

implies

$$(\mathrm{H}_N f)(t) = (\mathrm{H}_N g)(t) \qquad \text{for all } t \in \mathbb{T} \,.$$

(B) Convergence on a dense subset: The sequence $\{\mathrm{H}_N\}_{N \in \mathbb{Z}}$ satisfies

$$\lim_{N \to \infty} \left\| \mathrm{H}_N f - \widetilde{f} \right\|_\infty = 0 \qquad \text{for all } f \in \mathcal{C}^\infty(\mathbb{T}) \,.$$

(C) Generated by a stable sampling series: To the sequence $\{\mathrm{H}_N\}_{N \in \mathbb{N}}$ there corresponds a sequence of approximation operators $\mathrm{A}_N : \mathcal{B} \to \mathcal{B}$ such that

$$\lim_{N \to \infty} \|\mathrm{A}_N f - f\|_\infty = 0 \qquad \text{for all } f \in \mathcal{B}$$

and such that $\mathrm{H}_N f = \mathrm{H}\mathrm{A}_N f$ for all $N \in \mathbb{N}$.

Consequences & Properties

- A sequence $\{\mathrm{H}_N\}_{N\in\mathbb{N}}$ has property (A) if and only if to every $N \in \mathbb{N}$ there exists a finite set

$$\Lambda_N = \left\{\lambda_{1,N}, \lambda_{2,N}, \ldots, \lambda_{M_N,N}\right\} \quad \text{with} \quad M_N \in \mathbb{N} \quad \text{and} \quad \lambda_{n,N} \in \mathbb{T}$$

and functions $\{h_{n,N} \ : \ n = 1, \ldots, M_N\}$ in $\mathcal{B}$ such that

$$\left(\mathrm{H}_N f\right)(t) = \sum_{n=1}^{M_N} f(\lambda_{n,N})\, h_{n,N}(t) \qquad \text{for all } f \in \mathcal{B}\,.$$

- Then the approximation operators A_N in property (C) have the form

$$\left(\mathrm{A}_N f\right)(t) = \sum_{n=1}^{M_N} f(\lambda_{n,N})\, a_{n,N}(t)\,, \qquad t \in \mathbb{T}\,,$$

with functions $a_{n,N} \in \mathcal{B}$ such that $h_{n,N} = \mathrm{H}\, a_{n,N}$.

Example – Sampled (Conjugate) Fejér Mean

- Inserting the Fourier coefficients into the *Fejér mean* and exchanging the sum with the integral, one obtains the integral representation

$$\big(\mathrm{F}_N f\big)(t) = \frac{N}{2\pi} \int_{-\pi}^{\pi} f(\theta)\, \mathcal{F}_N(t-\theta)\, \mathrm{d}\theta \qquad (\Delta)$$

  with the so-called *Fejér kernel*

$$\mathcal{F}_N(\tau) = \left(\frac{\sin(N\tau/2)}{N\,\sin(\tau/2)} \right)^2 .$$

- Approximate the integral in (Δ) by its Riemann sum based on the rectangular integration rule yields the *sampled Fejér mean*

$$\big(\mathrm{S}_N f\big)(t) = \sum_{n=0}^{N-1} f\big(n\tfrac{2\pi}{N}\big)\, \mathcal{F}_N\big(t - n\tfrac{2\pi}{N}\big) \approx (\mathrm{F}_N f)(t) .$$

  It show the same approximation behavior as (Δ):

$$\lim_{N\to\infty} \big\|\mathrm{S}_N f - f\big\|_\infty = \lim_{N\to\infty} \big\|\mathrm{F}_N f - f\big\|_\infty = 0 \quad \text{for all} \quad f \in \mathcal{C}(\mathbb{T}) .$$

Example – Sampled Conjugate Fejér Mean

- Now we define the approximation operators $\mathrm{H}_N^{\mathcal{F}} := \mathrm{HS}_N$. This yields

$$\big(\mathrm{H}_N^{\mathcal{F}} f\big)(t) = \big(\mathrm{HS}_N f\big)(t) = \sum_{n=0}^{N-1} f\big(n\,\tfrac{2\pi}{N}\big)\, \widetilde{\mathcal{F}}_N\big(t - n\,\tfrac{2\pi}{N}\big)$$

with the *conjugate Fejér kernel* $\widetilde{\mathcal{F}}_N = \mathrm{H}\mathcal{F}_N$ given by

$$\widetilde{\mathcal{F}}_N(\tau) = \frac{N \sin\tau - \sin(N\tau)}{2\big[N\sin(\tau/2)\big]^2} = \frac{1}{N}\left(\frac{1}{\tan(\tau/2)} - \frac{\sin(N\tau)}{2N\sin^2(\tau/2)}\right) .$$

▷ By this construction, it is easy to verify that $\{\mathrm{H}_N^{\mathcal{F}}\}_{N\in\mathbb{N}}$ is indeed an approximation sequence with the desired property (A), (B), and (C).

▷ Replace the rectangular integration rule by any other integration method gives similar operators but with other kernels.

Outline

Weak Divergence of Hilbert transform Approximations

It is known that every sequence $\{\mathrm{H}_N\}_{N\in\mathbb{N}}$ with properties (A), (B), (C) diverges weakly on $\mathcal{B}$. More precisely, the following result was proven.

Theorem (Weak Divergence)

Let $\{\mathrm{H}_N\}_{N\in\mathbb{N}}$ be a sequence of operators with property (A), (B), and (C). Then there exists an $f_* \in \mathcal{B}$ such that

$$\limsup_{N\to\infty} \|\mathrm{H}_N f_*\|_\infty = \infty\,. \qquad (\square)$$

Moreover, the set of all $f_* \in \mathcal{B}$ for which $(\square)$ hold is a residual set in $\mathcal{B}$.

Remark

The proof is based on the Theorem of Banach-Steinhaus, showing that the operator norms $\|\mathrm{H}_N\|$ are not uniformly bounded.

On the calculation of the Hilbert transform from interpolated data
H. Boche and V. Pohl
IEEE Trans. Inform. Theory, vol. 54, no. 5 (May 2008), pp. 2358–2366

Conjecture - Strong Divergence of all Hilbert Transform Approximations from Sampled Data

Conjecture

Let $\{\mathrm{H}_N\}_{N\in\mathbb{N}}$ be an arbitrary sequence of linear approximation operators with properties (A), (B), and (C). Then there exists an $f_* \in \mathcal{B}$ such that

$$\lim_{N\to\infty} \left\|\mathrm{H}_N f_*\right\|_\infty = \infty\,. \qquad (\Delta)$$

Remark

We give 3 results which support this conjecture:

- Strong divergence of $\{\mathrm{H}_N\}_{N\in\mathbb{N}}$ on $\mathcal{C}(\mathbb{T}) \supset \mathcal{B}$.
- Strong divergence of the sampled Fejér means $\{\mathrm{H}_N^{\mathcal{F}}\}_{N\in\mathbb{N}}$. Even a stronger divergence result than (Δ).
- „Almost strong divergence“ for all approximation procedures with properties (A), (B), and (C).

Outline

Strong Divergence for Continuous Functions

Theorem (Strong divergence on $\mathcal{C}(\mathbb{T})$)

There exists a residual set $\mathcal{D} \subset \mathcal{C}(\mathbb{T})$ such that for every sequence $\{\mathrm{H}_N\}_{N\in\mathbb{N}}$ with properties (A), (B), and (C) holds

$$\lim_{N\to\infty} \left\|\mathrm{H}_N f\right\|_\infty = \infty \qquad \textit{for all } f \in \mathcal{D}\,.$$

Remark

The set $\mathcal{D}$ does not depend on the particular operator sequence $\{\mathrm{H}_N\}_{N\in\mathbb{N}}$ but it is universal in the sense that $\mathcal{D}$ is the same for all possible sequences $\{\mathrm{H}_N\}$.

Sampled Conjugate Fejér Means (SCFM)

We consider the particular sequence $\{\mathrm{H}_N^{\mathcal{F}}\}_{N\in\mathbb{N}}$ of the sampled conjugate Fejér means

$$\big(\mathrm{H}_N^{\mathcal{F}} f\big)(t) = \sum_{n=0}^{N-1} f\big(n\,\tfrac{2\pi}{N}\big)\,\widetilde{\mathcal{F}}_N\big(t - n\,\tfrac{2\pi}{N}\big)$$

with the *conjugate Fejér kernel*

$$\widetilde{\mathcal{F}}_N(\tau) = \frac{N\sin\tau - \sin(N\tau)}{2\big[N\sin(\tau/2)\big]^2} = \frac{1}{N}\left(\frac{1}{\tan(\tau/2)} - \frac{\sin(N\tau)}{2N\sin^2(\tau/2)}\right) .$$

Recall

- ▷ Obtained from the uniformly convergent Fejér means based on frequency samples (Fourier coefficients).
- ▷ Approximate integration by Riemann sums using a rectangular integration rule.

Strong Divergence of SCFM

Theorem

Let $\{\mathrm{H}_N^{\mathcal{F}}\}_{N\in\mathbb{N}}$ be the sequence sampled conjugate Fejér means (SCFM). There exists a function $f_* \in \mathcal{B}$ such that

$$\lim_{N\to\infty} \left(\mathrm{H}_N^{\mathcal{F}} f_*\right)(\pi) = \infty \ .$$

Remark

- This result implies the strong divergence of $\{\mathrm{H}_N^{\mathcal{F}}\}_{N\in\mathbb{N}}$:
  There exists an $f_* \in \mathcal{B}$ such that

$$\lim_{N\to\infty} \left\|\mathrm{H}_N^{\mathcal{F}} f_*\right\|_\infty = \infty \ .$$

- But the theorem shows even the strong divergence *at a fixed point* $\pi \in \mathbb{T}$.
- Similar to investigations of Erdős on the divergence of Lagrange interpolation.
- First example of pointwise strong divergence.

Divergent Kernels

The divergence behavior of the approximation series is determined by the properties of the kernel.

Corollary

Let $\{\mathrm{H}_N\}_{N\in\mathbb{N}}$ be a sequence with properties (A), (B), and (C) of the form

$$\big(\mathrm{H}_N f\big)(t) = \sum_{n=0}^{N-1} f\big(n\,\tfrac{2\pi}{N}\big)\,\widetilde{\mathcal{K}}_N\big(t - n\,\tfrac{2\pi}{N}\big)$$

and assume that the kernel $\widetilde{\mathcal{K}}_N$ has the following two properties

$$(i) \quad \widetilde{\mathcal{K}}_N(\tau) \geq 0 \qquad \text{for all} \quad 0 < \tau < \pi$$

$$(ii) \quad C(N) := \sum_{n=0}^{\lfloor N/2 \rfloor} \widetilde{\mathcal{K}}_N\big(\pi - n\tfrac{2\pi}{N}\big) \geq \frac{2}{\pi}\log(N+1) - C_0 \qquad \text{for all} \quad N \in \mathbb{N}$$

with a positive constant C_0 independent of N.
Then $\{\mathrm{H}\}_{N\in\mathbb{N}}$ diverges strongly on $\mathcal{B}$.

Strong Divergence of SCFM - Discussion

- SCFM - derived from conjugate Fejér mean due to numerical integration.
- Conjugate Fejér means are uniformly convergent $\Leftrightarrow$ SCFM strongly divergent.
- We used rectangular integration rule to derive SCFM.
- Other integration rules are possible (trapezoidal, Newton-Cotes, ...).
  - $\Rightarrow$ This yields approximation operators with property (A), (B), (C).
  - $\Rightarrow$ This yields other kernels.
  - ? Do these approximation method also diverge strongly?

Toward Strong Divergence

- Let $\{\mathrm{H}_N\}_{N\in\mathbb{N}}$ be a sequence with properties (A),(B) and (C). We want to show that there exists an $f_* \in \mathcal{B}$ such that

$$\lim_{N\to\infty} \|\mathrm{H}_N f_*\|_\infty = +\infty \,. \qquad \text{(SD)}$$

- Equivalently: To every $M > 0$ there exists an $N^{(1)} \in \mathbb{N}$ such that

$$\left\|\mathrm{H}_N f_*\right\|_\infty > M \qquad \text{for all} \quad N > N^{(1)} \,.$$

$\Rightarrow$ Thus, $\|\mathrm{H}_N f_*\|_\infty$ gets arbitrarily large on the infinite interval $[N^{(1)}, \infty)$.

Weaker Property – „Almost Strongly Divergent“

We show that for any sequence $\{\mathrm{H}_N\}_{N\in\mathbb{Z}}$ with properties (A), (B), (C) there exists a function $f_* \in \mathcal{B}$ such that $\|\mathrm{H}_N f_*\|_\infty$ gets *arbitrarily large* on *arbitrarily long intervals* $[N^{(1)}, N^{(2)}]$, i.e.

$$\|\mathrm{H}_N f_*\|_\infty > M \qquad \text{for all} \qquad N \in [N^{(1)}, N^{(2)}] \,.$$

Almost Strong Divergence

Theorem

Let $\{\mathrm{H}_N\}_{N\in\mathbb{N}}$ be a sequence of linear operators with properties (A), (B), and (C). Then there exists a function $f_* \in \mathcal{B}$ with the following property:
To all arbitrary natural numbers $M, N_0 \in \mathbb{N}$ and for every $\delta \in (0,1)$ there exist two natural numbers $N^{(1)} = N^{(1)}(M, N_0, \delta)$ and $N^{(2)} = N^{(2)}(M, N_0, \delta)$ with

$$N^{(2)} > N^{(1)} \geq N_0 \qquad \textit{and} \qquad \frac{N^{(2)} - N^{(1)}}{N^{(2)}} > 1 - \delta$$

such that $\|\mathrm{H}_N f_*\|_\infty > M$ for all $N \in [N^{(1)}, N^{(2)}]$.

Remark

- Let $\mathcal{D}(M, f_*) := \big\{N \in \mathbb{N} \ : \ \|\mathrm{H}_N f_*\|_\infty > M\big\}$. Then the above theorem implies
$$\limsup_{K\to\infty} \frac{\big|\mathcal{D}(M, f_*) \cap [1, 2, \ldots, K]\big|}{K} = 1\ .$$
- The theorem *implies not* the strong divergence of all $\{\mathrm{H}_N\}_{N\in\mathbb{N}}$.

Size of the Divergence Set – Banach-Steinhaus

Banach-Steinhaus Technique

Let $\{\mathrm{H}_N\}_{N\in\mathbb{N}}$ be a sequence of linear operators on $\mathcal{B}$ such that

$$\lim_{N\to\infty} \|\mathrm{H}_N f - \mathrm{H} f\|_{\mathcal{B}} = 0 \quad \text{for all} \quad f \in \mathcal{B}_0$$

in a dense subset $\mathcal{B}_0 \subset \mathcal{B}$, and such that

$$\limsup_{N\to\infty} \|\mathrm{H}_N f_*\|_\infty = \infty \quad \text{for some } f_* \in \mathcal{B}\ .$$

Then the set

$$\mathcal{D} = \left\{ f_* \in \mathcal{B}\ :\ \limsup_{N\to\infty} \left\|\mathrm{H}_N f_*\right\|_{\mathcal{B}} = \infty \right\}$$

is a residual set in $\mathcal{B}$.

- ▷ If there exists one function f_* such that $\mathrm{H}_N f_*$ diverges, then there exists a whole residual set of functions f for which $\mathrm{H}_N f$ diverges.

The Divergence Set for Almost Strong Divergence

Theorem

Let $\{\mathrm{H}_N\}_{N\in\mathbb{N}}$ be a sequence of linear operators with properties (A), (B), and (C), and denote by $\mathcal{D}_{\mathrm{H}}$ the set of all $f \in \mathcal{B}$ for which the following holds:
For arbitrary numbers $M \in \mathbb{N}$, $N_0 \in \mathbb{N}$, and $\delta \in (0,1)$ there exist numbers $N^{(1)} = N^{(1)}(M,\delta) \geq N_0$ and $N^{(2)} = N^{(2)}(M,\delta) > N^{(1)}$ with

$$\frac{N^{(2)} - N^{(1)}}{N^{(2)}} > 1 - \delta$$

such that

$$\|\mathrm{H}_N f\|_\infty > M \qquad \text{for all } N \in [N^{(1)}, N^{(2)}]\,.$$

Then $\mathcal{D}_{\mathrm{H}}$ is a residual set in $\mathcal{B}$.

- ▷ The set $\mathcal{D}_{\mathrm{H}}$ of all functions $f \in \mathcal{B}$ for which $\|\mathrm{H}_N f\|_{\mathcal{B}}$ gets arbitrarily large on arbitrarily long intervals is a residual set.
- ▷ **So almost strong divergence occurs basically for all functions $f \in \mathcal{B}$.**

Outline

Strong Divergence and Adaptive Methods

General setting

Given $\mathrm{T} : \mathcal{B}_1 \to \mathcal{B}_2$ a bounded linear operator.
Let $\{\mathrm{T}_N\}_{N\in\mathbb{Z}}$ a sequence with property (A), (B) and (C), such that

$$\limsup_{N\to\infty} \|\mathrm{T}_N f_*\|_{\mathcal{B}_2} = \infty \qquad \text{for some} \qquad f_* \in \mathcal{B}_1$$

but such that $\{\mathrm{T}_N\}_{N\in\mathbb{Z}}$ does *not diverge strongly*.

- ▷ To every $f \in \mathcal{B}_1$ there exists a subsequence $\{N_k(f)\}_{k\in\mathbb{N}}$ such that

$$\limsup_{k\to\infty} \|\mathrm{T}_{N_k(f)} f - \mathrm{T} f\|_{\mathcal{B}_2} < \infty \ .$$

- ▷ The corresponding subsequence $\{N_k(f)\}_{k\in\mathbb{N}}$ depends always on f $\Rightarrow \{\mathrm{T}_{N_k(f)}\}_{k\in\mathbb{N}}$ is an *approximation method adapted to the particular function* $f \in \mathcal{B}_1$.

Approximations with Finite Search Horizon

Goal

Find sequence $\{N_k(f)\}_{k\in\mathbb{N}}$ such that $\|\mathrm{T}_{N_k(f)} - \mathrm{T}f\|_{\mathcal{B}_2} < C_{\mathrm{u}}$ for each $k \in \mathbb{N}$.

Problem

The distance between two good indices N_k and N_{k+1} may be arbitrarily large.

Methods with finite search horizon

- Let $\{N_k\}_{k\in\mathbb{N}}$ be a given sequence of strictly monotonically increasing natural numbers.
- Given $f \in \mathcal{B}_1$ and choose

$$\widehat{N}_k(f) = \underset{N\in(N_k,N_{k+1}]}{\arg\min} \left\|\mathrm{T}_N f - \mathrm{T}f\right\|_{\mathcal{B}_2} , \qquad k = 1, 2, \ldots .$$

If the intervals $(N_k, N_{k+1}]$ are large enough, then we may hope to obtain a sequence $\{\widehat{N}_k(f)\}_{k\in\mathbb{N}}$ such that

$$\lim_{k\to\infty} \left\|\mathrm{T}_{\widehat{N}_k(f)} f - \mathrm{T}f\right\|_{\mathcal{B}_2} = 0 .$$

Existence of Methods with Finite Search Horizon

- From practical point of view, adaptive methods with finite search horizon are of importance.
- This is a stronger condition than *strong divergence* (infinite search horizon).

Problem 1

Let $\{\mathrm{T}_N\}_{N\in\mathbb{N}}$ be a given approximation method of $\mathrm{T} : \mathcal{B}_1 \to \mathcal{B}_2$.
Does there exist a strictly monotonically increasing sequence $\{N_k\}_{k\in\mathbb{Z}}$ in $\mathbb{N}$ such that for every $f \in \mathcal{B}_1$ there is a subsequence $\{\widehat{N}_k\}_{k\in\mathbb{N}}$ such that

$$\widehat{N}_k \in (N_k, N_{k+1}] \qquad \text{and} \qquad \inf_{k\in\mathbb{N}} \left\| \mathrm{T}_{\widehat{N}_k} f - \mathrm{T} f \right\|_{\mathcal{B}_2} < \infty \text{ ?}$$

Ul'yanovs Problem - Original Question

Consider the *Fourier series* for Lebesgue integrable functions on $\mathbb{T}$:

$$\big(\mathrm{S}_N f\big)(t) = \sum_{n=-N}^{N} \widehat{f}_n \, \mathrm{e}^{\mathrm{i}nt} \qquad \text{with} \qquad \widehat{f}_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) \, \mathrm{e}^{-\mathrm{i}nt} \, \mathrm{d}t \, .$$

Ul'yanovs Question

Does there exist a sequence $\{N_k\}_{k\in\mathbb{N}}$ such that the Fourier series of any $f \in L^1(\mathbb{T})$ possesses a subsequence $\{\mathrm{S}_{\widehat{N}_k(f)} f\}$ of its partial sums such that

$$\widehat{N}_k < N_k \qquad \text{and} \qquad \lim_{k\to\infty} (\mathrm{S}_{\widehat{N}_k} f)(t) = f(t) \quad \text{for almost all } t \in \mathbb{T} \text{ ?}$$

- The sequence $\{N_k\}_{k\in\mathbb{N}}$ characterizes how fast $\{\widehat{N}_k(f)\}_{k\in\mathbb{N}}$ has to grow such that the partial sums $\{\mathrm{S}_{\widehat{N}_k} f\}$ converge to the desired $f \in L^1(\mathbb{T})$.
- The subsequence $\big\{\widehat{N}_k(f)\big\}_{k\in\mathbb{N}}$ depends on the actual $f \in L^1(\mathbb{T})$.

Ul'yanovs Problem - Generalized Formulation

The question of Ul'yanov may be reformulated in our context as follows:

Ul'yanov-Type Problem

Let $\{\mathrm{T}_N\}_{N\in\mathbb{N}}$ be a given approximation method of $\mathrm{T} : \mathcal{B}_1 \to \mathcal{B}_2$.
Does there exist a strictly monotonically increasing sequence $\{N_k\}_{k\in\mathbb{Z}}$ in $\mathbb{N}$, such that for every $f \in \mathcal{B}_1$ there is a strictly monotonically increasing sequence $\{\widehat{N}_k\}_{k\in\mathbb{Z}}$ such that

$$\widehat{N}_k \leq N_k \qquad \text{and} \qquad \inf_{k\in\mathbb{N}} \left\| \mathrm{T}_{\widehat{N}_k} f - \mathrm{T} f \right\|_{\mathcal{B}_2} < \infty \, ?$$

- So how fast do the good approximation indices $\{\widehat{N}_k\}_{k\in\mathbb{N}}$ grow?
- Ul'yanov's problem: $\widehat{N}_k \leq N_k$ **contrary** Problem 1: $\widehat{N}_k \in (N_k, N_{k+1}]$
  - Ul'yanov: more freedom to adapt the subsequence $\{\widehat{N}_k(f)\}_{k\in\mathbb{N}}$
  - Problem 1: closer relation to practical adaptive methods.

Condition for the Existence of a Solution

To investigate concrete operators $\mathrm{T} : \mathcal{B}_1 \to \mathcal{B}_2$ and approximation sequences $\{\mathrm{T}_N\}_{N\in\mathbb{N}}$ the following Lemma will be useful

Lemma (Condition for Problem 1 to be solvable)

Problem 1 has no solution if and only if to every strictly monotonically increasing sequence $\{N_k\}_{k\in\mathbb{Z}}$ of natural numbers there exists a function $f \in \mathcal{B}_1$ such that

$$\limsup_{k\to\infty} \left(\min_{N\in(N_k,N_{k+1}]} \left\| \mathrm{T}_N f - \mathrm{T} f \right\|_{\mathcal{B}_2} \right) = \infty \,.$$

There is a close relation to the Ul'yanov-Type problem:

Theorem (Relation to Ul'yanov Type problem)

Problem 1 has a solution if and only if the Ul'yanov-Type Problem possess a solution.

Adaptive Methods - Size of the Divergence Sets

If Problem 1 is not solvable, then there exists *one* $f \in \mathcal{B}_1$ such that we can't find an adaptive convergent subsequence of $\{\mathrm{T}_N\}_{N\in\mathbb{N}}$.

- **?** Does there exists more such functions?
- **?** How large is he divergence set?

Definition: Divergence set

Let $\{\mathrm{T}\}_{N\in\mathbb{N}}$ be an approximation sequence, and let $\mathcal{N} = \{N_k\}_{k\in\mathbb{N}}$ be an arbitrary strictly monotonically increasing sequence of natural numbers. Then

$$\mathcal{D}_1(\{\mathrm{T}_N\},\mathcal{N}) := \Big\{ f \in \mathcal{B}_1 \ : \text{For every strictly monotonically increasing sequences } \{\widehat{N}_k\}_{k\in\mathbb{N}} \text{ with } \widehat{N}_k \in (N_k, N_{k+1}],\ k \in \mathbb{N} \text{ holds } \limsup_{k\to\infty} \left\| \mathrm{T}_{\widehat{N}_k} f - \mathrm{T} f \right\|_{\mathcal{B}_2} = \infty \Big\} .$$

The Divergence Sets are Residual

Theorem

If Problem 1 is not solvable for a given approximation sequence $\{\mathrm{T}_N\}_{N\in\mathbb{N}}$, then the divergence set $\mathcal{D}_1(\{\mathrm{T}_N\},\mathcal{N})$ is a residual set in $\mathcal{B}_1$ for any $\mathcal{N}$.

- If Problem 1 is not solvable for an operator sequence $\{\mathrm{T}_N\}_{N\in\mathbb{N}}$, then any adaptive approximation with finite search horizon diverges for almost all functions $f \in \mathcal{B}_1$.
- A similar result holds for the Ul'yanov-Type problem.

Theorem

If the Ul'yanov-Type problem is not solvable then the corresponding divergence set $\mathcal{D}_{\mathrm{U}}(\{\mathrm{T}_N\},\mathcal{N})$ is a residual set in $\mathcal{B}_1$ for any $\mathcal{N}$.

Outline

Problem 1 for Hilbert Transform Approximations

We consider again the concrete problem of the approximation of the Hilbert transform $\mathrm{H} : \mathcal{B} \to \mathcal{B}$ by a sequence $\{\mathrm{H}_N\}_{N\in\mathbb{N}}$ of operators with properties (A), (B) and (C).

Theorem

Let $\{\mathrm{H}_N\}_{N\in\mathbb{N}}$ be a given sequence of linear operators with properties (A), (B), and (C), and let $\{N_k\}_{k\in\mathbb{N}}$ be an arbitrary strictly monotonically increasing sequences of natural numbers. There exists a function $f_* \in \mathcal{B}$ such that

$$\limsup_{k\to\infty} \min_{N\in(N_k,N_{k+1}]} \left\|\mathrm{H}_N f_*\right\|_\infty = \infty \, .$$

- $\Rightarrow$ Problem 1 has no solution for our Hilbert transform approximations.
- $\Rightarrow$ The Ul'yanov-Type problem has no solution .

Corollary

There exists no adaptive approximation methods with a finite search horizon for our class of Hilbert transform approximations with properties (A), (B) and (C).

Outline

Summary and Conclusions

- **Weak divergence** is related to **non-adaptive approximation methods**.
- Strong divergence is related to the existence of adaptive approximation methods: **Strong divergence $\Rightarrow$ no adaptive methods**
- Modern signal processing is based on sampled data.
- We investigated approximation methods of the Hilbert transform from sampled data.
- **Conjecture:** All approximation methods of the Hilbert transform which are based on sampled data diverge strongly.

$\Rightarrow$ **There is no adaptive approximation method which is able to approximate the Hilbert transform based on samples of the signal.**

- However, there are non-adaptive, uniformly convergent approximation methods based on **analog signal processing**.
- Relation to interesting and long standing questions from Fourier analysis and approximation theory $\Rightarrow$ **Erdős, Ulyanov**

Thank You!

Questions?
Remarks?