# Identification over Channels with Feedback: Discontinuity Behavior and Super-Activation

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joint work with Holger Boche and H. Vincent Poor

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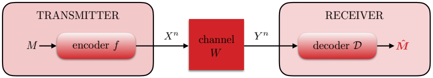
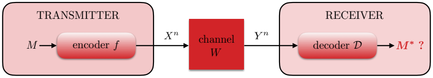
# Message Transmission vs. Identification

## Message Transmission




- Recover *exact message $M$*
  - $C(W) = \max_{P_X} I(P_X, W)$
  - Transmission message size: $|\mathcal{M}| = 2^{nR}$ (exponentially)

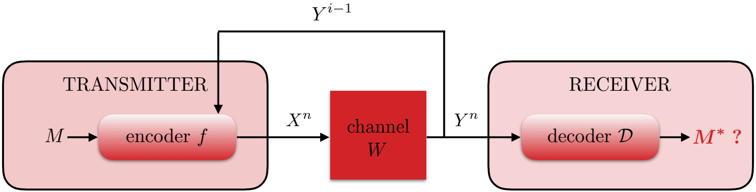
## Identification




- Identify if *particular message $M^*$ of interest* has been sent
  - $C^{\mathsf{ID}}(W) = \max_{P_X} I(P_X, W)$
  - Identification message size: $|\mathcal{M}| = \mathbf{2^{2^{nR}}}$ (double-exponentially)

R. Ahlswede and G. Dueck, "Identification via channels," *IEEE Trans. Inf. Theory*, vol. 35, no. 1, pp. 15–29, Jan. 1989

# Identification over Channels with Feedback




- Encoding function is now vector-valued

$$\boldsymbol{f} = (f^{(1)}, f^{(2)}, ..., f^{(n)})$$

with $f^{(i)} : \mathcal{Y}^{i-1} \to \mathcal{X}$ the encoding function at time instant $i$


——, "Identification in the presence of feedback–A discovery of new capacity formulas," *IEEE Trans. Inf. Theory*, vol. 35, no. 1, pp. 30–36, Jan. 1989

# Identification-Feedback (IDF) Capacity

*Theorem:* [Ahlswede-Dueck '89]

If the capacity $C(W)$ of a DMC $W$ satisfies $C(W) > 0$, then the *deterministic IDF capacity* is

$$C_f^{\mathsf{ID}}(W) = \max_{x \in \mathcal{X}} H(W(\cdot|x)).$$

If $C(W) = 0$, then

$$C_f^{\mathsf{ID}}(W) = 0.$$

*Theorem:* [Ahlswede-Dueck '89]

If the capacity $C(W)$ of a DMC $W$ satisfies $C(W) > 0$, then the *randomized IDF capacity* is

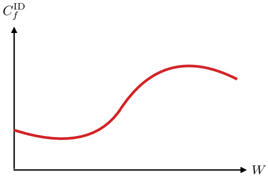
$$C_F^{\mathsf{ID}}(W) = \max_{P \in \mathcal{P}(\mathcal{X})} H(P \cdot W).$$

If $C(W) = 0$, then

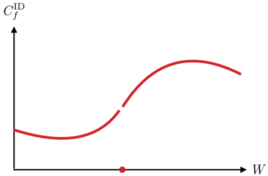
$$C_F^{\mathsf{ID}}(W) = 0.$$

*In the following we further study properties of the IDF capacity (for both deterministic and randomized encoding)*

## ❶ Continuity

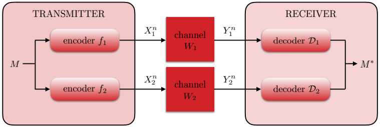



continuous

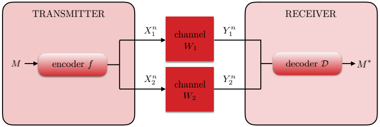



discontinuous

## ❷ Additivity

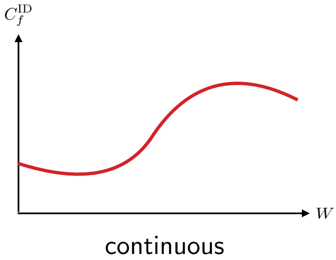


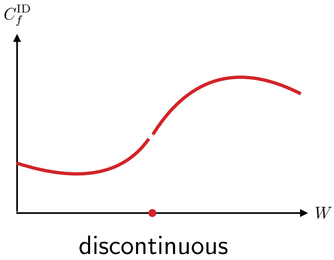

$$C^{\mathrm{ID}}(W_1) + C^{\mathrm{ID}}(W_2)$$




$$C^{\mathrm{ID}}(W_1 \otimes W_2)$$

# 1. Continuity of IDF capacity




- Need to understand whether the *performance of a communication system depends in a continuous way on the system parameters* or not
- ➠ Desirable would be a *continuous* behavior so that small changes in parameters result in small changes of the peformance only

# Distance

- To study the continuity of the IDF capacity, we need a concept of *distance*
- For two DMCs $W_1$, $W_2$ we define the *$d$-distance* between $W_1$ and $W_2$ based on the total variation distance as

$$d(W_1, W_2) \coloneqq \max_{x \in \mathcal{X}} \sum_{y \in \mathcal{Y}} \big| W_1(y|x) - W_2(y|x) \big|$$

# Continuity

*Definition:*

The continuity of the capacity $C_f^{\mathsf{ID}}(\cdot)$ is defined as follows.

1. The DMC $W$ is a *continuity point* of $C_f^{\mathsf{ID}}(\cdot)$ if for all sequences $\{W_n\}_{n=1}^{\infty}$ with
$$\lim_{n\to\infty} d(W_n, W) = 0 \tag{1}$$
we have $\lim_{n\to\infty} C_f^{\mathsf{ID}}(W_n) = C_f^{\mathsf{ID}}(W)$.
2. The DMC $W$ is a *discontinuity point* of $C_f^{\mathsf{ID}}(\cdot)$ if 1) does not hold, i.e., if there is a sequence $\{W_n\}_{n=1}^{\infty}$ that satisfies (1) but
$$\limsup_{n\to\infty} C_f^{\mathsf{ID}}(W_n) > \liminf_{n\to\infty} C_f^{\mathsf{ID}}(W_n)$$
is satisfied.
3. The capacity $C_f^{\mathsf{ID}}(\cdot)$ is a *continuous function* if all DMCs $W$ are continuity points according to 1).

# Discontinuity Points of IDF Capacity

- We further define the set

$$\mathcal{D}_f = \left\{ W : \; W \in \mathcal{N}_C \text{ and } \max_{x \in \mathcal{X}} H(W(\cdot|x)) > 0 \right\}$$

with

$$\mathcal{N}_C = \left\{ W : C(W) = 0 \right\}$$

the set of DMCs with zero capacity.

➠ We observe that these sets characterize all discontinuity points of the deterministic and randomized IDF capacities:

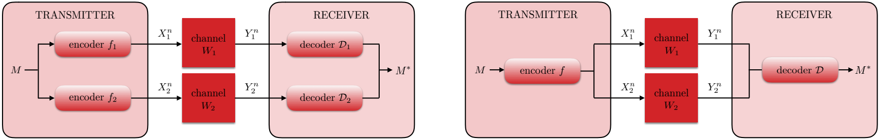
*Theorem (Deterministic IDF Capacity):*

$\mathcal{D}_f$ is the set of **discontinuity points** of $C_f^{\text{ID}}(\cdot)$.

*Theorem (Randomized IDF Capacity):*

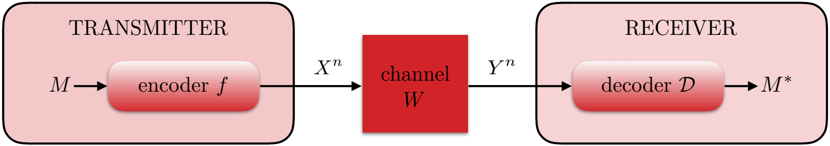
$\mathcal{D}_F = \mathcal{N}_C$ is the set of all **discontinuity points** of $C_F^{\text{ID}}(\cdot)$.

# 2. Additivity of IDF capacity



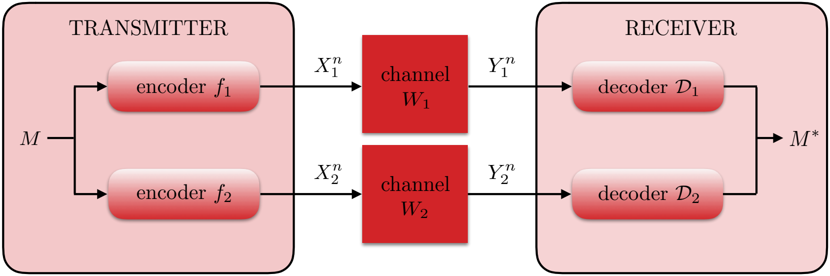

$$C^{\mathsf{ID}}(W_1) + C^{\mathsf{ID}}(W_2) \stackrel{?}{=} C^{\mathsf{ID}}(W_1 \otimes W_2)$$

# Capacity of DMC




- Capacity: $C(W_1)$

# Capacity of Parallel DMCs




- Independent encoding/decoding: $C(W_1) + C(W_2)$

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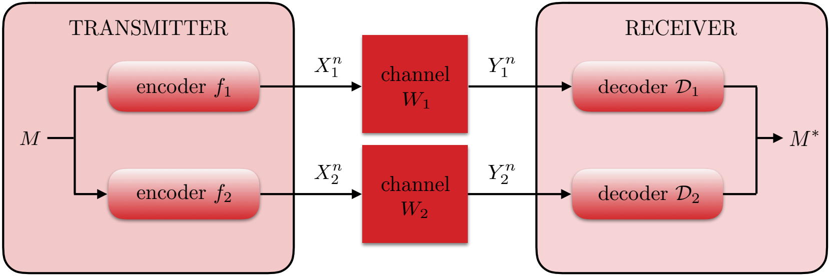


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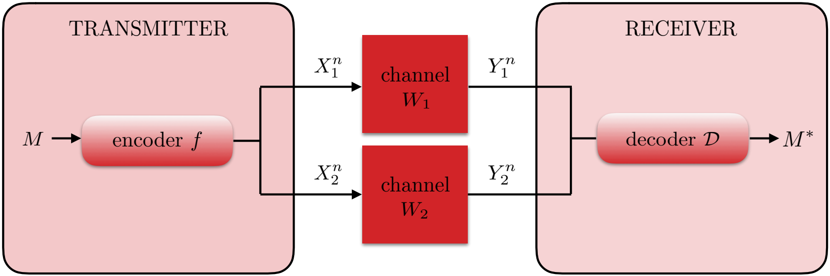
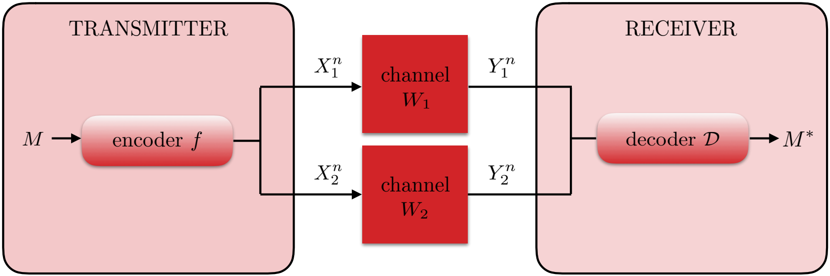



- Independent encoding/decoding: $C(W_1) + C(W_2)$
- Joint encoding/decoding: $C(W_1 \otimes W_2)$

$$C(W_1 \otimes W_2) = C(W_1) + C(W_2)$$

# Capacity of Parallel DMCs

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$$C(W_1 \otimes W_2) = C(W_1) + C(W_2)$$

# Zero Error Capacity

- Shannon conjectured in 1956 the zero-error capacity to be additive:

$$C_0(W_1 \otimes W_2) \stackrel{?}{=} C_0(W_1) + C_0(W_2)$$

> Theorem 4, of course, is analogous to known results for ordinary capacity C, where the product channel has the sum of the ordinary capacities and the sum channel has an equivalent number of letters equal to the sum of the equivalent numbers of letters for the individual channels. We conjecture but have not been able to prove that the equalities in Theorem 4 hold in general, not just under the conditions given.

C. E. Shannon, "The zero error capacity of a noisy channel," *IRE Trans. Inf. Theory*, vol. 2, no. 3, pp. 8–19, Sep. 1956

- Subsequently restated in 1979 by Lovász

L. Lovász, "On the Shannon capacity of a graph," *IEEE Trans. Inf. Theory*, vol. 25, no. 1, pp. 1–7, Jan. 1979

# Zero Error Capacity and AVCs

- Later disproved constructing explicit counter-examples with:

$$C_0(W_1 \otimes W_2) > C_0(W_1) + C_0(W_2)$$

- However, complete characterization is still an open problem

W. Haemers, "On some problems of Lovász concerning the Shannon capacity of a graph," *IEEE Trans. Inf. Theory*, vol. 25, no. 2, pp. 231–232, Mar. 1979

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- Since then non-additivity of the capacity has been observed for other scenarios as well

H. Boche and R. F. Schaefer, "Capacity results and super-activation for wiretap channels with active wiretappers," *IEEE Trans. Inf. Forensics Security*, vol. 8, no. 9, pp. 1482–1496, Sep. 2013

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# Deterministic IDF Capacity

- To characterize the case of super-additivity, we define the following region:

$$\mathcal{M}_f^{\text{supadd}} = \Big\{(W_1, W_2) : \max_{i=1,2}\{\min\{C(W_i), \Psi(W_i)\}\} > 0, \\ \min_{i=1,2}\{\min\{C(W_i), \Psi(W_i)\} = 0, \\ \min\{\Psi(W_1), \Psi(W_2)\} > 0\Big\}$$

with $\Psi(W) = \max_{x \in \mathcal{X}} H(W(\cdot|x))$

*Theorem:*

We have *super-additivity* in $(\hat{W}_1, \hat{W}_2)$ but no super-activation of $C_f^{\text{ID}}(\cdot)$ if and only if $(\hat{W}_1, \hat{W}_2) \in \mathcal{M}_f^{\textit{supadd}}$.

# Deterministic IDF Capacity (2)

- To study the phenomenon of super-activation and to characterize those points for which this is possible, we define

$$\mathcal{M}_f^{\mathsf{supact}} = \Big\{(W_1, W_2) : \min_{i=1,2}\{C(W_i), \Psi(W_i)\} = 0, \\ \max\{C(W_1), C(W_2)\} > 0, \\ \max\{\Psi(W_1), \Psi(W_2)\} > 0\Big\}$$

*Theorem:*

*Super-activation* of $C_f^{\mathsf{ID}}(\cdot)$ occurs if and only if $(\hat{W}_1, \hat{W}_2) \in \mathcal{M}_f^{\mathsf{supact}}$.

# Randomized IDF Capacity

***Theorem:***

Super-activation of $C_F^{\mathsf{ID}}(\cdot)$ is not possible.

- In contrast to the deterministic IDF capacity $C_f^{\mathsf{ID}}$ for which super-activation occurs
- However, $C_F^{\mathsf{ID}}(\cdot)$ is super-additive. For this purpose, we define

$$\mathcal{M}_F^{\mathsf{supadd}} = \Big\{(\hat{W}_1, \hat{W}_2) : \max_{i=1,2}\{\min\{C(\hat{W}_i), \Psi(\hat{W}_i)\}\} > 0, \\ \min_{i=1,2}\{\min\{C(\hat{W}_i), \Psi(\hat{W}_i)\} = 0, \\ \min\{\Psi(\hat{W}_1), \Psi(\hat{W}_2)\} > 0\Big\}.$$

***Theorem:***

We have *super-additivity* in $(\hat{W}_1, \hat{W}_2)$ if and only if $(\hat{W}_1, \hat{W}_2) \in \mathcal{M}_F^{\mathsf{supadd}}$.

# Conclusions

- Identification over channels with feedback has been considered and the IDF capacity has been further analyzed
- *Deterministic IDF capacity*
  - Discontinuity points characterized by $\mathcal{D}_f$
  - Super-additivity in $\mathcal{M}_f^{\text{supadd}}$
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- Outlook: Framework and techniques can be used to show that the IDF capacity is not computable on Turing machines

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