Positivity, Discontinuity and Finite Resources for Arbitrarily Varying Quantum Channels

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Topics (all with respect to AVQCs:)

1. CLASSICAL MESSAGE TRANSMISSION (CONTINUITY)
2. ENTANGLEMENT TRANSMISSION
3. STRONG SUBSPACE TRANSMISSION

Things you can do with a quantum channel

Let **n** be a quantum channel with input system S and an outputsystem R for some legitimate receiver.

The channel can be used to transmit particles carrying information.

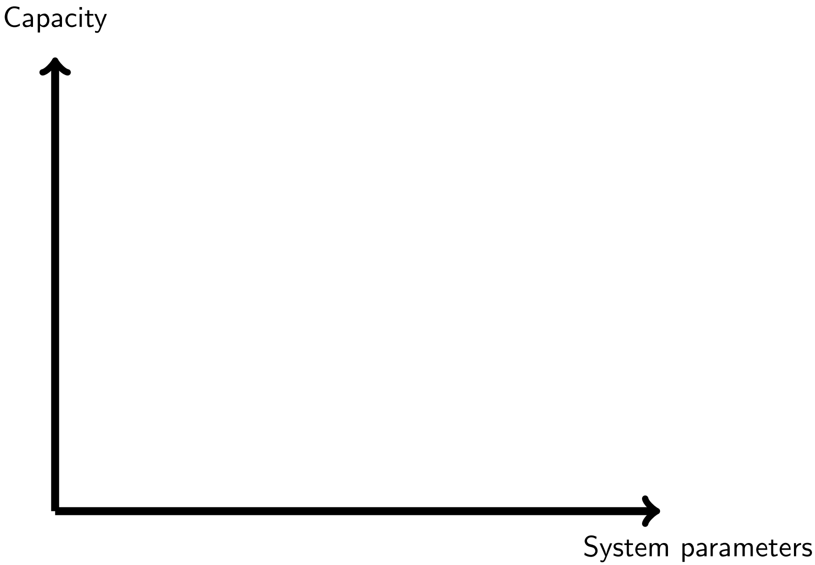
Different tasks can be carried out, leading to different mathematical criteria of 'successful transmission:

- message transmission (under average error criterion[1])
- message transmission (under maximal error criterion[2])
- entanglement transmission
- strong subspace transmission
- corresponding security criteria

[1] solved for AVCs in [CS89]

[2] connected to zero-error capacity [Ahl70]

Remarks on continuity of capacities



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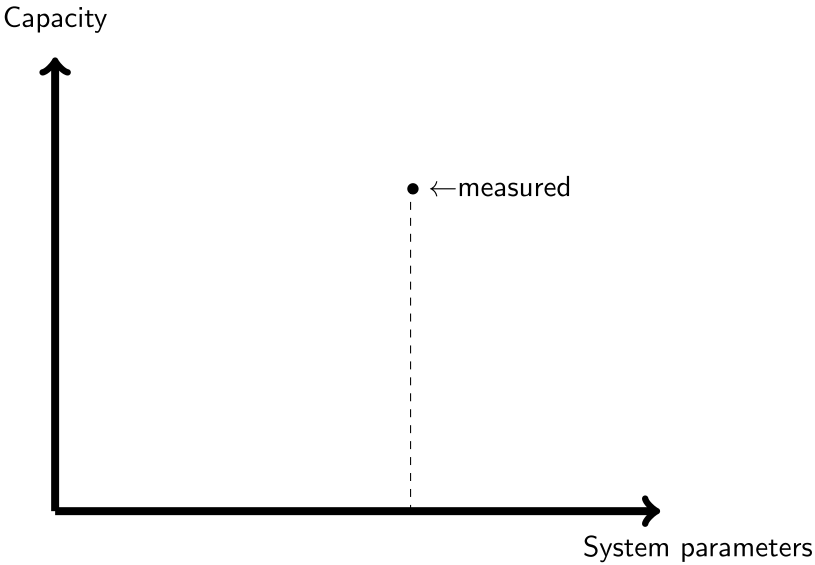

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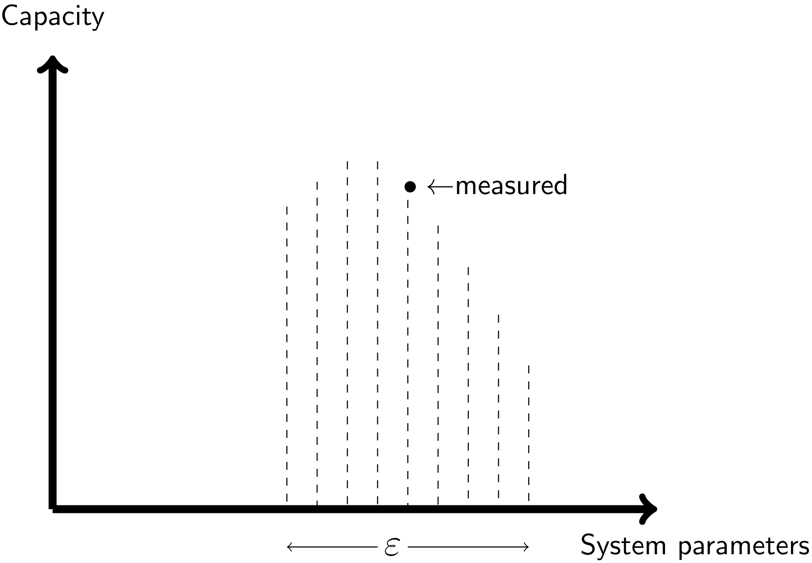

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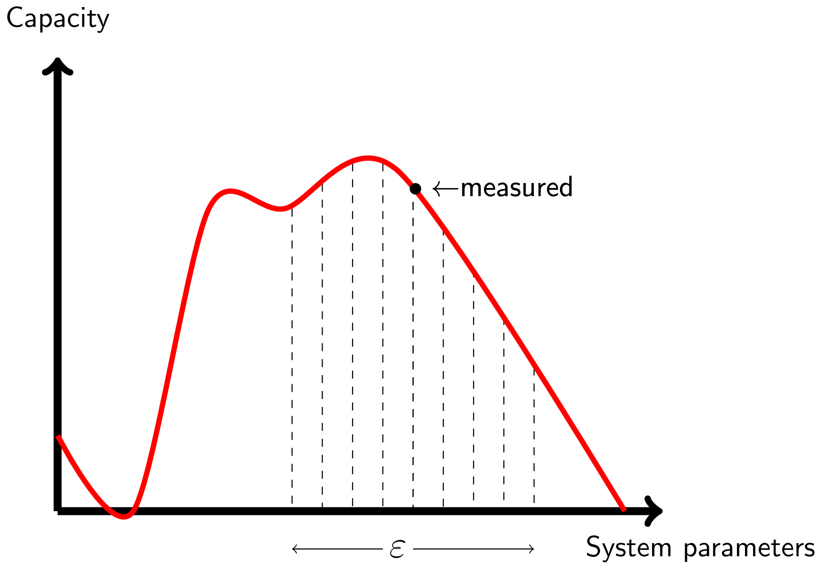

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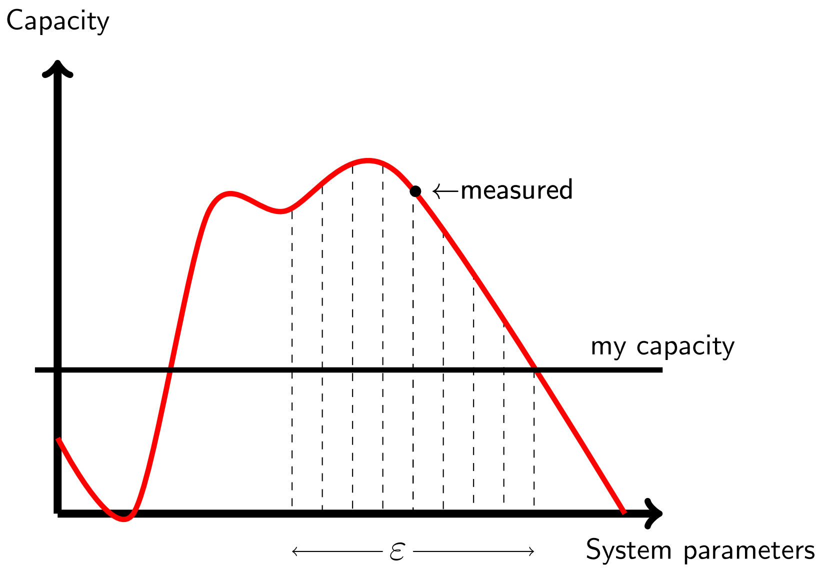

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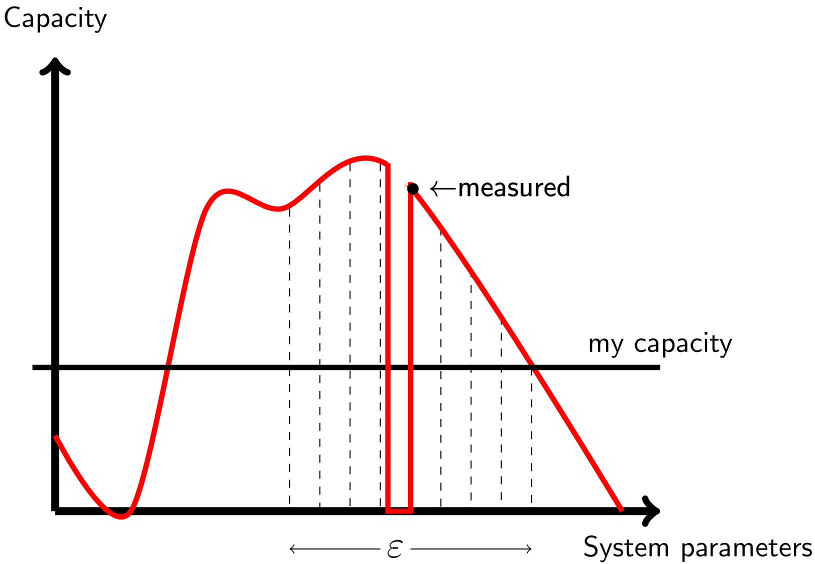
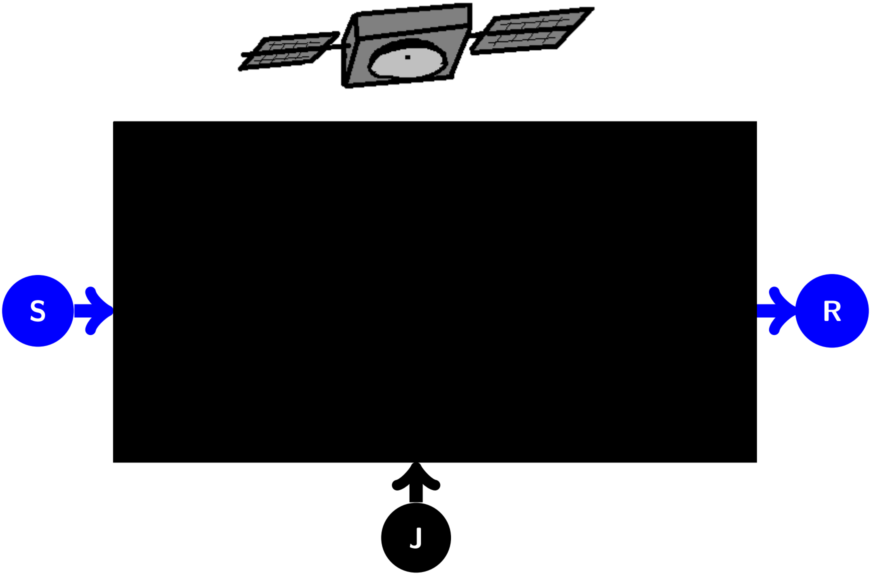


AVQC $\mathfrak{J} = \{\mathbf{n}_1, \mathbf{n}_2, \mathbf{n}_3\}$, no randomness, 10 channel uses



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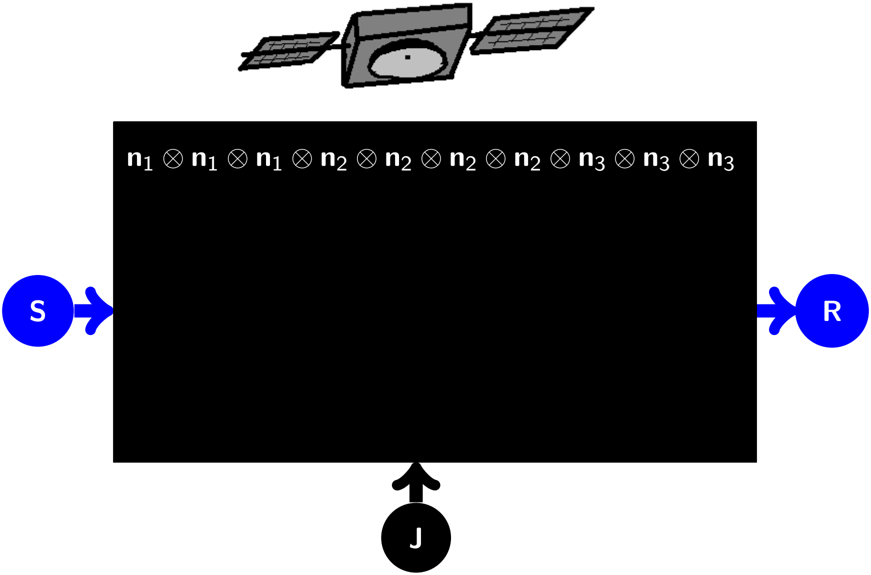

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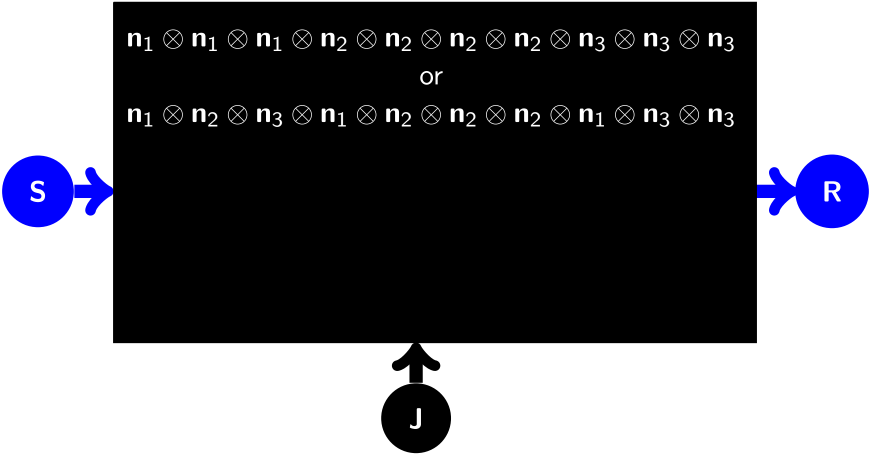

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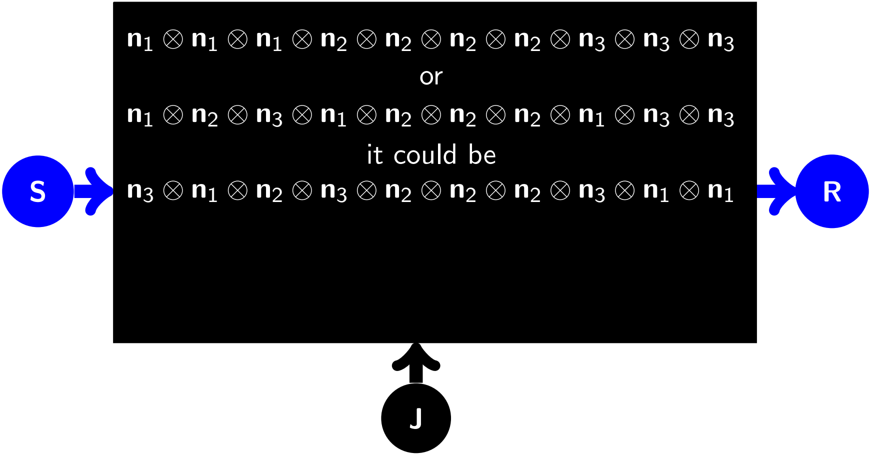

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$\mathbf{n}_1 \otimes \mathbf{n}_1 \otimes \mathbf{n}_1 \otimes \mathbf{n}_2 \otimes \mathbf{n}_2 \otimes \mathbf{n}_2 \otimes \mathbf{n}_2 \otimes \mathbf{n}_3 \otimes \mathbf{n}_3 \otimes \mathbf{n}_3$

or

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it could be

S $\rightarrow$ $\mathbf{n}_3 \otimes \mathbf{n}_1 \otimes \mathbf{n}_2 \otimes \mathbf{n}_3 \otimes \mathbf{n}_2 \otimes \mathbf{n}_2 \otimes \mathbf{n}_2 \otimes \mathbf{n}_3 \otimes \mathbf{n}_1 \otimes \mathbf{n}_1$ $\rightarrow$ R

or something completely different?

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$\uparrow$

J

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$$\vdots$$

J

S

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$\vdots$

$\uparrow$

J

AVQC $\mathfrak{I} = \{\mathbf{n}_1, \mathbf{n}_2, \mathbf{n}_3\}$ with randomness, 10 channel uses

S

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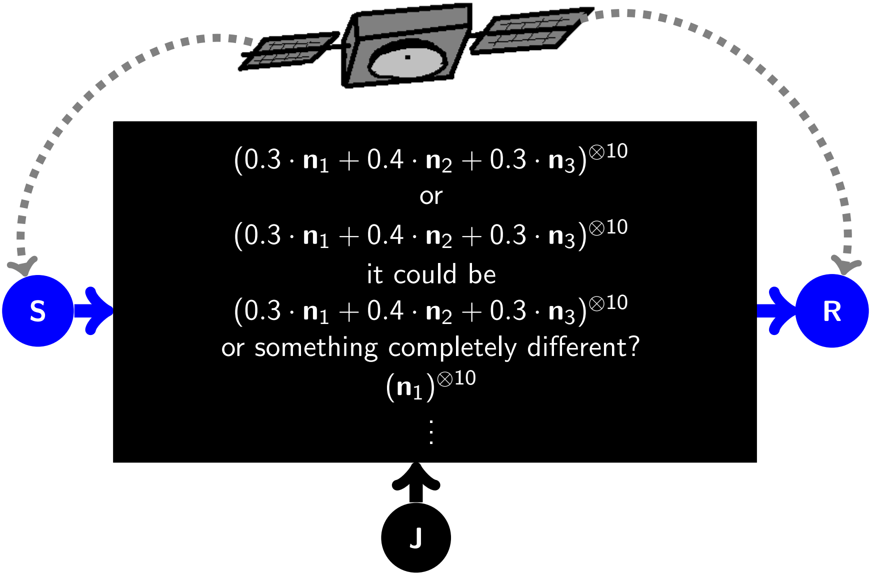
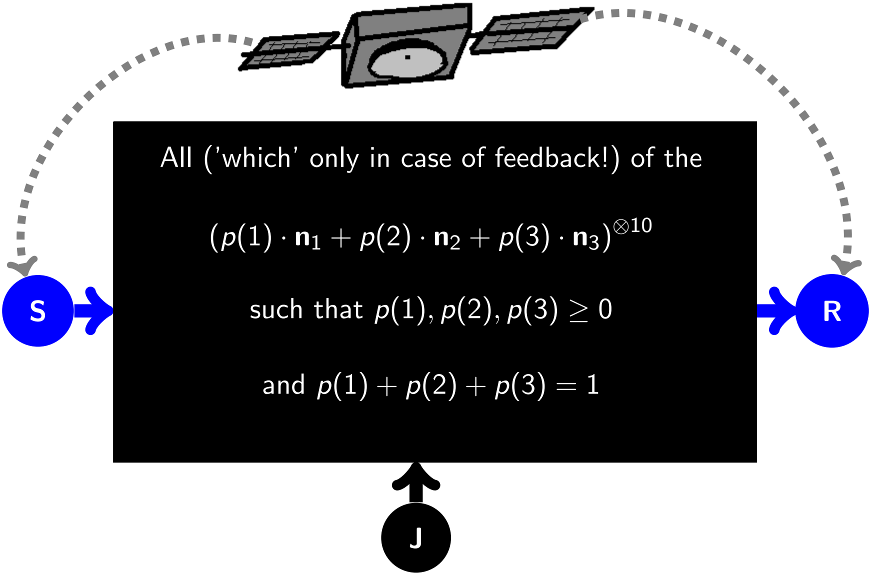


Shared randomness: AVQC → compound channel



Basics

- The quantum systems $\mathcal{H}, \mathcal{K}$ under consideration are modeled on finite dimensional Hilbert spaces labelled by the same letters $\mathcal{H}, \mathcal{K}$.
- Quantum channels from $\mathcal{H}$ to $\mathcal{K}$ are modeled by completely positive trace preserving maps. The set of all channels from $\mathcal{H}$ to $\mathcal{K}$ is written $CPTPM(\mathcal{H}, \mathcal{K})$.
- If the input system is a finite set $\mathbf{X}$ instead of a quantum system, the channel is called a 'cq-channel'. The set of all channels with input set $\mathbf{X}$ and output system $\mathcal{H}$ is labeled $CQ(\mathbf{X}, \mathcal{H})$.

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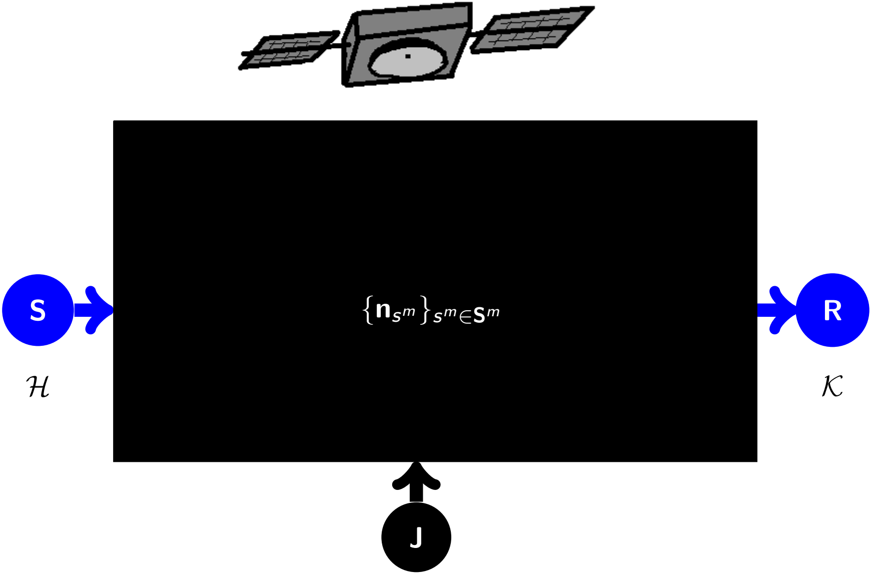
- Throughout, $\mathfrak{I}$ denotes a finite subset: $\mathfrak{I} \subset CPTPM(\mathcal{H}, \mathcal{K})$.
- Throughout, $\mathfrak{I} = \{\mathbf{n}_s\}_{s \in \mathbf{S}}$ also denotes the arbitrarily varying quantum channel which is generated by it: $(\{\mathbf{n}_{s^m}\}_{s^m \in \in \mathbf{S}^m})_{m \in \mathbb{N}}$.
- For m channel uses, the possible channels are $\{\mathbf{n}_{s^m}\}_{s^m \in \mathbf{S}^m}$, where

$$\mathbf{S}^m := \mathbf{S} \times \ldots \times \mathbf{S}, \qquad \mathbf{n}_{s^m} := \mathbf{n}_{s_1} \otimes \ldots \otimes \mathbf{n}_{s_m}$$

AVQC, no shared randomness, m channel uses



Message transmission under channel uncertainty

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- Sender S has message set $[M] = \{1, \dots, M\}$. He wants to send them to receiver R. Transmission takes place over one of the channels from the set $\{\mathbf{n}_s\}_{s \in \mathbf{S}} \subset CPTPM(\mathcal{H}, \mathcal{K})$. Neither sender nor receiver knows the index $s \in \mathbf{S}$.
- S uses the encoding $\mathcal{P} \in CQ([M], \mathcal{H})$, R tries to guess the message - he measures the output system with a POVM $\mathbf{D} \in \mathcal{M}_M$:

$$\mathcal{M}_M := \{(D(1), \dots, D(M)) : D(k) \geq 0 \ \forall k, \ \sum_k D(k) = \mathbb{1}_{\mathcal{K}}\}.$$

- Probability of (perhaps wrongly) guessing k when k' was sent over channel $\mathbf{n}_s$:

$$\mathrm{tr}\{ \ D(k) \cdot \mathbf{n}_s(\mathcal{P}(k')) \ \}.$$

- Measure of successful transmission:

$$\min_{s \in \mathbf{S}} \frac{1}{M} \sum_k \mathrm{tr}\{D(k)\mathbf{n}_s(\mathcal{P}(k))\} \ \in [0, 1].$$

Entropic quantities

Asymptotic system performance is described by entropic quantities:

Definition (Von Neumann Entropy)

Let $\rho \in \mathcal{S}(\mathcal{H})$ be a state. Its von Neumann entropy is

$$S(\rho) := -\mathrm{tr}\,\rho \log \rho$$

Definition (Ensemble)

Any finite alphabet $\mathcal{X}$, probability distribution p on $\mathcal{X}$ and set $\{\rho_x\}_{x \in \mathcal{X}} \subset \mathcal{S}(\mathcal{H})$ defines an ensemble $E := \{p(x), \rho_x\}_{x \in \mathcal{X}}$

Definition (Holevo Quantity)

Let $\mathbf{n}$ be a channel and $E = \{p(x), \rho_x\}_{x \in \mathcal{X}}$ an ensemble. Then

$$\chi(E, \mathbf{n}) := S(\mathbf{n}(\rho)) - \sum_x p(x) S(\mathbf{n}(\rho_x)),$$

Definition: Message transmission capacities

- Let $m \in \mathbb{N}$. An (m, M_m) random code for message transmission over $\mathfrak{I}$ is a probability distribution γ_m on a finite subset $\{\mathcal{P}_i\}_{i=1}^{\Gamma_m} \times \{\mathbf{D}_j\}_{j=1}^{\Gamma_m}$ of $CQ([M_m], \mathcal{H}^{\otimes m}) \times \mathcal{M}_{M_m}$.
- $R \geq 0$ is called achievable with random codes if there exists a sequence $(\gamma_m)_{m\in\mathbb{N}}$ of random codes satisfying both

$$1) \quad \liminf_{m\to\infty} \min_{s^m \in \mathbf{S}^m} \sum_{i,j=1}^{|\Gamma_m|} \gamma_m(i,j) \frac{1}{M_m} \sum_{k=1}^{M_m} \mathrm{tr}\{D_j(k) \mathbf{n}_{s^m}(\mathcal{P}_i(k))\} = 1$$

$$2) \quad \liminf_{m\to\infty} \frac{1}{m} \log M_m \geq R.$$

- The corresponding random message transmission capacity is

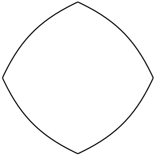
$$\overline{C}_{\mathrm{ran}}(\mathfrak{I}) := \sup\{R : R \text{ is achievable with random codes}\}$$

- The capacity without randomness is

$$\overline{C}_{\mathrm{det}}(\mathfrak{I}) := \sup\left\{ R : \begin{array}{l} \text{R is achievable with random codes} \\ \text{such that } |\Gamma_m| = 1 \ \forall m \in \mathbb{N} \end{array} \right\}$$

Old results

For every finite $\mathfrak{I} = \{\mathbf{n}_s\}_{s\in\mathbf{S}}$, define
$conv(\mathfrak{I}) := \{\mathbf{n}_p = \sum_{s\in\mathbf{S}} p(s)\mathbf{n}_s |\ p \in \mathfrak{P}(\mathbf{S})\}$.



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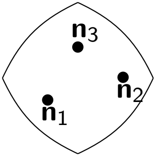

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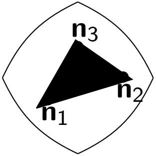

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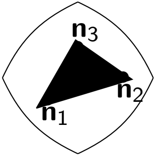

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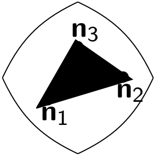


Theorem (Dichotomy for Message Transmission)

For every finite AVQC $\mathfrak{J}$ we have

1. $\overline{C}_{\mathrm{ran}}(\mathfrak{J}) = \lim_{m\to\infty} \frac{1}{m} \max_E \min_{\mathbf{n}\in conv(\mathfrak{J})} \chi(E, \mathbf{n}^{\otimes m})$
2. *Either* $\overline{C}_{\mathrm{det}}(\mathfrak{J}) = 0$ *or else* $\overline{C}_{\mathrm{det}}(\mathfrak{J}) = \overline{C}_{\mathrm{ran}}(\mathfrak{J})$.

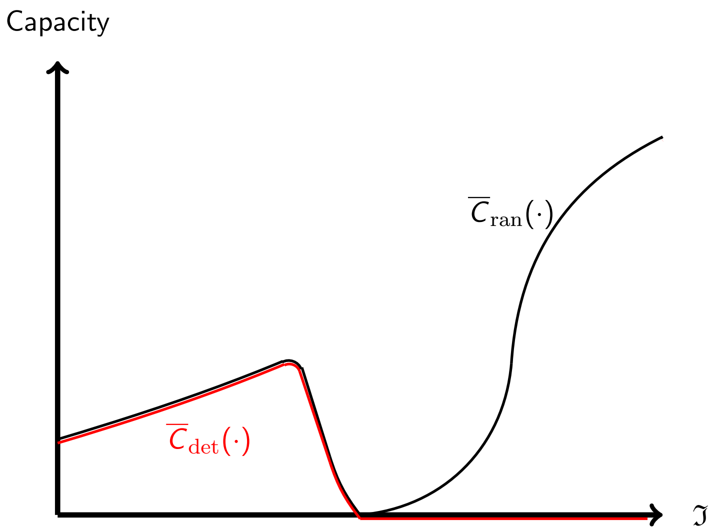
Theorem ([ABBN13])

A finite AVQC $\{\mathbf{n}_s\}_{s\in\mathbf{S}}$ has $\overline{C}_{\mathrm{det}}(\mathfrak{J}) = 0$ if and only if it satisfies for all $m \in \mathbb{N}$: For all $\rho, \sigma \in \mathcal{S}(\mathcal{H}^{\otimes m})$ there is $p, q \in \mathfrak{P}(\mathbf{S}^m)$ such that

$$\sum_{s^m\in\mathbf{S}^m} p(s^m)\mathbf{n}_{s^m}(\rho) = \sum_{s^m\in\mathbf{S}^m} q(s^m)\mathbf{n}_{s^m}(\sigma) \qquad (*)$$

$(*)$ An AVQC with this property is called 'm-symmetrizable'

New Result I: Randomness helps!



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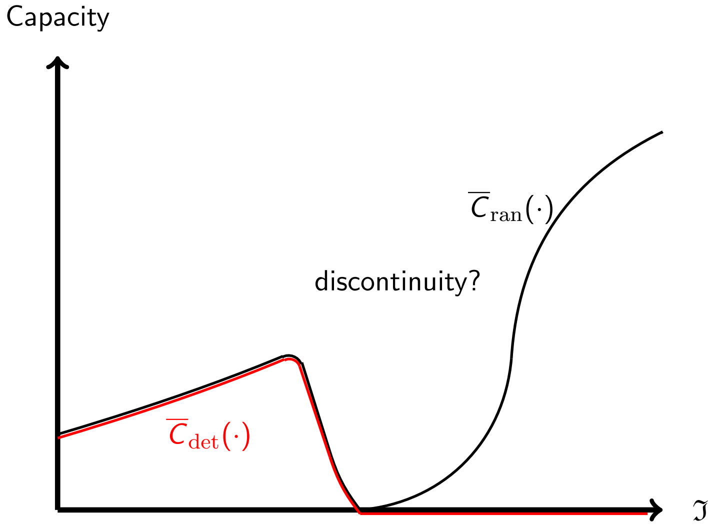

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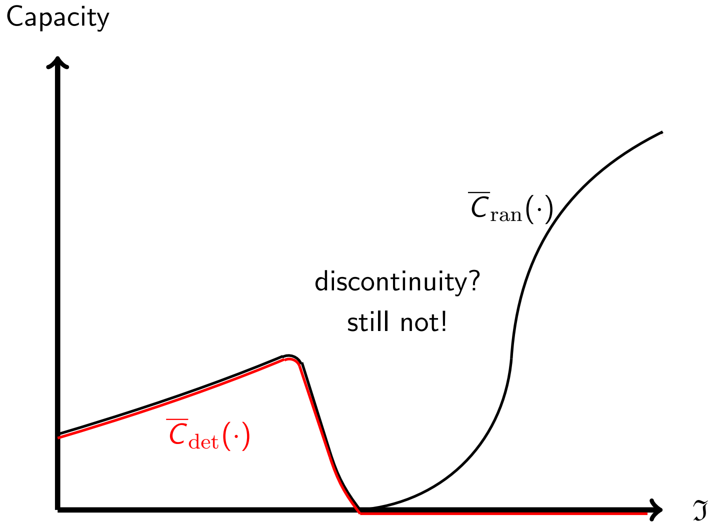


New result I: Randomness helps!

Theorem

❶ *Let $\mathfrak{I}$ consist of entanglement breaking channels that have the special form $\mathbf{n}_s(\rho) := \sum_{x\in\mathbf{X}} \mathrm{tr}\{\rho M_x\}\rho_{s,x}$, $s \in \mathbf{S}$, for some finite set $\mathbf{S}$ and POVM $\{M_i\}_{i=1}^{M}$ on $\mathcal{H}$. The following is true:*

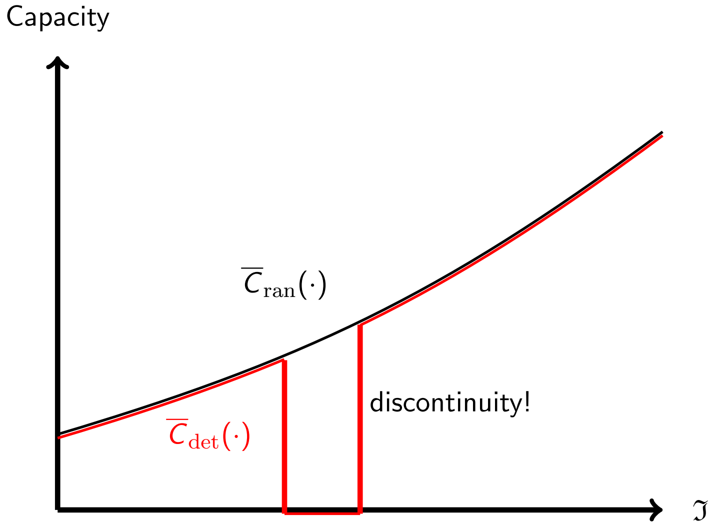
If there are probability distributions $\{p_x\}_{x\in\mathbf{X}} \subset \mathfrak{P}(\mathbf{S})$ such that

$$\sum_{s\in\mathbf{S}} p_{x'}(s)\rho_{s,x} = \sum_{s\in\mathbf{S}} p_x(s)\rho_{s,x'} \qquad \forall x, x' \in \mathbf{X},$$

then it holds $\overline{C}_{\mathrm{det}}(\mathfrak{I}) = 0$.

❷ *There exists an example of an AVQC satisfying the above conditions which additionally has the property $\overline{C}_{\mathrm{ran}}(\mathfrak{I}) > 0$.*

New result II: Discontinuity



New result II: $\overline{C}_{\mathrm{det}}$ is discontinuous

Theorem (Discontinuity of $\overline{C}_{\mathrm{det}}$)

The function $\overline{C}_{\mathrm{det}}$ is discontinuous on $\{\mathfrak{I} \subset \mathcal{C}(\mathcal{H}, \mathcal{K}) : |\mathfrak{I}| < \infty\}$.

New result III: The property $C_{\mathrm{det}}(\mathfrak{I}) > 0$ is stable

(the set where $C_{\mathrm{det}} > 0$ is open)

Theorem (Positivity of $\overline{C}_{\mathrm{det}}$ is stable)

Let $\mathfrak{I}$ be a finite AVQC satisfying $\overline{C}_{\mathrm{det}}(\mathfrak{I}) > 0$. There exists $\delta_0 > 0$ such that for all finite AVQCs $\mathfrak{I}'$ satisfying $D_{\Diamond}(\mathfrak{I}, \mathfrak{I}') \leq \delta_0$ it holds $\overline{C}_{\mathrm{det}}(\mathfrak{I}') > 0$.

($D_{\Diamond}$ denotes the Hausdorff-distance induced by the diamond norm)

Randomness helps: Idea of proof

$$\mathcal{H} \xrightarrow[\text{(entanglement breaking)}]{\text{measurement}} \mathbf{X} \xrightarrow[\text{classical-quantum channel}]{\text{arbitrarily varying}} \mathcal{K}$$

- Entanglement breaking channel $\Rightarrow$ one-shot random capacity
- Entangled signal states from $\mathcal{H}$ $\rightarrow$ random encoding for *cq* part
- Message transmission capacity of an AVcqC does not benefit from randomization at the encoder (proof: straightforward as in [AB07])
- It follows $\overline{C}_{\mathrm{det}}(\mathfrak{I}) = 0$ (by symmetrizability of the cq-part).
- To prove that $\overline{C}_{\mathrm{ran}}(\mathfrak{I}) > 0$: Take fixed product state encoding. Use results of [AB07] again $\Rightarrow$ for specific choice $\rho_{1,1} = |e_1\rangle\langle e_1|$, $\rho_{1,2} = \rho_{2,1} = |e_3\rangle\langle e_3|$, $\rho_{2,2} = |e_2\rangle\langle e_2|$, $\mathbf{X} = \mathbf{S} = \{1, 2\}$, $M_1 = |e_1\rangle\langle e_1|$, $M_2 = |e_2,\rangle\langle e_2|$:

$$\overline{C}_{\mathrm{ran}}(\mathfrak{I}) \geq \min_{\mathbf{n} \in \mathrm{conv}(\mathfrak{I})} \chi(\{\tfrac{1}{2}, \{|e_i\rangle\langle e_i\}_{i=1}^{2}\}, \mathbf{n}) \geq 1/2$$

Discontinuity: Idea of proof

- Take the same channel $\mathfrak{I}$ as before. Augment it by a tiny bit of 'identity': $Id(X) = X$ embeds the matrices from $\mathbb{C}^2$ into those on $\mathbb{C}^3$.

$$\mathbf{n}_{s,\lambda} := (1-\lambda)Id + \lambda \mathbf{n}_s, \qquad \mathfrak{I}_\lambda := \{\mathbf{n}_{s,\lambda}\}_{s \in \mathbf{S}}$$

- Obviously, $\lim_{\lambda \to 1} D_\Diamond(\mathfrak{I}_\lambda, \mathfrak{I}) = 0$
- We know that $\mathfrak{I}_1 = \{\mathbf{n}_{s,\lambda}\}_{s \in \mathbf{S}}$ satisfies $C_{\mathrm{det}}(\mathfrak{I}_1) = 0$.
- It is easy to show that $\mathfrak{I}_\lambda$ is non-symmetrizable for all $\lambda \in [0, 1)$
- It follows $\overline{C}_{\mathrm{ran}}(\mathfrak{I}_\lambda) = C_{\mathrm{det}}(\mathfrak{I}_\lambda)$ for all $\lambda \in [0, 1)$ (here, one uses the dichotomy-result!)
- But $C_{\mathrm{ran}}(\mathfrak{I}_1) \geq 1/2$, whence $C_{\mathrm{ran}}(\mathfrak{I}_\lambda) \geq 1/4$ for $\lambda \approx 1$
- Thus $C_{\mathrm{det}}(\mathfrak{I}_\lambda) \geq 1/4$ for $\lambda \approx 1$, but $C_{\mathrm{det}}(\mathfrak{I}_1) = 0$

Stability: Idea of proof

- For every $m \in \mathbb{N}$ define, on finite sets $\{\mathbf{n}_i\}_{i\in\mathcal{I}} \subset \mathcal{C}(\mathcal{H}^{\otimes m}, \mathcal{K}^{\otimes m})$ of channels, a function F_m through

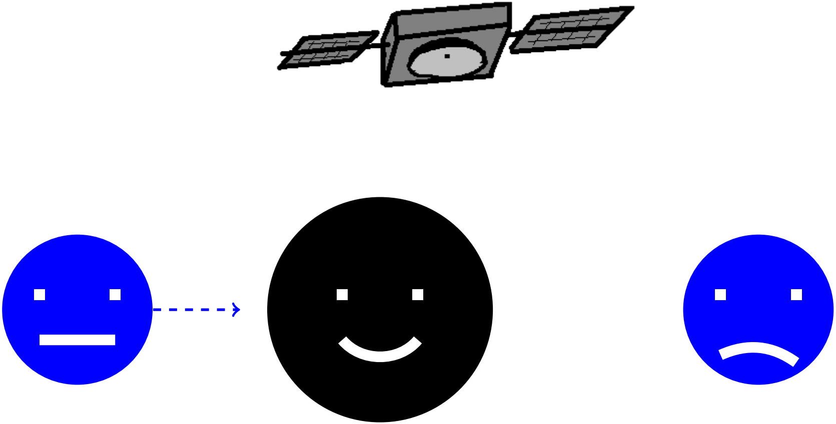
$$\{\mathbf{n}_i\}_{i\in\mathcal{I}} \mapsto \max_{\rho,\sigma\in\mathcal{S}(\mathcal{H}^{\otimes m})} \min_{q,p\in\mathfrak{P}(\mathcal{I})} \|\sum_{i\in\mathcal{I}}(p(i)\mathbf{n}_i(\rho) - q(i)\mathbf{n}_i(\sigma))\|_1.$$

- It holds $\overline{C}_{\mathrm{det}}(\mathfrak{I}) > 0 \Leftrightarrow \exists\ m \in \mathbb{N}: \ F_m(\mathfrak{I}) > 0$.
  (Use the connection between symmetrizability and $\overline{C}_{det}$)

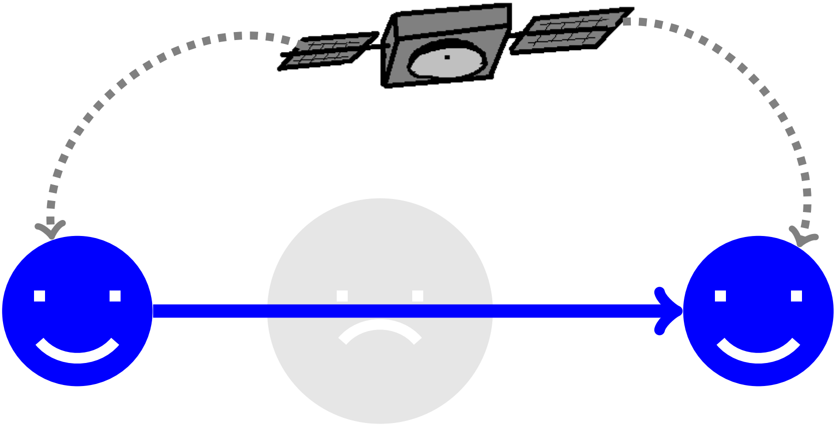
- Show that $F_m(\mathfrak{I}) \approx F_m(\mathfrak{I}')$ if $\mathfrak{I} \approx \mathfrak{I}'$.

- If $C_{\mathrm{det}}(\mathfrak{I}) > 0$ then there is $m \in \mathbb{N}$ such that $F_m(\mathfrak{I}) > 0$, but then choosing $\mathfrak{I}' \approx \mathfrak{I}$ ensures $F_m(\mathfrak{I}') > 0$, whence $C_{\mathrm{det}}(\mathfrak{I}') > 0$.

Conclusion



Shared randomness:
Can increase the capacity and stabilize the system



Finite error, finite resources

If instead of

$$\liminf_{m\to\infty} \min_{s^m\in\mathbf{S}^m} \sum_{i,j=1}^{|\Gamma_m|} \gamma_m(i,j)\frac{1}{M_m}\sum_{k=1}^{M_m} \mathrm{tr}\{D_j(k)\mathbf{n}_{s^m}(\mathcal{P}_i(k))\} = 1$$

on requires for some $\lambda \in (0,1)$ only

$$\liminf_{m\to\infty} \min_{s^m\in\mathbf{S}^m} \sum_{i,j=1}^{|\Gamma_m|} \gamma_m(i,j)\frac{1}{M_m}\sum_{k=1}^{M_m} \mathrm{tr}\{D_j(k)\mathbf{n}_{s^m}(\mathcal{P}_i(k))\} \geq 1-\lambda$$

then the corresponding capacity $\overline{C}_{\mathrm{ran}}(\mathfrak{I},\lambda)$ can be achieved using only a finite amount K of shared random bits, and K scales as $K \approx 1/\lambda$.
The number of channel uses needed to achieve an error smaller than λ scales as $\log(1/\lambda)$.

See our paper for an exact statement.

Entanglement fidelity (entanglement transmission)

- Sender $\mathfrak{S}$ has access to one part of pure entangled state $|\psi\rangle\langle\psi| \in \mathcal{S}(\mathcal{F} \otimes \mathcal{F})$.
- He wishes to transmit his half to receiver $\mathfrak{R}$ by use of the channel $\mathbf{n} \in CPTPM(\mathcal{H}, \mathcal{K})$.
- $\mathfrak{S}$ uses the encoding map $\mathcal{P} \in CPTPM(\mathcal{F}, \mathcal{H})$
- and $\mathfrak{R}$ the decoding map $\mathcal{R} \in CPTPM(\mathcal{K}, \mathcal{F})$.
- Measure of success:

$$\langle \psi, Id_{\mathcal{F}} \otimes \mathcal{R} \circ \mathbf{n} \circ \mathcal{P}(|\psi\rangle\langle\psi|)\psi\rangle =: F_e(\rho, \mathcal{R} \circ \mathbf{n} \circ \mathcal{P}) \in [0, 1],$$

where $\rho := Id_{\mathcal{F}} \otimes \mathrm{tr}_{\mathcal{F}}(|\psi\rangle\langle\psi|)$ (the marginal state).

Remark: For arbitrary (sub) spaces $\mathcal{F}$, $\pi_{\mathcal{F}}$ denotes the state supported only on $\mathcal{F}$ satisfying $\mathrm{spec}(\mathcal{F}) = \frac{1}{\dim \mathcal{F}}$.

Minimum fidelity (strong subspace transmission)

- Sender $\mathfrak{S}$ controls the system $\mathcal{F}$.
- He wants to make sure that he can send arbitrary states $\rho \in \mathcal{S}(\mathcal{F})$ to the receiver $\mathfrak{R}$ by use of a channel $\mathbf{n} \in CPTPM(\mathcal{H}, \mathcal{K})$.
- $\mathfrak{S}$ uses the encoding $\mathcal{P} \in CPTPM(\mathcal{F}, \mathcal{H})$
- and $\mathfrak{R}$ the decoding $\mathcal{R} \in CPTPM(\mathcal{K}, \mathcal{F})$.
- Measure describing how well the channel $\mathcal{R} \circ \mathbf{n} \circ \mathcal{P}$ preserves the states sent by $\mathfrak{S}$:

$$\min_{x \in S(\mathcal{F})} \langle x, \mathcal{R} \circ \mathbf{n} \circ \mathcal{P}(|x\rangle\langle x|)x\rangle =: F_{\min}(\mathcal{F}, \mathcal{R} \circ \mathbf{n} \circ \mathcal{P}) \ \in [0, 1].$$

Remark: $S(\mathcal{F}) = \{x \in \mathcal{F} : \|x\| = 1\}$ is the unit sphere on $\mathcal{F}$.

Coherent information

- Measure of how much noise is induced by $\mathbf{n}$: For arbitrary $\rho \in \mathcal{S}(\mathcal{H})$, $\mathbf{n} \in CPTPM(\mathcal{H}, \mathcal{K})$ it is given by

  $$I_c(\rho, \mathbf{n}) := S(\mathbf{n}(\rho)) - S(Id \otimes \mathbf{n}(|\psi\rangle\langle\psi|)).$$

  Here, and only for a brief moment, S denotes von Neumann Entropy.

Entanglement transmission

- A (deterministic) (m, k_m)-code for the AVQC $\mathfrak{I} = \{\mathbf{n}_s\}_{s\in\mathbf{S}}$ is a triple $(\mathcal{F}_m, \mathcal{P}^m, \mathcal{R}^m)$, where

$$\begin{aligned} \mathcal{F}_m \quad &- \quad \text{Hilbert space of dimension } \dim \mathcal{F}_m = k_m \\ \mathcal{P}^m \in CPTPM(\mathcal{F}_m, \mathcal{H}^{\otimes m}) \quad &- \quad \text{the encoding} \\ \mathcal{R}^m \in CPTPM(\mathcal{K}^{\otimes m}, \mathcal{F}_m) \quad &- \quad \text{the decoding} \end{aligned}$$

- $R \geq 0$ is a an achievable rate for entanglement transmission over the AVQC $\mathfrak{I}$ if there is a sequence of (m, k_m)-codes with

$$\liminf_{m\to\infty} \frac{1}{m} \log k_m \geq R,$$
$$\lim_{m\to\infty} \inf_{s^m\in\mathbf{S}^m} F_e(\pi_{\mathcal{F}_m}, \mathcal{R}^m \circ \mathbf{n}_{s^m} \circ \mathcal{P}^m) = 1.$$

- Entanglement transmission capacity $\mathcal{A}_{\text{det}}(\mathfrak{I})$ of $\mathfrak{I}$:

$$\mathcal{A}_{\text{det}}(\mathfrak{I}) := \sup \left\{ R : \begin{array}{l} R \text{ is achievable entanglement} \\ \text{transmission rate for } \mathfrak{I} \end{array} \right\}$$

Entanglement transmission using random codes

- A (random) (m, k_m)-random code for the AVQC $\mathfrak{I} = \{\mathbf{n}_s\}_{s \in \mathbf{S}}$ is a probability distribution μ on a finite set Γ together with a set of triples $(\mathcal{F}_m, \mathcal{P}^m_\gamma, \mathcal{R}^m_\gamma)$ for each $\gamma \in \Gamma$, where

$$\begin{aligned} \mathcal{F}_m \quad &- \quad \text{Hilbert space of dimension } \dim \mathcal{F}_m = k_m \\ \mathcal{P}^m_\gamma \in CPTPM(\mathcal{F}_m, \mathcal{H}^{\otimes m}) \quad &- \quad \text{the encoding} \\ \mathcal{R}^m_\gamma \in CPTPM(\mathcal{K}^{\otimes m}, \mathcal{F}_m) \quad &- \quad \text{the decoding} \end{aligned}$$

- $R \geq 0$ achievable: $\exists$ sequence of (m, k_m)-random codes with

$$\liminf_{m \to \infty} \frac{1}{m} \log k_m \geq R,$$

$$\lim_{m \to \infty} \inf_{s^m \in \mathbf{S}^m} \sum_{\gamma \in \Gamma} \mu(\gamma) F_e(\pi_{\mathcal{F}_m}, \mathcal{R}^m_\gamma \circ \mathbf{n}_{s^m} \circ \mathcal{P}^m_\gamma) = 1.$$

- Randomness-assisted entanglement transmission capacity of $\mathfrak{I}$:

$$\mathcal{A}_{\text{rand}}(\mathfrak{I}) := \sup \left\{ R : \begin{array}{l} R \text{ is achievable entanglement transm.} \\ \text{rate for } \mathfrak{I} \text{ (with random codes)} \end{array} \right\}$$

Strong subspace transmission

- A (deterministic) (m, k_m)-code for the AVQC $\mathfrak{I} = \{\mathbf{n}_s\}_{s \in \mathbf{S}}$ is a triple $(\mathcal{F}_m, \mathcal{P}^m, \mathcal{R}^m)$, where

$$\begin{aligned} \mathcal{F}_m &\ - \ \text{Hilbert space of dimension } \dim \mathcal{F}_m = k_m \\ \mathcal{P}^m &\in CPTPM(\mathcal{F}_m, \mathcal{H}^{\otimes m}) \ - \ \text{the encoding} \\ \mathcal{R}^m &\in CPTPM(\mathcal{K}^{\otimes m}, \mathcal{F}_m) \ - \ \text{the decoding} \end{aligned}$$

- $R \geq 0$ is a an achievable rate for strong subspace transmission over the AVQC $\mathfrak{I}$ if there is a sequence of (m, k_m)-codes with

$$\liminf_{m \to \infty} \frac{1}{m} \log k_m \geq R,$$
$$\lim_{m \to \infty} \inf_{s^m \in \mathbf{S}^m} F_{\min}(\mathcal{F}_m, \mathcal{R}^m \circ \mathbf{n}_{s^m} \circ \mathcal{P}^m) = 1.$$

- Strong subspace transmission capacity $\mathcal{A}_{s,\mathrm{det}}(\mathfrak{I})$ of $\mathfrak{I}$:

$$\mathcal{A}_{s,\mathrm{det}}(\mathfrak{I}) := \sup \left\{ R : \begin{array}{l} R \text{ is achievable strong subspace} \\ \text{transmission rate for } \mathfrak{I} \end{array} \right\}$$

An equivalence

Theorem

For every arbitrarily varying quantum channel defined through a subset $\mathfrak{I} \subset CPTPM(\mathcal{H}, \mathcal{K})$ it holds:

$$\mathcal{A}_{\mathsf{s},\mathrm{det}}(\mathfrak{I}) = \mathcal{A}_{\mathrm{det}}(\mathfrak{I}), \qquad \mathcal{A}_{\mathsf{s},\mathrm{rand}}(\mathfrak{I}) = \mathcal{A}_{\mathrm{rand}}(\mathfrak{I})!$$

Comparison to classical quantities

Strong subspace transmission is considered an analogue to the maximal error[1] criterion,

entanglement transmission as the analogue to the average error[2] criterion.

For classical arbitrarily varying channels, the capacities for message transmission under average -and maximal error probability criterion are NOT identical!
(There, at least one has to explicitly assume that randomized encoding schemes are used in order to get identical capacities)

[1] average error of a code is given by $$\frac{1}{M}\sum_{i=1}^{M}\mathrm{tr}\{D_i\mathbf{n}(\rho_i)\},$$

[2] maximal error of a code is given by $$\max_{1\le i\le M}\mathrm{tr}\{D_i\mathbf{n}(\rho_i)\}.$$

Why THAT analogue?

- For every $\mathcal{M} \in CPTPM(\mathcal{F}_m, \mathcal{F}_m)$, $\dim(\mathcal{F}_m)$ arbitrary, it holds

$$1 - \min_{\rho \in \mathcal{S}(\mathcal{F}_m)} F_e(\rho, \mathcal{M}) \leq 4\sqrt{1 - F_{\min}(\mathcal{F}_m, \mathcal{M})} \leq 4\sqrt{\|\mathcal{M} - id_{\mathcal{F}_m}\|_\infty}$$

$$\leq 4\sqrt{\|\mathcal{M} - id_{\mathcal{F}_m}\|_{cb}} \leq 8(1 - \min_{\rho \in \mathcal{S}(\mathcal{F}_m)} F_e(\rho, \mathcal{M}))^{1/4}$$

[KW04]

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$$\leq 4\sqrt{\|\mathcal{M} - id_{\mathcal{F}_m}\|_{cb}} \leq 8(1 - \min_{\rho \in \mathcal{S}(\mathcal{F}_m)} F_e(\rho, \mathcal{M}))^{1/4}$$

[KW04]

- For the normalized Haar measure μ on $S(\mathcal{F}_m)$:

$$\int \langle x, \mathcal{M}(|x\rangle\langle x|)x\rangle d\mu(x) = \frac{\dim(\mathcal{F}_m) \cdot F_e(\pi_{\mathcal{F}_m}, \mathcal{M}) + 1}{\dim(\mathcal{F}_m) + 1}.$$

[HHH99, N02]

Why THAT analogue?

- For every $\mathcal{M} \in CPTPM(\mathcal{F}_m, \mathcal{F}_m)$, $\dim(\mathcal{F}_m)$ arbitrary, it holds

$$1 - \min_{\rho \in \mathcal{S}(\mathcal{F}_m)} F_e(\rho, \mathcal{M}) \leq 4\sqrt{1 - F_{\min}(\mathcal{F}_m, \mathcal{M})} \leq 4\sqrt{\|\mathcal{M} - id_{\mathcal{F}_m}\|_\infty}$$
$$\leq 4\sqrt{\|\mathcal{M} - id_{\mathcal{F}_m}\|_{cb}} \leq 8(1 - \min_{\rho \in \mathcal{S}(\mathcal{F}_m)} F_e(\rho, \mathcal{M}))^{1/4}$$

[KW04]

- For the normalized Haar measure μ on $S(\mathcal{F}_m)$:

$$\int \langle x, \mathcal{M}(|x\rangle\langle x|) x\rangle d\mu(x) = \frac{\dim(\mathcal{F}_m) \cdot F_e(\pi_{\mathcal{F}_m}, \mathcal{M}) + 1}{\dim(\mathcal{F}_m) + 1}.$$

[HHH99, N02]

- And *no* function $f : [0,1] \to [0,1]$ with $\lim_{x \to 1} f(x) = 0$ satisfies

$$\|\mathcal{M} - id_{\mathcal{F}_m}\|_\infty \leq f(F_e(\pi_{\mathcal{F}_m}, \mathcal{M})) \qquad \forall\ \mathcal{M}$$

[KW04]

Quantum-Ahlswede dichotomy for AVQCs

Theorem (Quantum - Ahlswede[3] dichotomy (proven in [ABBN13]))

For the AVQC defined by $\mathfrak{I} = \{\mathbf{n}_s\}_{s\in\mathbf{S}} \subset CPTPM(\mathcal{H},\mathcal{K})$:

$$\mathcal{A}_{\text{random}}(\mathfrak{I}) = \lim_{m\to\infty}\frac{1}{m}\max_{\rho\in\mathcal{S}(\mathcal{H}^{\otimes m})}\inf_{\mathbf{n}\in conv(\mathfrak{I})} I_c(\rho,\mathbf{n}^{\otimes m})$$

Either $C_{\text{det}}(\mathfrak{I}) = 0$ *or* $\mathcal{A}_{\text{det}}(\mathfrak{I}) = \mathcal{A}_{\text{random}}(\mathfrak{I})$.

$$conv(\mathfrak{I}) = \left\{\mathbf{n}_{\mathfrak{q}} \,\middle|\, \mathbf{n}_{\mathfrak{q}} = \sum_{s\in\mathbf{S}'}\mathfrak{q}(s)\mathbf{n}_s,\ \mathfrak{q}\in\mathfrak{P}(\mathbf{S}'),\ \mathbf{S}'\subset\mathbf{S},\ |\mathbf{S}'|<\infty\right\}.$$

$\mathfrak{P}(\mathbf{S}')$ - set of probability distributions on $\mathbf{S}'$.

[3] Find its ancestor, the classical Ahlswede-Dichotomie, in [Ahl78]

Conjectures

Conjecture ([ABBN13, BN13])

First, there exist AVQVs $\mathfrak{I}$ *for which*

$$\overline{C}_{\mathrm{random}}(\mathfrak{I}) > \overline{C}_{\mathrm{det}}(\mathfrak{I}).$$

Second, for every AVQC $\mathfrak{I}$ *it holds*

$$\mathcal{A}_{\mathrm{random}}(\mathfrak{I}) = \mathcal{A}_{\mathrm{det}}(\mathfrak{I}).$$

- First conjecture: Solved in [BN–]
- Second conjecture: Still open.

Conclusion

- Message transmission: Shared randomness assisted capacity is continuous
- Message transmission: Shared randomness assisted capacity can be strictly larger than unassisted capacity
- Message transmission: $\overline{C}_{\mathrm{det}}$ is discontinuous
- Entanglement transmission is equivalent to strong subspace transmission
- If our conjecture turns out to be true then the unassisted entanglement transmission capacity is continuous

THANK YOU

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