Foundation of Digital Signal Processing: Signal Spaces, System Representation, and Quantization Effects

Holger Boche

Lehrstuhl für Theoretische Informationstechnik
Technische Universität München

Strobl11

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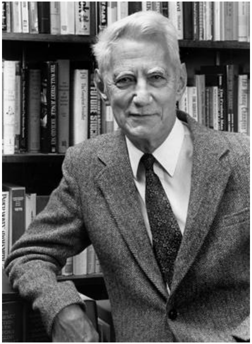
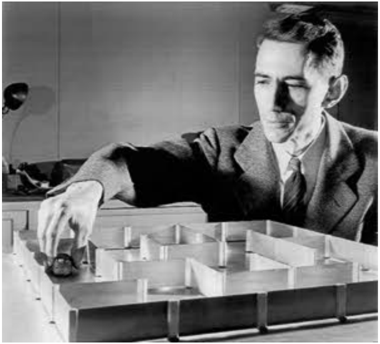
Outline

Claude E. Shannon (1916-2001)

Shannon – Founder of the Information Age

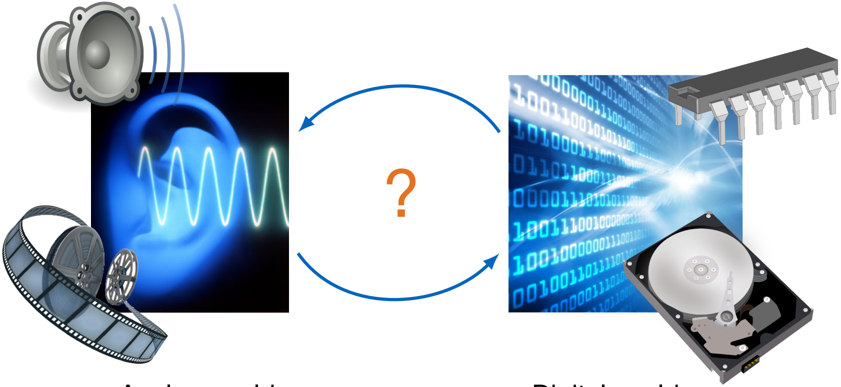




From: [IEEE Information Theory Society]

Today's ‘‘digital world’’ is based on his theoretical work!

Analog World Versus Digital World



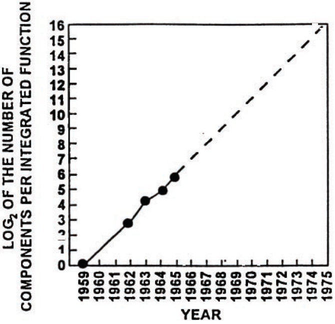
Analog world   Digital world

Moore’s Law

Moore’s Law

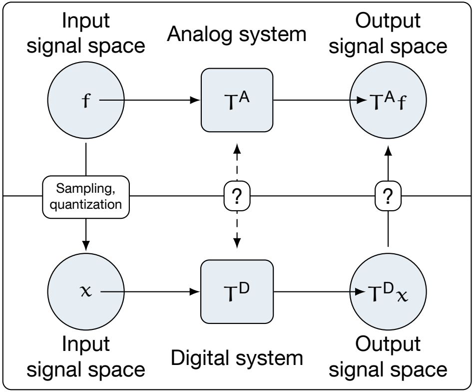
The complexity of electronic circuits, measured in the number of transistors on a chip, doubles approximately every two years.

- It was postulated in 1965 by Gordon Moore.
- Based on the development between 1959 and 1965, he originally stated a doubling every year. (Corrected in 1975 to the current two year statement.)
- This remarkable progress in technology could be observed over the last 50 years.
- The miniaturization cannot be continued forever due to physical limitations.



From: G. Moore, ‘‘Cramming More Components onto Integrated Circuits,’’ Electronics Magazine, 38, 1965.

Analog World Versus Digital World



Analog world

Digital world

Richard Feynman

''Richard Feynman was one of the 20th century's most influential physicists . . . '' [The Observer, Sunday 15 May 2011]

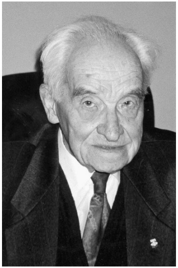
He discusses the problem of transmitting a function of time and writes in this context:

''Consideration of such a problem will bring us on to consider the famous *Sampling Theorem*, another baby of Claude Shannon.''

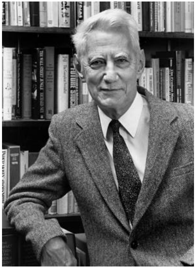
R. P. Feynman, *Feynman Lectures on Computation*, A. J. Hey and R. W. Allen, Eds. Penguin Books, 1999.

Whittaker-Kotel’nikov-Shannon Sampling Series

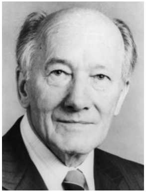
Edmund T. Whittaker



Vladimir A. Kotel’nikov



Claude E. Shannon



Herbert Raabe

$$\sum_{k=-\infty}^{\infty} f(k) \frac{\sin(\pi(t-k))}{\pi(t-k)}$$

Kinnosuke Ogura

Outline

Signal Reconstruction

We analyze the local and global convergence behavior of the sampling series

$$\sum_{k=-\infty}^{\infty} f(t_k)\phi_k(t).$$

for the Paley-Wiener space $\mathcal{PW}_{\pi}^{1}$.

- ϕ_k, $k \in \mathbb{Z}$, are certain reconstruction functions
- $\{t_k\}_{k\in\mathbb{Z}}$ is the sequence of real sampling points

We assume that $t_0 = 0$, and

$$\ldots < t_{-N} < \ldots < t_{-1} < t_0 < t_1 < \ldots < t_N < \ldots.$$

- The sampling points $\{t_k\}_{k\in\mathbb{Z}}$ are the zeros of sine-type functions.

Signal Spaces

- $\mathcal{B}_\sigma$ is the set of all entire functions f with the property that for all $\epsilon > 0$ there exists a constant $C(\epsilon)$ with $|f(z)| \leqslant C(\epsilon) \exp\big((\sigma + \epsilon)|z|\big)$ for all $z \in \mathbb{C}$.

Definition (Bernstein Space)

The Bernstein space $\mathcal{B}_\sigma^p$ consists of all signals in $\mathcal{B}_\sigma$, whose restriction to the real line is in $L^p(\mathbb{R})$, $1 \leqslant p \leqslant \infty$.
$\mathcal{B}_{\sigma,0}^\infty$ denotes the set of all signals in $f \in \mathcal{B}_\sigma^\infty$ that satisfy $\lim_{|t|\to\infty} f(t) = 0$.

Signal Spaces

Definition (Paley-Wiener Space)

For $1 \leqslant p \leqslant \infty$ we denote by $\mathcal{PW}_\sigma^p$ the Paley-Wiener space of functions f with a representation $f(z) = \frac{1}{2\pi} \int_{-\sigma}^{\sigma} g(\omega)\, e^{iz\omega}\, d\omega$, $z \in \mathbb{C}$, for some $g \in L^p[-\sigma, \sigma]$.

The norm for $\mathcal{PW}_\sigma^p$ is given by $\|f\|_{\mathcal{PW}_\sigma^p} = \left(\frac{1}{2\pi} \int_{-\sigma}^{\sigma} |\hat{f}(\omega)|^p \, d\omega \right)^{1/p}$.

- We focus on $\mathcal{PW}_\pi^1$.
- $\mathcal{PW}_\pi^p \supset \mathcal{PW}_\pi^s$ for $1 \leqslant p < s \leqslant \infty$; $\|f\|_\infty \leqslant \|f\|_{\mathcal{PW}_\pi^1}$.
- $\mathcal{PW}_\pi^2$ is the space of bandlimited function with finite energy.

Operational Meaning of Signal Spaces

Motivated by Feichtinger's axiomatic approach to characterize function spaces with desirable properties.

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$\mathcal{PW}_\pi^1$ and the convergence of wide-sense stationary (WSS) stochastic processes X

- Mean-square error: $\mathbb{E}\left|X(t) - \sum_{k=-N}^{N} X(t_k)\phi_k(t)\right|^2$

Let $T > 0$. We have

$$\lim_{N\to\infty} \max_{t\in[-T,T]} \left|f(t) - \sum_{k=-N}^{N} f(t_k)\phi_k(t)\right| = 0$$

for all $f \in \mathcal{PW}_\pi^1$, if and only if

$$\lim_{N\to\infty} \max_{t\in[-T,T]} \mathbb{E}\left|X(t) - \sum_{k=-N}^{N} X(t_k)\phi_k(t)\right|^2 = 0$$

for an important subclass bandlimited wide-sense stationary processes (I-processes).

Sine-Type Functions

Definition

An entire function f of exponential type π is said to be of sine type if

(i) the zeros of f are separated, and

(ii) there exist positive constants A, B, and H such that

$$A\, e^{\pi|y|} \leqslant |f(x+iy)| \leqslant B\, e^{\pi|y|}$$

whenever x and y are real and $|y| \geqslant H$.

Example

$\sin(\pi z)$ is a function of sine type and its zeros are $t_k = k$, $k \in \mathbb{Z}$.

Zeros of Sine Type Functions and Complete Interpolating Sequences

Definition (Complete Interpolating Sequence)

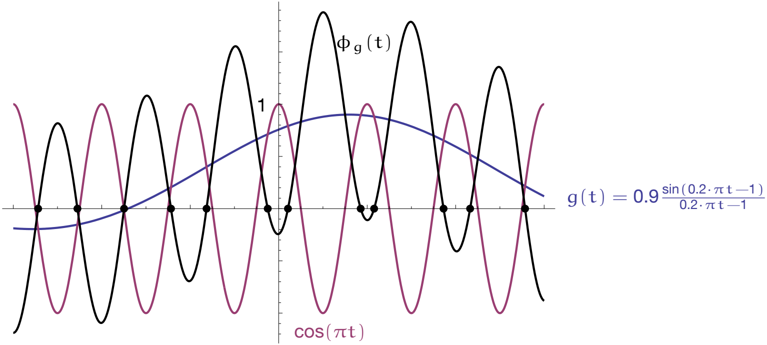
We say that $\{t_k\}_{k\in\mathbb{Z}}$ is a complete interpolating sequence for $\mathcal{PW}_\pi^2$ and coefficient space l^2 if the interpolation problem $f(t_k) = c_k$, $k \in \mathbb{Z}$, has exactly one solution $f \in \mathcal{PW}_\pi^2$ for every sequence $\{c_k\}_{k\in\mathbb{Z}} \in l^2$.

Lemma

If $\{t_k\}_{k\in\mathbb{Z}}$ is the set of zeros of a function of sine type, then the system $\{e^{i\omega t_k}\}_{k\in\mathbb{Z}}$ is a Riesz basis for $L^2[-\pi,\pi]$, and $\{t_k\}_{k\in\mathbb{Z}}$ is a complete interpolating sequence for $\mathcal{PW}_\pi^2$.

Construction of Sampling Patterns

- real-valued signal $g \in \mathcal{PW}_\pi^1$, $\|g\|_{\mathcal{PW}_\pi^1} < 1$
  $\Rightarrow \phi_g(t) = g(t) - \cos(\pi t)$ is a function of sine type.
- The zeros $\{t_k\}_{k\in\mathbb{Z}}$ of ϕ_g are all real, because we assumed that g is real-valued and $\|g\|_{\mathcal{PW}_\pi^1} < 1$.
  $\rightarrow$ method to construct arbitrarily many sampling patterns $\{t_k\}_{k\in\mathbb{Z}}$.



Reconstruction Functions

If $\{t_k\}_{k\in\mathbb{Z}}$ are the zeros of a function of sine type, then the product

$$\phi(z) = z \lim_{N\to\infty} \prod_{\substack{|k|\leqslant N \\ k\neq 0}} (1 - z/t_k)$$

converges uniformly on $|z| \leqslant R$ for all $R < \infty$ and ϕ is an entire function of exponential type π.

$$\phi_k(t) = \frac{\phi(t)}{\phi'(t_k)(t - t_k)}$$

is the unique function in $\mathcal{PW}_\pi^2$ that solves the interpolation problem

$$\phi_k(t_l) = \begin{cases} 1 & l = k \\ 0 & l \neq k \end{cases}$$

and $\{\phi_k\}_{k\in\mathbb{Z}}$ is a Riesz basis for $\mathcal{PW}_\pi^2$.

Local Uniform Convergence

Theorem

Let ϕ be a function of sine type, whose zeros $\{t_k\}_{k\in\mathbb{Z}}$ are all real. Then we have

$$\lim_{N\to\infty} \max_{t\in[-T,T]} \left| f(t) - \sum_{k=-N}^{N} f(t_k)\phi_k(t) \right| = 0$$

for all $T > 0$ and all $f \in \mathcal{PW}_\pi^1$.

- The sampling series is locally uniformly convergent.

Remark

- The same result holds even for $f \in \mathcal{B}_{\pi,0}^\infty$.

Brown's Theorem

Theorem (Brown)

For all $f \in \mathcal{PW}_\pi^1$ *and* $T > 0$ *fixed we have*

$$\lim_{N\to\infty} \max_{t\in[-T,T]} \left| f(t) - \sum_{k=-N}^{N} f(k) \frac{\sin(\pi(t-k))}{\pi(t-k)} \right| = 0.$$

The Shannon sampling series is uniformly convergent on all compact subsets of $\mathbb{R}$ for the space $\mathcal{PW}_\pi^1$.

Global Uniform Convergence

Theorem

Let ϕ *be a function of sine type, whose zeros* $\{t_k\}_{k\in\mathbb{Z}}$ *are all real. Then, for all* $0 < \beta < 1$ *and all* $f \in \mathcal{PW}^1_{\beta\pi}$*, we have*

$$\lim_{N\to\infty} \max_{t\in\mathbb{R}} \left| f(t) - \sum_{k=-N}^{N} f(t_k)\phi_k(t) \right| = 0.$$

- If oversampling is used the sampling series is globally uniformly convergent.

Local Convergence with Oversampling

- For $\mathcal{B}^{\infty}_{\beta\pi}$, $0 < \beta < 1$, we have local uniform convergence.

Theorem

Let ϕ be a function of sine type, whose zeros $\{t_k\}_{k\in\mathbb{Z}}$ are all real. Then, for all $T > 0$, $0 < \beta < 1$, and all $f \in \mathcal{B}^{\infty}_{\beta\pi}$, we have

$$\lim_{N\to\infty} \max_{t\in[-T,T]} \left| f(t) - \sum_{k=-N}^{N} f(t_k)\phi_k(t) \right| = 0.$$

Conjecture 1.1

Conjecture (Local Convergence Behavior for Complete Interpolating Sequences)

There exist a complete interpolating sequence $\{t_k^1\}_{k\in\mathbb{Z}}$, $f_1 \in \mathcal{PW}_\pi^1$, *and* $t_1 \in \mathbb{R}$, *such that*

$$\limsup_{N\to\infty} \left| f_1(t_1) - \sum_{k=-N}^{N} f_1(t_k)\phi_k^1(t_1) \right| = \infty.$$

Remark

If this conjecture is true, it shows that the zero sequences of sine-type functions have very nice properties.

Zeros of Sine Type Functions and Complete Interpolating Sequences

Theorem (Avdonin and Joó)

If $\{t_k\}_{k\in\mathbb{Z}} \subset \mathbb{R}$ *is a complete interpolating sequence then there exists* $d \in (0, 1/4)$ *and a sine-type function with zeros* $\{\mu_k\}_{k\in\mathbb{Z}}$ *such that*

$$d(t_{k-1} - t_k) \leqslant \mu_k - t_k \leqslant d(t_{k+1} - t_k)$$

for all $k \in \mathbb{Z}$*.*

Global Convergence Behavior without Oversampling

- Two positive results for $\mathcal{PW}_\pi^1$:
  1) local uniform convergence when no oversampling is used,
  2) global uniform convergence when oversampling is used.
- Global convergence behavior without oversampling?

- Recent result: For $\mathcal{PW}_\pi^1$ and a large class of reconstruction processes a globally bounded signal reconstruction is impossible if the samples are taken equidistantly at Nyquist rate.
- Non-equidistant sampling $\rightarrow$ additional degree of freedom, which may help to improve the convergence behavior.

A Subclass of the Functions of Sine Type

Definition

By $\mathcal{S}$ we denote the set of all entire functions ϕ with separated real zeros $\{t_k\}_{k\in\mathbb{Z}}$ that have a representation as Fourier-Stieltjes integral in the form

$$\phi(t) = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{i\omega t}\, d\mu(\omega),$$

where $\mu(\omega)$ is a real function of bounded variation on the interval $[-\pi, \pi]$ and has a jump discontinuity at each endpoint.

- $\mathcal{S}$ is a subclass of the functions of sine type.

Global Convergence Behavior without Oversampling

Theorem

Let ϕ be a function of sine type in $\mathcal{S}$, whose zeros $\{t_k\}_{k\in\mathbb{Z}}$ are all real. Then there exists a signal $f_1 \in \mathcal{PW}_\pi^1$ such that

$$\limsup_{N\to\infty} \max_{t\in\mathbb{R}} \left| f_1(t) - \sum_{k=-N}^{N} f_1(t_k)\phi_k(t) \right| = \infty.$$

Global Convergence Behavior of the Shannon Sampling Series without Oversampling

A special case of the previous theorem concerns the global convergence behavior of the Shannon sampling series:

Corollary

There exists a signal $f_1 \in \mathcal{PW}_\pi^1$ *such that*

$$\limsup_{N\to\infty} \max_{t\in\mathbb{R}} \left| f_1(t) - \sum_{k=-N}^{N} f_1(k) \frac{\sin(\pi(t-k))}{\pi(t-k)} \right| = \infty.$$

Conjecture 1.2

Conjecture (Global Convergence Behavior for Complete Interpolating Sequences)

For each complete interpolating sequences $\{t_k\}_{k\in\mathbb{Z}}$ *there exists a* $f_1 \in \mathcal{PW}_\pi^1$ *such that*

$$\limsup_{N\to\infty} \left(\sup_{t\in\mathbb{R}} \left| f_1(t) - \sum_{k=-N}^{N} f_1(t_k)\phi_k(t) \right| \right) = \infty.$$

I-Processes

Definition

We call a bandlimited wide-sense stationary process X I-process if its correlation function R_X has the representation

$$R_X(\tau) = \frac{1}{2\pi} \int_{-\pi}^{\pi} S_X(\omega)\, e^{i\omega\tau}\, d\omega,$$

for some non-negative $S_X \in L^1[-\pi, \pi]$.

Stochastic Processes: Local Convergence Behavior

Theorem

Let ϕ be a function of sine type, whose zeros $\{t_k\}_{k\in\mathbb{Z}}$ are all real. Then, for all I-processes X and all $T > 0$, we have

$$\lim_{N\to\infty} \max_{t\in[-T,T]} \mathbb{E}\left|X(t) - \sum_{k=-N}^{N} X(t_k)\phi_k(t)\right|^2 = 0.$$

- We have a good local convergence behavior for I-processes.

Stochastic Processes: Global Convergence Behavior without Oversampling

- There exist I-processes such that the global mean-square approximation error increases unboundedly.

Theorem

Let ϕ be a function of sine type in $\mathcal{S}$, whose zeros $\{t_k\}_{k\in\mathbb{Z}}$ are all real. Then there exists an I-process X_1 such that

$$\limsup_{N\to\infty} \sup_{t\in\mathbb{R}} \mathbb{E} \left| X_1(t) - \sum_{k=-N}^{N} X_1(t_k)\phi_k(t) \right|^2 = \infty.$$

Stochastic Processes: Global Convergence Behavior with Oversampling

- Similar to the deterministic case, oversampling improves the global convergence behavior of the series for I-processes.

Theorem

Let ϕ be a function of sine type, whose zeros $\{t_k\}_{k\in\mathbb{Z}}$ are all real. Then, for all $0 < \beta < 1$ and all I-processes X, whose power spectral density $S_X(\omega)$ is supported in $[-\beta\pi, \beta\pi]$, we have

$$\sup_{N\in\mathbb{N}} \sup_{t\in\mathbb{R}} \mathbb{E} \left| \sum_{k=-N}^{N} X(t_k)\phi_k(t) \right|^2 < \infty.$$

Idea of Sampling-Based Signal Processing

- In many applications the task is to reconstruct some transformation Tf of $f \in \mathcal{PW}_\pi^1$ and not f itself.

Key idea of sampling-based signal processing:

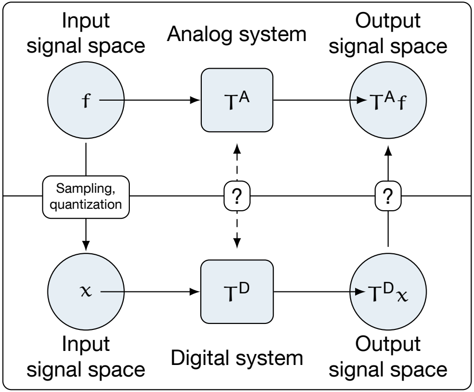
- Not the whole signal is used to calculate some transformation of the signal, but only the samples of the signal.
  $\rightarrow$ Calculate Tf from the samples of f
- Corresponds to the natural situation in digital signal processing, where only the samples of the signal are available.

The question:

- Is it always possible to calculate Tf from the samples of f?

Sampling-based signal processing should be potentially possible because f, as a bandlimited signal, is uniquely determined by its samples.

Analog World Versus Digital World



Analog world

Digital world

Stable Linear Time Invariant Systems

A linear system $T : \mathcal{PW}_\pi^1 \to \mathcal{PW}_\pi^1$ is called stable linear time invariant (LTI) system if:

- T is bounded, i.e., $\|T\| = \sup_{\|f\|_{\mathcal{PW}_\pi^1} \leqslant 1} \|Tf\|_{\mathcal{PW}_\pi^1} < \infty$ and
- T is time invariant, i.e., $\big(Tf(\cdot - a)\big)(t) = (Tf)(t - a)$ for all $f \in \mathcal{PW}_\pi^1$ and $t, a \in \mathbb{R}$.

The Hilbert transform H and the low-pass filter are stable LTI systems.

Example (Hilbert transform)

The Hilbert transform $\tilde{f}$ of a signal $f \in \mathcal{PW}_\pi^1$ is defined by

$$\tilde{f}(t) = (Hf)(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} -i \operatorname{sgn}(\omega) \hat{f}(\omega)\, e^{i\omega t}\, d\omega,$$

where sgn denotes the signum function.

Representation of Stable LTI Systems

- For every stable LTI system $T : \mathcal{PW}_\pi^1 \to \mathcal{PW}_\pi^1$ there is exactly one function $\widehat{h}_T \in L^\infty[-\pi, \pi]$ such that

$$(Tf)(t) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \widehat{h}_T(\omega) \widehat{f}(\omega) \, e^{i\omega t} \, d\omega$$

  for all $f \in \mathcal{PW}_\pi^1$, and the integral is absolutely convergent.
- Every $\widehat{h}_T \in L^\infty[-\pi, \pi]$ defines a stable LTI system $T : \mathcal{PW}_\pi^1 \to \mathcal{PW}_\pi^1$.

The operator norm $\|T\| := \sup_{\|f\|_{\mathcal{PW}_\pi^1} \leqslant 1} \|Tf\|_{\mathcal{PW}_\pi^1}$ is given by $\|T\| = \|\widehat{h}_T\|_\infty$.

System Approximation

System approximation process:

$$\sum_{k=-N}^{N} f(t_k)(T\phi_k)(t)$$

- $T : \mathcal{PW}_{\pi}^{1} \to \mathcal{PW}_{\pi}^{1}$ is a stable LTI system.
- $\phi_k \in \mathcal{PW}_{\pi}^{2}$, $k \in \mathbb{Z}$, are reconstruction functions.
- f is a signal in $\mathcal{PW}_{\pi}^{1}$.

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- f is a signal in $\mathcal{PW}_\pi^1$.

Theorem

Let $\{t_k\}_{k\in\mathbb{Z}} \subset \mathbb{R}$ be a complete interpolating sequence for $\mathcal{PW}_\pi^2$ and ϕ_k, $k \in \mathbb{Z}$, the corresponding reconstruction functions. Then, for all $t \in \mathbb{R}$ there exists a stable LTI system T_1 with continuous $\hat{h}_{T_1}$ and a signal $f_1 \in \mathcal{PW}_\pi^1$ such that

$$\limsup_{N\to\infty} \left| (T_1 f_1)(t) - \sum_{k=-N}^{N} f_1(t_k)(T_1\phi_k)(t) \right| = \infty.$$

Conjecture 2.1

Conjecture

The divergence remains even if oversampling is applied:

Let $\{t_k\}_{k\in\mathbb{Z}} \subset \mathbb{R}$ *be a complete interpolating sequence and* $0 < \beta < 1$. *Then, for all* $t \in \mathbb{R}$ *there exists a stable LTI system* T_1 *and a signal* $f_1 \in \mathcal{PW}^1_{\beta\pi}$ *such that*

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If this conjecture is true, it implies a no go result for sampling based signal processing.

General Sampling Functionals

- Sampling of the signal f corresponds to a point evaluation of f at the sampling points $\{t_k\}_{k\in\mathbb{Z}}$.
- It is also possible to consider more general linear functionals $c_k : \mathcal{PW}^1_\pi \to \mathbb{C}$, $k \in \mathbb{Z}$.
- For example, functionals that also take the signal values in the proximity of the sampling points into account.

New approximation process:

$$\sum_{k=-N}^{N} c_k(f)\,(T\phi_k)(t).$$

- In the classical sampling approach the functionals are given by $c_k(f) = f(t_k)$, $k \in \mathbb{Z}$.

Conjecture 2.2

For approximation processes that use the general evaluation functionals we have the following conjecture.

Conjecture

Let $\sigma < \pi$. There exists a sequence of continuous linear functionals c_k, $k \in \mathbb{Z}$, on $\mathcal{PW}_\pi^1$ such that for all stable LTI systems T and all $f \in \mathcal{PW}_\sigma^1$ we have

$$\lim_{N\to\infty} \sup_{t\in\mathbb{R}} \left| (Tf)(t) - \sum_{k=-N}^{N} c_k(f)\,(T\phi_k)(t) \right| = 0.$$

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If this conjecture is true, it shows that for all stable LTI systems a digital implementation is possible using general sampling functionals.

- It would be interesting to find suitable functionals.

Outline

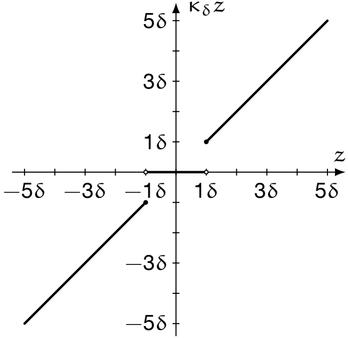
Signal Processing Under Quantization and Thresholding

- The principle of digital signal processing relies on the fact that certain bandlimited signals can be perfectly reconstructed from their samples.
- Reconstruction of the signal: $\{f(k)\}_{k \in \mathbb{Z}} \to f$
- Approximation of a transformation: $\{f(k)\}_{k \in \mathbb{Z}} \to Tf$
- Perfect reconstruction only possible if the sample values are known exactly.
- Not given in practical applications, because samples are disturbed (quantizers with limited resolution, thresholding effects).

The Threshold Operator Θ_δ

- The threshold operator Θ_δ sets all signal values, whose absolute value is smaller than some threshold $\delta > 0$ to zero.
- For continuous functions $f : \mathbb{R} \to \mathbb{C}$:

$$(\Theta_\delta f)(t) = \kappa_\delta f(t),\ t \in \mathbb{R}, \qquad \text{where} \qquad \kappa_\delta z = \begin{cases} z & |z| \geqslant \delta \\ 0 & |z| < \delta \end{cases}$$



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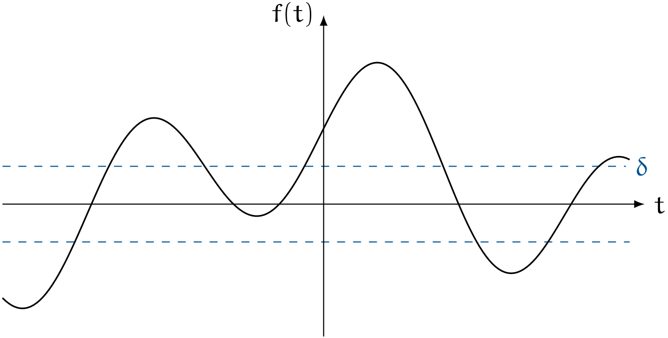
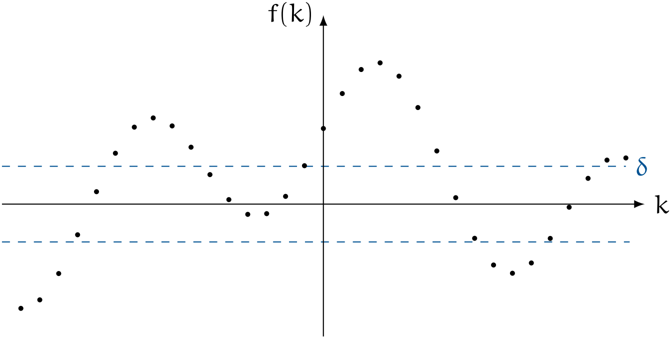


The Threshold Operator Θ_δ

- The threshold operator Θ_δ sets all signal values, whose absolute value is smaller than some threshold $\delta > 0$ to zero.
- For continuous functions $f : \mathbb{R} \to \mathbb{C}$:

$$(\Theta_\delta f)(t) = \kappa_\delta f(t),\ t \in \mathbb{R}, \qquad \text{where} \qquad \kappa_\delta z = \begin{cases} z & |z| \geqslant \delta \\ 0 & |z| < \delta \end{cases}$$



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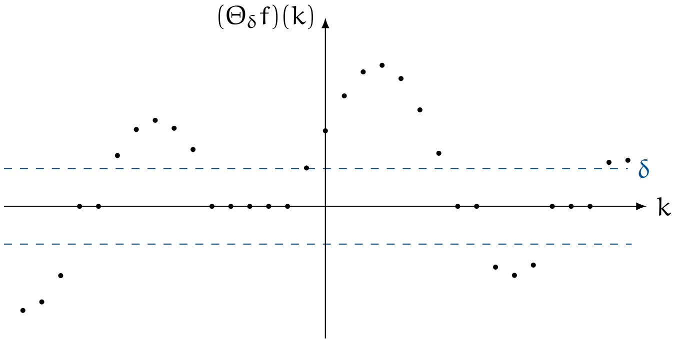
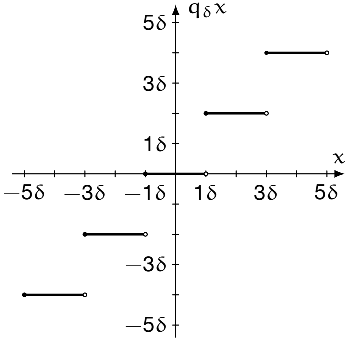


The Quantization Operator Υ_δ

- 2δ is the quantization step size
- For continuous functions $f : \mathbb{R} \to \mathbb{C}$:
  $(\Upsilon_\delta f)(t) = q_\delta f(t)$, $t \in \mathbb{R}$, where $q_\delta z = \left\lfloor \frac{\operatorname{Re} z}{2\delta} + \frac{1}{2} \right\rfloor 2\delta + \left\lfloor \frac{\operatorname{Im} z}{2\delta} + \frac{1}{2} \right\rfloor 2\delta i$



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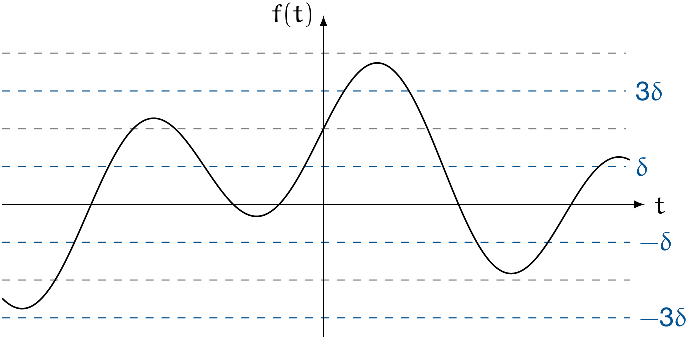

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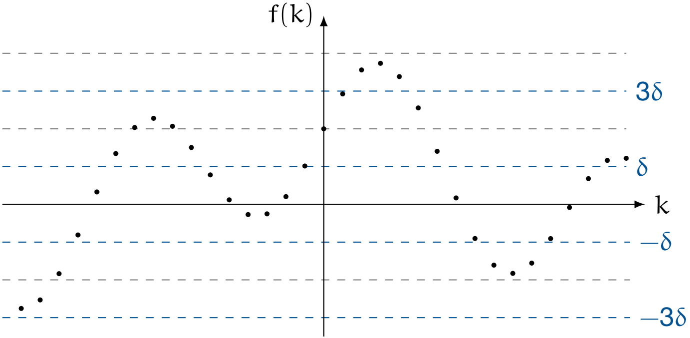

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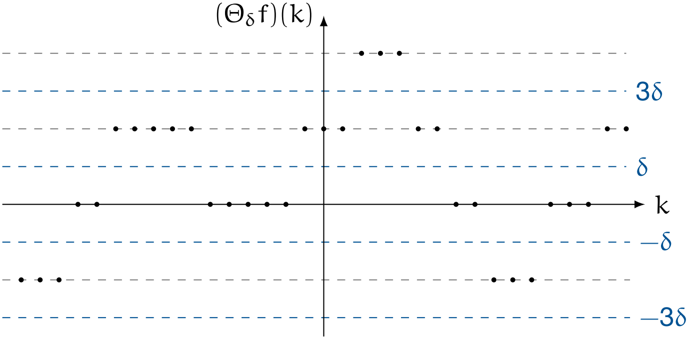

The Reconstruction Process A_δ

- The threshold operator is applied on the samples $\{f(k)\}_{k \in \mathbb{Z}}$ of signals $f \in \mathcal{PW}_\pi^1$.
- The resulting samples $\{(\Theta_\delta f)(k)\}_{k \in \mathbb{Z}}$ are used to build an approximation

$$(A_\delta f)(t) := \sum_{k=-\infty}^{\infty} (\Theta_\delta f)(k) \frac{\sin(\pi(t-k))}{\pi(t-k)} = \sum_{\substack{k=-\infty \\ |f(k)| \geqslant \delta}}^{\infty} f(k) \frac{\sin(\pi(t-k))}{\pi(t-k)}$$

  of the original signal f.
- We have $\lim_{|t| \to \infty} f(t) = 0$ (Riemann-Lebesgue lemma)
  $\Rightarrow$ the series has only finitely many summands
  $\Rightarrow A_\delta f \in \mathcal{PW}_\pi^2 \subset \mathcal{PW}_\pi^1$.

Since the series uses all ‘‘important’’ samples of the signal, one could expect $A_\delta f$ to have an approximation behavior similar to the Shannon sampling series.

Properties of the Reconstruction Process A_δ

1. For every $\delta > 0$, A_δ is a non-linear operator.
2. For every $\delta > 0$, the operator $A_\delta : (\mathcal{PW}_\pi^1, \|\cdot\|_{\mathcal{PW}_\pi^1}) \to (\mathcal{PW}_\pi^2, \|\cdot\|_{\mathcal{PW}_\pi^2})$ is discontinuous.
3. For some $f \in \mathcal{PW}_\pi^1$, the operator A_δ is also discontinuous with respect to δ.

The non-linearity of the threshold operator makes the analysis difficult.

Approximation of Stable LTI Systems

- In many applications the task is to reconstruct some transformation Tf of $f \in \mathcal{PW}_{\pi}^{1}$ and not f itself.
- The goal is to approximate the desired transformation Tf of a signal f by an approximation process, which uses only the samples of the signal that are disturbed by the threshold operator.

System Approximation under Thresholding

- If the samples $\{f(k)\}_{k\in\mathbb{Z}}$ are known perfectly we can use

$$\sum_{k=-N}^{N} f(k)\, T(\text{sinc}(\,\cdot - k))(t) = \sum_{k=-N}^{N} f(k) h_T(t-k)$$

  to obtain an approximation of Tf.

- Here: samples are disturbed. $\rightarrow$ Approximate Tf by

$$(T_\delta f)(t) := (TA_\delta f)(t) = \sum_{k=-\infty}^{\infty} (\Theta_\delta f)(k) h_T(t-k)$$

- Goal: small approximation error
  Since

$$|(T_\delta f)(t) - (Tf)(t)| \leqslant |(T_\delta f)(t)| + \|T\|\, \|f\|_{\mathcal{PW}_\pi^1}$$

  it is interesting how large $\sup_{\|f\|_{\mathcal{PW}_\pi^1}\leqslant 1} |(T_\delta f)(t)|$ can get.

Pointwise Stability

- The following theorem gives a necessary and sufficient condition for $\sup_{\|f\|_{\mathcal{PW}_\pi^1} \leqslant 1} |(T_\delta f)(t)|$ to be finite.

Theorem

Let T *be a stable LTI system,* $0 < \delta < 1/3$*, and* $t \in \mathbb{R}$*. Then we have*

$$\sup_{\|f\|_{\mathcal{PW}_\pi^1} \leqslant 1} |(T_\delta f)(t)| < \infty$$

if and only if

$$\sum_{k=-\infty}^{\infty} |h_T(t-k)| < \infty. \qquad (*)$$

- Note that (*) is nothing else than the BIBO stability condition for discrete-time systems.

Pointwise Convergence

Corollary

Let T *be a stable LTI system,* $0 < \delta < 1/3$*, and* $t \in \mathbb{R}$*. If*

$$\sum_{k=-\infty}^{\infty} |h_T(t-k)| < \infty \qquad (*)$$

then we have

$$\lim_{\delta \to 0} \sup_{f \in \mathcal{PW}_\pi^1} |(Tf)(t) - (T_\delta f)(t)| = 0.$$

- If (*) is fulfilled, then we have a good pointwise approximation behavior because the approximation error converges to zero as the threshold δ goes to zero.

Example: Ideal Low-Pass Filter

Even for common stable LTI systems like the ideal low-pass filter there are problems because (*) is not fulfilled.

Example

T_L: ideal low-pass filter, $h_{T_L}(t) = \sin(\pi t)/(\pi t)$
$\rightarrow \sum_{k=-\infty}^{\infty} |h_{T_L}(t-k)| = \infty$ for all $t \in \mathbb{R} \setminus \mathbb{Z}$

For $t \in \mathbb{R} \setminus \mathbb{Z}$ and $0 < \delta < 1/3$,

$$\sup_{\|f\|_{\mathcal{PW}_\pi^1} \leqslant 1} |(T_{L,\delta} f)(t)| = \sup_{\|f\|_{\mathcal{PW}_\pi^1} \leqslant 1} \left| \sum_{\substack{k=-\infty \\ |f(k)| \geqslant \delta}}^{\infty} f(k) \frac{\sin(\pi(t-k))}{\pi(t-k)} \right| = \infty.$$

Global Stability

We can also give a necessary and sufficient condition for the uniform boundedness on the whole real axis.

Theorem

Let T *be a stable LTI system and* $0 < \delta < 1/3$. *We have*

$$\sup_{\|f\|_{\mathcal{PW}_\pi^1} \leqslant 1} \|T_\delta f\|_\infty < \infty$$

if and only if

$$\sup_{0 \leqslant t \leqslant 1} \sum_{k=-\infty}^{\infty} |h_T(t-k)| < \infty$$

if and only if

$$\int_{-\infty}^{\infty} |h_T(\tau)| \, d\tau < \infty. \qquad (**)$$

- Note that (**) is nothing else than the BIBO stability condition for continuous-time systems.

Global Uniform Convergence

Corollary

Let T *be a stable LTI system and* $0 < \delta < 1/3$*. If*

$$\int_{-\infty}^{\infty} |h_T(\tau)|\, d\tau < \infty. \qquad (**)$$

then we have

$$\lim_{\delta \to \infty} \sup_{f \in \mathcal{PW}_{\pi}^{1}} \|Tf - T_{\delta} f\|_{\infty} = 0.$$

- This shows the good global approximation behavior of $T_{\delta} f$ if (**) is fulfilled.

Threshold Tending to Zero

Theorem

There exists a signal $f_1 \in \mathcal{PW}_\pi^1$ *such that*

$$\limsup_{\delta \to 0} \left| \sum_{\substack{k=-\infty \\ |f_1(k)| \geqslant \delta}}^{\infty} f_1(k) \frac{\sin(\pi(t-k))}{\pi(t-k)} \right| = \infty$$

for all $t \in \mathbb{R} \setminus \mathbb{Z}$*.*

Remark

Much more difficult behavior compared to the Shannon sampling series $\sum_{k=-N}^{N} f(k) \frac{\sin(\pi(t-k))}{\pi(t-k)}$
$\to$ local uniform convergence (Brown's theorem).

Conjecture 3.1

Conjecture

Let $\{t_k\}_{k\in\mathbb{Z}} \subset \mathbb{R}$ *be a complete interpolating sequence. Then, there exists a signal* $f_1 \in \mathcal{PW}_\pi^1$ *such that for all* $t \in \mathbb{R} \setminus \{t_k\}_{k\in\mathbb{Z}}$ *we have*

$$\limsup_{\delta\to 0} \left| \sum_{\substack{k=-\infty \\ |f_1(t_k)|\geqslant\delta}}^{\infty} f_1(t_k)\phi_k(t) \right| = \infty.$$

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Open Problems:

- Is a stable system implementation under quantization possible if oversampling is applied?
- If Conjecture 2.2 (general sampling functionals) is true, what are the consequences for quantization?

Outline

Introduction

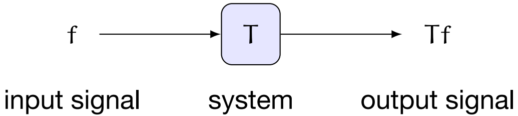
- Filters are a widely used tool in signal processing and system theory.
- Descriptive when the signals are treated in the frequency domain.
- Filters can be categorized according to their passband:
  - low-pass type, high-pass type, band-pass type, band-stop type
- All signals that have only frequencies within the passband are not disturbed by the filter.

We use the term system instead of filter because filters are often assumed to be linear and time-invariant, and we do not want to restrict our analysis a priori to systems with those properties.

The Analyzed Systems

- We analyze band-pass type systems operating on bounded bandlimited signals.
- Important in all applications where the peak value of the signal has to be controlled.
- In wireless communication systems the peak value of the transmitted signals has to be bounded by some constant in order that the power amplifier does not overload (clipping of the signal).

Efficient Band-Pass Type System



Band-pass type systems should be efficient in the sense that

P1) $\text{range}(T) \subseteq \mathcal{B}^{\infty}_{[\omega_1,\omega_2]}$,

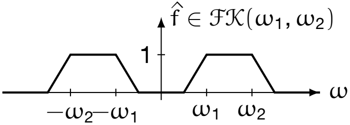
P2) $Tf = f$ for all $f \in \mathcal{B}^{\infty}_{[\omega_1,\omega_2]}$,

P3) $T : \mathcal{B}^{\infty}_{\pi} \to \mathcal{B}^{\infty}_{[\omega_1,\omega_2]}$ is bounded.

An Alternative Definition of Band-Pass Signals

Definition

For $0 \leqslant \omega_1 < \omega_2 < \infty$ let
$\mathcal{K}(\omega_1, \omega_2) = \left\{ f \in L^1(\mathbb{R}) : \hat{f}(\omega) = 1 \text{ for } |\omega| \in [\omega_1, \omega_2] \right\}$



Definition

The space $\mathcal{B}^\infty_{[\omega_1,\omega_2]}$ consists of all signals $f \in L^\infty(\mathbb{R})$ that fulfill $f(t) = \int_{-\infty}^{\infty} f(\tau) K(t-\tau)\, d\tau$ for all $t \in \mathbb{R}$ and all $K \in \mathcal{K}(\omega_1, \omega_2)$.

Note that we have $\mathcal{B}^\infty_{\omega_2} = \mathcal{B}^\infty_{[0,\omega_2]}$, according to this definition.

No Linear Realization of Efficient Band-Pass Type Systems

Theorem

Let $0 \leqslant \omega_1 < \omega_2 \leqslant \pi$ *with* $w_2 - w_1 < \pi$*. There exists no linear operator* T *defined on* $\mathcal{B}_\pi^\infty$ *with the properties*

1. $\text{range}(T) \subseteq \mathcal{B}_{[\omega_1,\omega_2]}^\infty$ *and*
2. $Tf = f$ *for all* $f \in \mathcal{B}_{[\omega_1,\omega_2]}^\infty$
3. $T : \mathcal{B}_\pi^\infty \to \mathcal{B}_{[\omega_1,\omega_2]}^\infty$ *is bounded*

Consequently, a linear realization of efficient band-pass type systems for the signal space $\mathcal{B}_\pi^\infty$ cannot exist.

- The result is very general, because there are many conceivable realizations.
- For example we do not restrict the systems to be time-invariant.

Non-Linear Realization of Efficient Band-Pass Type Systems

- Now we drop the requirement that the system is linear.
- The following theorem shows that a non-linear realization of efficient band-pass type systems is possible for the space $\mathcal{B}_\pi^\infty$.

Theorem

Let $0 \leqslant \omega_1 < \omega_2 \leqslant \pi$*. There exists an operator* T *defined on* $\mathcal{B}_\pi^\infty$ *with the properties*

1. $\mathrm{range}(T) \subseteq \mathcal{B}_{[\omega_1,\omega_2]}^\infty$
2. $Tf = f$ *for all* $f \in \mathcal{B}_{[\omega_1,\omega_2]}^\infty$*, and*
3. $\|Tf\|_\infty \leqslant 2\|f\|_\infty$ *for all* $f \in \mathcal{B}_\pi^\infty$*.*

Comparing Signals in the Frequency Domain

Definition

We say that $f \in \mathcal{B}_\pi^\infty$ and $g \in \mathcal{B}_\pi^\infty$ agree on the open frequency interval (ω_1, ω_2), $-\infty < \omega_1 < \omega_2 < \infty$, if

$$\int_{-\infty}^{\infty} f(\tau)h(\tau)\, d\tau = \int_{-\infty}^{\infty} g(\tau)h(\tau)\, d\tau$$

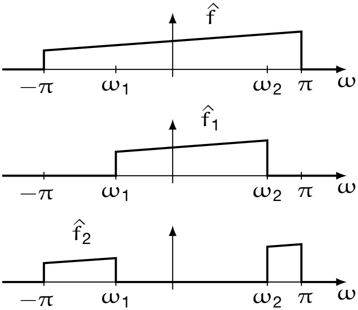
for all $h \in L^1(\mathbb{R})$ with $\hat{h}(\omega) = 0$ for all $\omega \in \mathbb{R} \setminus (\omega_1, \omega_2)$.

- For $f \in \mathcal{B}_\pi^2$ this definition is equivalent to the definition that uses the Fourier transform.
- Makes only a statement about what it means that two signals agree on open sets of frequencies.

Frequency Splitting for $\mathcal{B}_\pi^2$

- For $\mathcal{B}_\pi^2$ it is possible to split a signal with respect to its frequency content.
- The signal f_1 is given by

$$f_1(t) = \frac{1}{2\pi} \int_{\omega_1}^{\omega_2} \hat{f}(\omega)\, e^{i\omega t}\, d\omega.$$



For every signal $f \in \mathcal{B}_\pi^2$ and every frequency interval $[\omega_1, \omega_2]$ it is possible to split f into two signals $f_1 \in \mathcal{B}_\pi^2$ and $f_2 \in \mathcal{B}_\pi^2$ such that f agrees with f_1 on the frequency interval $[\omega_1, \omega_2]$ and with f_2 on the frequency interval $[-\pi, \pi] \setminus [\omega_1, \omega_2]$.

Frequency Splitting for $\mathcal{B}_\pi^\infty$?

Question

Given $f \in \mathcal{B}_\pi^\infty$. Can we find a decomposition $f = f_1 + f_2$ with $f_1 \in \mathcal{B}_{\omega_1}^\infty$, $0 < \omega_1 < \pi$, and $f_2 \in \mathcal{B}_\pi^\infty$, such that f and f_1 agree on the frequency interval $(-\omega_1, \omega_1)$?

If "yes":

- It would immediately follow that f_2 agrees with the zero function on the frequency interval $(-\omega_1, \omega_1)$ and that $f_2 \in \mathcal{B}_{[\omega_1,\pi]}^\infty$.
- f_1 would be the low-pass part of f, which agrees with f on the open frequency interval $(-\omega_1, \omega_1)$.
- f_2 would be the band-pass part of f, which agrees with f on the open set of frequencies $(-\pi, -\omega_1) \cup (\omega_1, \pi)$.

No Frequency Splitting for $\mathcal{B}_\pi^\infty$

Theorem

Let $0 < \omega_1 < \pi$*. There exists a signal* $f \in \mathcal{B}_{\pi,0}^\infty$ *such that there exists no signal* $f_1 \in \mathcal{B}_{\omega_1}^\infty$ *such that*

$$\int_{-\infty}^{\infty} f(\tau)h(\tau)\, d\tau = \int_{-\infty}^{\infty} f_1(\tau)h(\tau)\, d\tau$$

for all $h \in \mathcal{B}_{\omega_1}^1$*.*

- A frequency splitting is not possible for signals in $\mathcal{B}_\pi^\infty$.
- This signal theoretic result implies that there exists no filter—regardless of how complicated the realization is made—that can perform this task.
- Result is also true for the band-pass case.

Approximate Frequency Splitting

For $0 < \omega_1 < \infty$, $\delta > 0$, and $1 < \kappa < \infty$ let $\overline{\mathcal{K}}(\omega_1, \delta, \kappa)$ denote the set of all functions $K \in L^1(\mathbb{R})$, whose Fourier transform fulfills $\|\hat{K}\|_\infty \leqslant \kappa$, $\hat{K}(\omega) = 1$ for $|\omega| \leqslant \omega_1$, and $\hat{K}(\omega) = 0$ for $|\omega| > \omega_1 + \delta$.

For $K \in \overline{\mathcal{K}}(\omega_1, \delta, \kappa)$ we define the system $\Psi_K : \mathcal{B}^\infty_{\pi,0} \to \mathcal{B}^\infty_{\omega_1+\delta,0}$ by

$$(\Psi_K f)(t) = \int_{-\infty}^{\infty} f(\tau) K(t-\tau)\, d\tau.$$

- For signals in $\mathcal{B}^2_\pi$, $\hat{K}$ has the meaning of a transfer function.
- Relaxation of P1).

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Theorem

For all $0 < \omega_1 < \pi$ *and* $1 < \kappa < \infty$ *we have*

$$\liminf_{\delta \to 0} \inf_{K \in \overline{\mathcal{K}}(\omega_1, \delta, \kappa)} \|\Psi_K\| = \infty.$$

Conclusion

- We analyzed the convergence behavior of sampling series for signal reconstruction and system approximation.
- It was shown that quantization and thresholding impair the convergence of the sampling series.
- For the space $\mathcal{B}_\pi^\infty$, a linear realization of efficient band-pass type systems and frequency splitting are not possible

Happy birthday Hans!