On the Approximability of the Hilbert Transform

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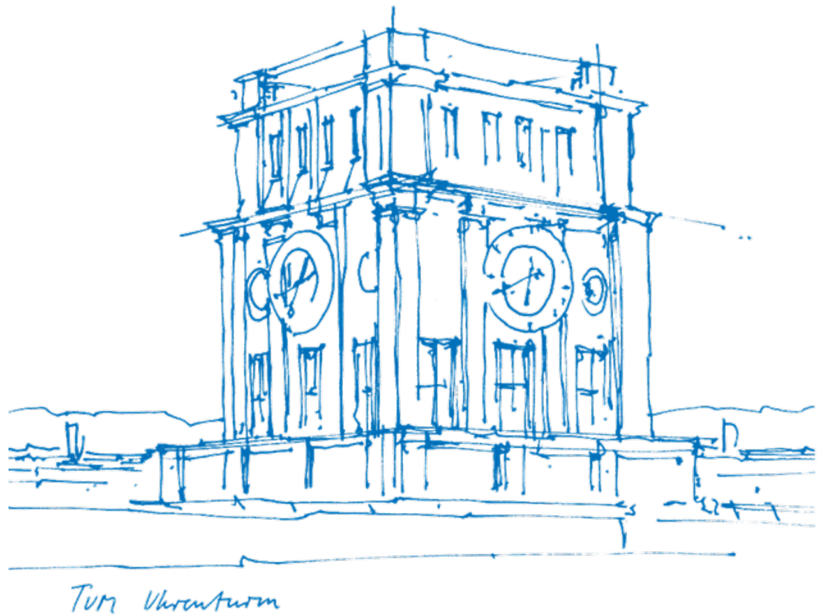

Introduction – The Hilbert Transform

- We consider continuous functions f on $\mathbb{T} = [-\pi, \pi]$ with $f(-\pi) = f(\pi)$
- Assume f can be represented by its Fourier series

$$f(t) = \sum_{n=-\infty}^{\infty} c_n(f)\,\mathrm{e}^{\mathrm{i}nt} \qquad \text{with Fourier coefficients} \qquad c_n(f) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\tau)\,\mathrm{e}^{-\mathrm{i}n\tau}\,\mathrm{d}\tau$$

- Its harmonic *conjugate* $\widetilde{f}$ is given by

$$\widetilde{f}(t) = (\mathrm{H}f)(t) = -\mathrm{i}\sum_{n=-\infty}^{\infty} \operatorname{sgn}(n)\,c_n(f)\,\mathrm{e}^{\mathrm{i}nt} \qquad \text{with} \qquad \operatorname{sgn}(n) = \begin{cases} -1 & : \ n < 0 \\ 0 & : \ n = 0 \\ 1 & : \ n > 0 \end{cases}$$

  such that

$$f(t) + \mathrm{i}\widetilde{f}(t) = c_0(f) + 2\sum_{n=1}^{\infty} c_n(f)\,\mathrm{e}^{\mathrm{i}nt}\,.$$

- The transformation $\mathrm{H} : f \mapsto \widetilde{f}$ is known as Hilbert transform.

$$\widetilde{f}(t) = (\mathrm{H}f)(t) = \lim_{\varepsilon\to 0}\frac{1}{2\pi}\int_{\varepsilon\le|t-\tau|\le\pi} \frac{f(\tau)}{\tan([t-\tau]/2)}\,\mathrm{d}\tau\,, \qquad t \in [-\pi,\pi)\,. \tag{HT}$$

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Hilbert Transform – Importance and Properties

- In physics H is known as Kramers–Kronig relation.
- It is related to causality:
  - The real and imaginary part of a causal signal is related by the Hilbert transform.
  - The phase of a causal signal is determined by its amplitude – phase retrieval.
  - Prediction and estimation of stationary time series – spectral factorization.
- Analytic tool in information theory
  - Broadband quantum channels[1] – here the Hilbert transform defines the complex structure of the quantum mechanical system corresponding to the quantum Gaussian channel.
  - Hilbert transform techniques were used in the elementary solution of the Kadison–Singer problem[2]

Properties

- Hilbert transform is bounded mapping $\mathrm{H} : L^p(\mathbb{T}) \to L^p(\mathbb{T})$, $1 < p < \infty$.
- The Hilbert transform is a bounded mapping $\mathrm{H} : L^\infty(\mathbb{T}) \to BMO$.
- $\mathrm{H} : \mathscr{C}(\mathbb{T}) \to \mathscr{C}(\mathbb{T})$ is not bounded but $\mathrm{H} : \mathscr{C}^\alpha(\mathbb{T}) \to \mathscr{C}(\mathbb{T})$ is bounded
- For $f \in \mathscr{C}(\mathbb{T})$, we have $\widetilde{f} = \mathrm{H}f \in L^p(\mathbb{T})$ for every $1 \leq p < \infty$ but $\widetilde{f} = \mathrm{H}f \notin \mathscr{C}(\mathbb{T})$, in general.

[1] A. S. Holevo, "The classical capacity of quantum Gaussian gauge–covariant channels: Beyond i.i.d," *IEEE Inf. Theory Soc. Newsletter*, vol. 66, no. 4, Dec. 2016.

[2] D. A. Singer, "The solution of the Kadison–Singer Problem," *Minerva Lectures*, Princeton University, March 2016.

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Example – Causal Linear Systems

▷ Input-output relation of a linear system S

$$y_n = \sum_{k=0}^{\infty} c_k x_{n-k}\,, \qquad n \in \mathbb{Z}$$

Input sequence	:	$\{x_n\}_{n\in\mathbb{Z}} \subset \mathbb{C}$
Output sequence	:	$\{y_n\}_{n\in\mathbb{Z}} \subset \mathbb{C}$
Impulse response of S	:	$\{c_n\}_{n\in\mathbb{Z}} \subset \mathbb{C}$

▷ $c_k = 0$ for all $k < 0 \Rightarrow$ causal system

▷ Take discrete-time Fourier transform (DTFT) of the input-output relation yields

$$Y(\omega) = C(\omega)\, X(\omega)\,, \qquad \omega \in [-\pi, \pi)$$

with the *transfer function* $C(\omega)$ of S

$$C(\omega) = \sum_{k=0}^{\infty} c_k \, \mathrm{e}^{\mathrm{i}k\omega} \quad = \quad \Re\,[C(\omega)] + \mathrm{i}\Im\,[C(\omega)] = |C(\omega)|\, \mathrm{e}^{\mathrm{i}\arg[C(\omega)]}$$

▷ Because S is causal, we have

$$\Im\,[C(\omega)] = \mathrm{H}(\Re\,[C(\omega)]) \qquad \text{and} \qquad \arg\,[C(\omega)] = \mathrm{H}(\log|C(\omega)|)\,.$$

– So $C(\omega)$ is already uniquely determined by its real part $\Re\,[C(\omega)]$ or by its amplitude $|C(\omega)|$.
– The corresponding imaginary part or phase can be calculated using the Hilbert transform.

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Problem Statement

Given a subset $\mathscr{B} \subset \mathscr{C}(\mathbb{T})$ of continuous functions on $\mathbb{T}$

▷ Does there exists an algorithm which is able to calculate the Hilbert transform

$$\widetilde{f}(t) = (\mathrm{H}f)(t) = \lim_{\varepsilon\to 0} \frac{1}{2\pi} \int_{\varepsilon \leq |t-\tau| \leq \pi} \frac{f(\tau)}{\tan([t-\tau]/2)} \mathrm{d}\tau \,, \qquad t \in [-\pi,\pi)\,. \tag{HT}$$

on a digital computer for every $f \in \mathscr{B}$?

▷ Is it possible to characterize subspaces $\mathscr{B}$ for which such algorithm do exist and for which they do not exist?

- digital computer $\Rightarrow$ the calculation of $(\mathrm{H}f)(t)$ is based on only finitely many samples $\{f(t_n)\}_{n=1}^{N}$ of the given function f at a certain sampling set $T_N = \{t_n\}_{n=1}^{N} \subset \mathbb{T}$.
- Then only an approximation $\mathrm{H}_N : \{f(t_n)\}_{n=1}^{N} \mapsto \widetilde{f}_N$ of the Hilbert transform $\widetilde{f} = \mathrm{H}f$ can be determined.
- Problem: Design a sequence $\{\mathrm{H}_N\}_{N=1}^{\infty}$ of operators H_N (each H_N is concentrated on T_N) such that

$$\lim_{N\to\infty} \left\| \mathrm{H}_N f - \mathrm{H}f \right\|_{\infty} = \lim_{N\to\infty} \max_{t\in[-\pi,\pi)} \left| (\mathrm{H}_N f)(t) - (\mathrm{H}f)(t) \right| = 0 \qquad \text{for all } f \in \mathscr{B}\,.$$

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Outline of the Paper

1. We introduce a scale of Banach space $\{\mathscr{B}_\beta\}_{\beta\geq 0}$ of continuous functions of finite energy.
   - These are „good“ for the Hilbert transform.
   - The parameter $\beta \geq 0$ characterizes the energy concentration of the signals.
2. We introduce a class of sampling based Hilbert transform approximations $\{\mathrm{H}_N\}_{N\in\mathbb{N}}$.
   - This class is characterizes by three simple axioms.
   - This class contains basically all practically relevant Hilbert transform approximation methods.
3. Divergence results for the spaces $\mathscr{B}_\beta$ with $\beta \leq 1$.
   - For these spaces, there exists no Hilbert transform approximation in our class.
4. Convergence results for spaces $\mathscr{B}_\beta$ with $\beta > 1$.
   - For these spaces, there always exist a Hilbert transform approximation in our class.
   - Simple examples of convergent methods can be found.

A Family of Sobolev-like Signal Spaces of Finite Energy

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Signal Spaces of Finite Energy

▷ Let $f \in \mathscr{C}(\mathbb{T})$ with *Fourier series* representation

$$f(t) = \frac{a_0(f)}{2} + \sum_{n=1}^{\infty} a_n(f)\cos(nt) + b_n(f)\sin(nt) \qquad \text{with} \qquad \begin{array}{l} a_n(f) = \frac{1}{\pi}\int_{\mathbb{T}} f(\tau)\cos(n\tau)\,\mathrm{d}\tau \\ b_n(f) = \frac{1}{\pi}\int_{\mathbb{T}} f(\tau)\sin(n\tau)\,\mathrm{d}\tau \end{array} .$$

▷ For any $\beta \geq 0$ we define a seminorm on $\mathscr{C}(\mathbb{T})$

$$\|f\|_\beta = \left(\sum_{n=1}^{\infty} n(1+\log n)^\beta \left[|a_n(f)|^2 + |b_n(f)|^2\right]\right)^{1/2} .$$

▷ Therewith, we define a family of Sobolev-like Banach spaces $\{\mathscr{B}_\beta\}_{\beta\geq 0}$ by

$$\mathscr{B}_\beta = \left\{ f \in \mathscr{C}(\mathbb{T}) \,:\, \widetilde{f} \in \mathscr{C}(\mathbb{T}) \text{ and } \|f\|_\beta < \infty \right\}$$

and equip it with the norm

$$\|f\|_{\mathscr{B}_\beta} = \max(\|f\|_\infty, \|\widetilde{f}\|_\infty, \|f\|_\beta) .$$

Remarks

- $\beta \geq 0$ characterizes the smoothness of the functions $f \in \mathscr{B}_\beta$: As larger β as smoother f.

$$\mathscr{B}_{\beta'} \hookrightarrow \mathscr{B}_\beta \hookrightarrow \mathscr{B}_0 \hookrightarrow \mathscr{C}(\mathbb{T}) \qquad \text{for all} \qquad \beta' \geq \beta \geq 0 .$$

- $\|\cdot\|_0$ corresponds to the norm in Sobolev space $H^{1/2}(\mathbb{T}) = W^{1/2,2}(\mathbb{T})$
- $\|f\|_0$ is the (Dirichlet) energy of f.
- The Hilbert transform is well defined and bounded on $\mathscr{B}_\beta$: $\|\mathrm{H}\|_{\mathscr{B}_\beta \to \mathscr{B}_\beta} = 1$.

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Spaces of Smooth Functions: Motivation

For sufficiently smooth functions, there are standard procedures to obtain the desired sequences $\{\mathrm{H}_N\}_{N\in\mathbb{N}}$ of Hilbert transform approximations:

- ▷ Assume f belongs to a Sobolev space $H^s(\mathbb{T}) = W^{s,\frac{1}{2}}(\mathbb{T})$ with $s > 1/2$.
- ▷ Sobolev embedding shows that f is Hölder continuous of index $0 < \alpha < s - 1/2$, i.e. $f \in \mathscr{C}^{\alpha}(\mathbb{T})$.
- ▷ Assume $\mathscr{T}_N = \{t_1, \dots, t_N\}$ is a sampling set with mesh size $r_N = \min_{n\neq m}|t_n - t_m|$.
- ▷ There is a unique interpolating function f_N which is continuous, piecewise linear, and which satisfies
$$f_N(\lambda_n) = f(\lambda_n) \qquad \text{for all } \lambda_n \in \Lambda_N\,.$$
- ▷ Since $f \in \mathscr{C}^{\alpha}(\mathbb{T})$ it follows that for all $0 < \alpha' < \alpha$
$$\|f - f_N\|_{\mathscr{C}^{\alpha'}(\mathbb{T})} \to 0 \qquad \text{as} \qquad r_N \to 0\,.$$
- ▷ Since it is known that $\mathrm{H} : \mathscr{C}^{\alpha'}(\mathbb{T}) \to \mathscr{C}(\mathbb{T})$ is bounded, we set $\widetilde{f}_N = \mathrm{H} f_N$ and obtain
$$\left\|\widetilde{f}_N - \widetilde{f}\right\|_\infty = \|\mathrm{H}(f_N - f)\|_\infty \leq \|\mathrm{H}\|\; \|f - f_N\|_{\mathscr{C}^{\alpha'}(\mathbb{T})} \to 0 \qquad \text{as } r_N \to 0\,.$$

Remark:

- Procedure fails for $s \leq 1/2$ because Sobolev embedding yields no longer Hölder continuity.
- Is this failure a particular property of the above procedure?

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Relation to the Dirichlet Problem

Dirichlet Problem on the Unit Circle

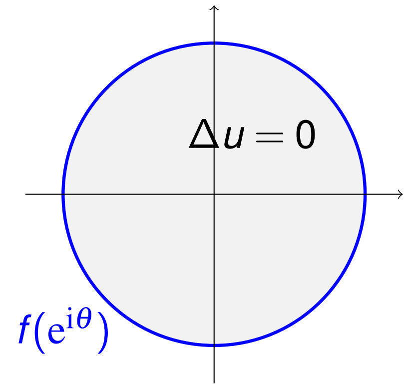
Let f be a given function on the unit circle $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$. We look for an u inside the unit circle $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ such that

1. $\frac{\partial^2 u}{\partial x^2}(z) + \frac{\partial^2 u}{\partial y^2}(z) = (\Delta u)(z) = 0$ for all $z = x + \mathrm{i}y \in \mathbb{D}$
2. $u(\mathrm{e}^{\mathrm{i}t}) = f(\mathrm{e}^{\mathrm{i}t})$ for all $t \in \mathbb{T} = [-\pi, \pi)$

Dirichlet's Principle

The solution of the Dirichlet problem can be obtained by minimizing the Dirichlet energy

$$D(u) = \frac{1}{2\pi} \iint_{\mathbb{D}} \|(\mathrm{grad}\, u)(z)\|^2_{\mathbb{R}^2} \,\mathrm{d}z = \sum_{n=-\infty}^{\infty} |n| \, |c_n(f)|^2 = \|f\|^2_{H^{1/2}}$$



- The boundary function of solutions of the Dirichlet problem belongs to the Sobolev space $H^{1/2}$.
- If f is additionally in $\mathscr{B}$ then $f \in \mathscr{B}_0$.

Axiomatic for Hilbert Transform Approximations

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Hilbert Transform Approximations – Axiomatic

Definition: Let $\beta \geq 1$ be arbitrary and let $\mathrm{H} = \{\mathrm{H}_N\}_{N\in\mathbb{N}}$ be a sequence of mappings $\mathrm{H}_N : \mathscr{B}_\beta \to \mathscr{C}(\mathbb{T})$ with the associated functionals $\Phi_N(f) = \|\mathrm{H}_N(f)\|_\infty$. We say that H satisfies Axiom

(A) if Φ_N is lower semicontinuous for every $N \in \mathbb{N}$ and if for every $N \in \mathbb{N}$ there exists a finite subset $T_N \subset \mathbb{T}$ such that for arbitrary $f_1, f_2 \in \mathscr{B}_\beta$

$$\begin{aligned} f_1(t_n) &= f_2(t_n) && \text{for all } t_n \in T_N \\ \text{implies} \quad (\mathrm{H}_N f_1)(t) &= (\mathrm{H}_N f_2)(t) && \text{for all } t \in \mathbb{T}\,. \end{aligned}$$

(B) if there exists a dense subset $\mathscr{M} \subset \mathscr{B}_\beta$ such that

$$\lim_{N\to\infty} \left\|\mathrm{H}_N(f) - \mathrm{H}f\right\|_\infty = 0 \qquad \text{for all } f \in \mathscr{M}\,.$$

Remarks

- Axiom (A) requires that the approximation $\widetilde{f}_N = \mathrm{H}_N(f)$ is uniquely determined by the values of f on the finite sampling set $T_N \subset \mathbb{T}$. So (A) ensures that $\mathrm{H}_N(f)$ is computable on a digital computer.
- Axiom (B) describes $\{\mathrm{H}_N\}$ as a sequence which approximates the Hilbert transform, namely it requires that $\mathrm{H}_N(f)$ converges to $\mathrm{H}f$ at least for all f from a dense subset of $\mathscr{B}_\beta$.
- The operators H_N may be non-linear.
- The dense set $\mathscr{M}$ in Axiom (B) needs not to have a linear structure.

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Example – Sampled Fourier series

- Let $f \in \mathscr{B}_\beta$ arbitrary with Fourier series representation

$$f(t) = \frac{a_0(f)}{2} + \sum_{n=1}^{\infty} a_n(f)\cos(nt) + b_n(f)\sin(nt)\,, \qquad t \in \mathbb{T}$$

and with the Fourier coefficients

$$a_n(f) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(\tau)\cos(n\tau)\,\mathrm{d}\tau \qquad \text{and} \qquad b_n(f) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(\tau)\sin(n\tau)\,\mathrm{d}\tau\,. \tag{1}$$

Then its conjugate function $\widetilde{f} = \mathrm{H}f$ is given by

$$\widetilde{f}(t) = (\mathrm{H}f)(t) = \sum_{n=1}^{\infty} a_n(f)\sin(nt) - b_n(f)\cos(nt)\,, \qquad t \in \mathbb{T}$$

- We define approximation operators $\mathrm{H}_N : \mathscr{B}_\beta \to \mathscr{C}(\mathbb{T})$ as follows
  - For every $N \in \mathbb{N}$ we define the uniform sampling set

$$T_N = \left\{ t_{N,k} = \tfrac{k-N}{N}\pi \ :\ k = 0,1,2,\dots,2N-1 \right\}\,.$$

  - For every $N \in \mathbb{N}$ we approximate the integrals in (1) by its Riemann sums

$$a_{N,n}(f) = \tfrac{1}{2N}\textstyle\sum_{k=0}^{2N-1} f\big(t_{N,k}\big)\cos\big(n\,t_{N,k}\big) \qquad \text{and} \qquad b_{N,n}(f) = \tfrac{1}{2N}\textstyle\sum_{k=0}^{2N-1} f\big(t_{N,k}\big)\sin\big(n\,t_{N,k}\big)$$

  - Therewith, we define for every $N \in \mathbb{N}$ the approximation operators

$$(\mathrm{H}_N f)(t) := \sum_{n=1}^{N} a_{N,n}(f)\sin(nt) - b_{N,n}(f)\cos(nt)\,, \qquad t \in \mathbb{T}\,.$$

Example – Sampled Fourier series (cont.)

▷ Inserting the Fourier coefficients into the sum, a closed form representations is obtained

$$(\mathrm{H}_N f)(t) = \sum_{k=0}^{2N-1} f\big(t_{N,k}\big)\, \mathscr{D}_N\big(t - t_{N,k}\big) \qquad \text{with} \qquad \mathscr{D}_N(t) = \frac{1}{N}\sum_{n=1}^{N-1} \sin(nt)\,.$$

Remarks

- $\{\mathrm{H}_N\}_{N\in\mathbb{N}}$ satisfies Axiom (A) – each H_N is concentrated on the finite sampling set $T_N = \{t_{N,k}\}_{k=0}^{2N-1}$.
- $\{\mathrm{H}_N\}_{N\in\mathbb{N}}$ satisfies Axiom (B) – $\mathrm{H}_N f$ converges to $\mathrm{H}f$ for all polynomials.
- All operators H_N are even continuous $\Rightarrow$ the associated functionals are lower semicontinuous.

- Other linear approximation methods satisfying Axioms (A) and (B) are obtained by
  - using other summation methods (Fejér, Cesáro, ...)
  - using other numerical integration methods to approximate the exact Fourier coefficients.

- If the Fourier coefficients $\{a_n(f)\}_{n\in\mathbb{N}}$ and $\{b_n(f)\}_{n\in\mathbb{N}}$ are perfectly known, the above the partial conjugate Fourier (or Fejér) series would converge for all $f \in \mathscr{B}_\beta$
  $\Rightarrow$ Convergence problems are due to the sampling based form of the approximation operators.

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Divergence Results

for $\mathscr{B}_\beta$ with $0 \leq \beta \leq 1$

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Divergence of Sampling Based Approximations

Theorem

Let $0 \leq \beta \leq 1$ be arbitrary and let $\boldsymbol{H} = \{H_N\}_{N\in\mathbb{N}}$ be a sequence of mappings $\mathrm{H}_N : \mathscr{B}_\beta \to \mathscr{C}(\mathbb{T})$ which satisfies Axioms (A) and (B). Then

$$\mathscr{R}_\beta(\boldsymbol{H}) = \left\{f \in \mathscr{B}_\beta \;:\; \limsup_{N\to\infty} \left\|\mathrm{H}_N(f)\right\|_\infty = +\infty\right\}$$

is a residual set in $\mathscr{B}_\beta$.

Remarks

- In particular, there always exists an $f \in \mathscr{B}_\beta$ such that

  $$\limsup_{N\to\infty} \|\mathrm{H}f - \mathrm{H}_N(f)\|_{\mathscr{B}_\beta} = +\infty\,.$$

  No matter how we choose the sampling based (Axiom A) approximation operators $\boldsymbol{H} = \{\mathrm{H}_N\}_{N\in\mathbb{N}}$ there always exist functions $f \in \mathscr{B}_\beta$ such that $\mathrm{H}_N(f)$ diverges.
- This result includes even even non-linear approximation operators $\mathrm{H}_N : \mathscr{B}_\beta \to \mathscr{C}(\mathbb{T})$.
  Similar result for linear methods was already proven previously[3]
- Divergence occurs in particular on the Sobolev space $\mathscr{B}_0 = H^{1/2}(\mathbb{T}) = W^{1/2,2}(\mathbb{T})$ of signals with finite Dirichlet energy, and even on some smoother subspaces $\mathscr{B}_\beta$ with $\beta > 0$.

[3] H. Boche, V. Pohl, ISIT 2017.

Ingredients of the Proof

Interpolation Lemma:

Let $0 \leq \beta \leq 1$ be arbitrary and let $T = \{t_n\}_{n=1}^{N} \subset \mathbb{T}$ be a finite sampling set. Then for every $\varepsilon > 0$ the following statement is true: To every $g \in \mathscr{C}(\mathbb{T})$ there exists an $f \in \mathscr{B}_\beta$ such that

$$\begin{aligned} &(a) \quad f(t_n) = g(t_n) \qquad \text{for all } t_n \in T \\ &(b) \quad \|f\|_{\mathscr{B}_\beta} \leq (1+\varepsilon)\,\|g\|_\infty\,. \end{aligned}$$

- ▷ So the operators H_N can't distinguisch between $f \in \mathscr{B}_\beta$ and $f \in \mathscr{C}(\mathbb{T})$.
- ▷ Using that the Hilbert transform $\mathrm{H} : \mathscr{C}(\mathbb{T}) \to \mathscr{C}(\mathbb{T})$ is unbounded.

Generalized uniform boundedness principle:

Let $\mathscr{B}$ be a Banach space and let $\mathbf{\Phi}$ be a family of lower semicontinuous functionals on $\mathscr{B}_\beta$ such that there exists a set $S \subset \mathscr{B}_\beta$ of second category so that

$$\sup_{\varphi \in \mathbf{\Phi}} \varphi(f) = M(f) < +\infty \qquad \text{for all } f \in S\,.$$

Then there exist a constant $M_{\mathbf{\Phi}} < \infty$, an $f_0 \in \mathscr{B}_\beta$, and a $\delta > 0$ such that for all $f \in \mathscr{B}_\delta(f_0, \mathscr{B}_\beta) = \left\{ f \in \mathscr{B}_\beta \;:\; \|f - f_0\|_{\mathscr{B}_\beta} < \delta \right\}$ always $\varphi(f) \leq M_{\mathbf{\Phi}}$ for all $\varphi \in \mathbf{\Phi}$.

- ▷ Necessary to include non-linear operators in our analysis.

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Convergence Results for $\mathscr{B}_\beta$ with $\beta > 1$

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Spaces with Convergent Approximation Methods

Theorem

For any $\beta > 1$ there exit sequences $\{\mathrm{H}_N\}_{N\in\mathbb{N}}$ of bounded linear operators $\mathrm{H}_N : \mathscr{B}_\beta \to \mathscr{C}(\mathbb{T})$ which satisfy Axioms (A) and (B) such that

$$\lim_{N\to\infty} \left\| \mathrm{H}_N f - \mathrm{H} f \right\|_\infty = 0 \qquad \text{for all } f \in \mathscr{B}_\beta \,.$$

- If the energy of the signals is sufficiently concentrated then there always exist sampling based approximation methods which converges for all signals in the space $\mathscr{B}_\beta$ with $\beta > 1$.
- There even exist linear approximation methods.
- Theorem can be proved by a constructing particular method, e.g. the sampled Fourier series considered at the beginning.

Characterization of Convergent Method

Theorem

Let $\beta > 1$ and let $\{\mathrm{H}_N\}_{N\in\mathbb{N}}$ be a sequence of bounded linear operators $\mathrm{H}_N : \mathscr{B}_\beta \to \mathscr{B}_\beta$ such that

1. *For every $n \in \mathbb{N}$ holds*

$$\lim_{N\to\infty} \left\| \mathrm{H}_N[\cos(n\cdot)] - \sin(n\cdot) \right\|_\infty = 0 \qquad \textit{and} \qquad \lim_{N\to\infty} \left\| \mathrm{H}_N[\sin(n\cdot)] + \cos(n\cdot) \right\|_\infty = 0\,.$$

2. *There exists a constant C such that*

$$\max\left(\left\| \mathrm{H}_N[\cos(n\cdot)] \right\|_\infty, \left\| \mathrm{H}_N[\sin(n\cdot)] \right\|_\infty \right) \leq C \qquad \textit{for all } N \in \mathbb{N}\,.$$

Then one has

$$\lim_{N\to\infty} \left\| \mathrm{H}_N f - \mathrm{H} f \right\|_\infty = 0 \qquad \textit{for all } f \in \mathscr{B}_\beta\,.$$

Thus, if an approximation method $\{\mathrm{H}_N\}_{N\in\mathbb{N}}$

- ▷ converges for the sine- and cosine functions (i.e. for the pure frequencies), and
- ▷ if the approximations of the pure frequencies are uniformly bounded

then the method $\mathrm{H}_N f$ converges to $\mathrm{H} f$ for all $f \in \mathscr{B}_\beta$ with $\beta > 1$.

Example: The sampled Fourier series considered at the beginning.

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Conclusions and Outlook

- We introduced a scale of Sobolev-like Banach spaces $\mathscr{B}_\beta$, $\beta \geq 0$ of functions
  - which are continuous with a continuous conjugate
  - with finite (Dirichlet) energy
  - with different energy concentration, characterized by β
- In the scale $\{\mathscr{B}_\beta\}_{\beta \geq 0}$, we characterized precisely those spaces on which
  - there do not exist any sampling based Hilbert transform approximations: $\beta \in [0, 1]$
  - there do exist sampling based Hilbert transform approximations: $\beta > 1$.
- For $\beta > 1$ even very simple approximations methods (sampled conjugate Fourier series) work.
- Based on our framework, one can show that there exists no Turing computable method to determine the Hilbert transform on the spaces $\mathscr{B}_\beta$ with $0 \leq \beta \leq 1$.

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- Based on our framework, one can show that there exists no Turing computable method to determine the Hilbert transform on the spaces $\mathscr{B}_\beta$ with $0 \leq \beta \leq 1$.

Thank You! – Questions?