Banach–Steinhaus Theory Revisited: Lineability, Spaceability, and Related Questions for Phase Retrieval Problems for Functions with Finite Energy

Holger Boche

joint work with Volker Pohl and Ullrich J. Mönich

Technische Universität München
Lehrstuhl für Theoretische Informationstechnik

March 16, 2016
Workshop on Harmonic Analysis, Graphs and Learning
Hausdorff Research Institute for Mathematics, Bonn

Outline

Introduction

- Hans Feichtinger's questions on convolution sum representations of stable systems for Paley–Wiener spaces
  - Signal space structure of signals/systems with "exceptional behavior"
- Standard phase retrieval problem in infinite dimensions
- Connections to Anders Hansen's talk on solvability complexity index and Curt McMullen's solution of Smale's conjecture

Curt McMullen, **Families of rational maps and iterative root-finding algorithms,** Annals of Mathematics, 125, pp. 467–493, 1987.

Outline

Approximation

General task: Approximate a bounded linear operator T by a sequence of operators $\{T_N\}_{N\in\mathbb{N}}$.

We consider the following setting:

- B_1, B_2: Banach spaces
- $T\colon B_1 \to B_2$: bounded linear operator
- $\{T_N\}_{N\in\mathbb{N}}$: sequence of bounded linear operators mapping from B_1 into B_2

Question: Does $T_N f$ converges to Tf in the norm of B_2 for all $f \in B_1$?

Banach–Steinhaus Theorem

Question: Does $T_N f$ converges to Tf in the norm of B_2 for all $f \in B_1$?

The Banach–Steinhaus theorem gives the answer.

Banach–Steinhaus Theorem

Question: Does $T_N f$ converges to Tf in the norm of B_2 for all $f \in B_1$?

The Banach–Steinhaus theorem gives the answer.

Answer is “yes” if and only if

1. there exists a constant C_2 such that $\|T_N\|_{B_1 \to B_2} \leqslant C_2$ for all $N \in \mathbb{N}$, and
2. $T_N f \to Tf$ for all f from a dense subspace of B_1.

Further important fact:
If 1 does not hold then we have have divergence for all functions from a residual subset of B_1 (condensation of singularities).

$$D_{UB} = \left\{ f \in B_1 \colon \limsup_{N \to \infty} \|T_N f\|_{B_2} = \infty \right\}$$

Set of Divergence Creating Functions

$$\mathrm{D}_{\mathsf{UB}} = \left\{ f \in \mathrm{B}_1 \colon \limsup_{N\to\infty} \|T_N f\|_{\mathrm{B}_2} = \infty \right\}$$

- Does the set D_{UB} have further interesting structural properties?
- Does it have a linear structure?

Lineability

Difficult to show linear structure for D_{UB}.

The set of convergence has always a linear structure, i.e., is a linear subspace:

- f_1, f_2 such that $T_N f_1$ and $T_N f_2$ converge
- $T_N(f_1 + f_2)$ converges

D_{UB} has no linear structure:

- g any functions such that $T_N g$ diverges
- $g_1 = f_1 + g$, $g_2 = f_1 - g$
- $T_N g_1$ and $T_N g_2$ diverge
- But $T_N(g_1 + g_2) = T_N(2f_1)$ converges

Basic Definition: Lineability and Spaceability

The zero function plays a special role.

Lineability:
A subset S of a Banach space X is said to be lineable if $S \cup \{0\}$ contains an infinite dimensional subspace.

Spaceability:
A subset S of a Banach space X is said to be spaceable if $S \cup \{0\}$ contains a closed infinite dimensional subspace of X.

Lineability for Approximation Processes

Extension of the Banach–Steinhaus theorem.

Theorem 1

Let B_1 *and* B_2 *be two Banach spaces and* $\mathrm{T}\colon \mathrm{B}_1 \to \mathrm{B}_2$ *a bounded linear operator. Further, let* $\{\mathrm{T}_N\}_{N\in\mathbb{N}}$ *be a sequence of bounded linear operators, mapping from* B_1 *into* B_2*, with:*

1. $\limsup_{N\to\infty} \|\mathrm{T}_N\|_{\mathrm{B}_1\to\mathrm{B}_2} = \infty$*, and*
2. *there exists a dense subset* $\mathcal{K}$ *of* B_1 *such that* $\lim_{N\to\infty} \|\mathrm{T}f - \mathrm{T}_N f\|_{\mathrm{B}_2} = 0$ *for all* $f \in \mathcal{K}$*.*

Then the set

$$\mathrm{D}_{UB} = \left\{ f \in \mathrm{B}_1\colon \limsup_{N\to\infty} \|\mathrm{T}_N f\|_{\mathrm{B}_2} = \infty \right\}$$

is lineable.

Remarks:

- The zero function is not in D_{UB}.
- There exists an infinite dimensional subspace $\mathcal{U}_{\mathrm{UB}} \subset \mathrm{D}_{\mathrm{UB}} \cup \{0\}$ such that $\limsup_{N\to\infty} \|\mathrm{T}_N f\|_{\mathrm{B}_2} = \infty$ for all $f \in \mathcal{U}_{\mathrm{UB}}$, $f \not\equiv 0$.

Key Lemma

Lemma

Let B_1 *and* B_2 *be two Banach spaces and* $T\colon B_1 \to B_2$ *a bounded linear operator. Further, let* $\{T_N\}_{N\in\mathbb{N}}$ *be a sequence of bounded linear operators, mapping from* B_1 *into* B_2*, with:*

1. $\limsup_{N\to\infty} \|T_N\|_{B_1\to B_2} = \infty$*, and*
2. *there exists a dense subset* $\mathcal{K}$ *of* B_1 *such that* $\lim_{N\to\infty} \|Tf - T_N f\|_{B_2} = 0$ *for all* $f \in \mathcal{K}$.

Then there exist a sequence of finitely linearly independent functions $\{\varphi_n\}_{n\in\mathbb{N}} \subset B_1$ *with*

3. $\|\varphi_n\|_{B_1} = 1$ *for all* $n \in \mathbb{N}$,

and a constant $C_3 > 0$*, such that for all* $n \in \mathbb{N}$ *there exists a sequence of natural numbers* $\{N_k(n)\}_{k\in\mathbb{N}}$ *with:*

4. $\limsup_{k\to\infty} \|T_{N_k(n)} \varphi_n\|_{B_2} = \infty$*, and*
5. $\sup_{k\in\mathbb{N}} \|T_{N_k(m)} \varphi_n\|_{B_2} \leqslant C_3$ *for all* $m \neq n$.

Sketch of Proof (Theorem 1)

Intuition of the proof:

Under the assumptions of the theorem, every Banach space is decomposable into two subspaces.

$$\mathcal{B} = \mathcal{V} + \mathcal{U}$$

$\mathcal{V}$: functions with convergence
$\mathcal{U}$: functions with divergence

The axiom of choice is used to show the decomposition.
In the decomposition $\mathcal{U}$ is not unique.

Using a Hamel basis for $\mathcal{U}$, we can show that for all elements from $\mathcal{U}$, except 0, we have unbounded divergence.
Employing the key lemma we show that $\mathcal{U}$ is infinite dimensional.

In general, $\mathcal{V}$ and $\mathcal{U}$ are not closed.

Spaceability for Approximation Processes

Theorem 2

Let B_1 *and* B_2 *be two Banach spaces and* $\mathrm{T}\colon \mathrm{B}_1 \to \mathrm{B}_2$ *a bounded linear operator. Further, let* $\{\mathrm{T}_\mathrm{N}\}_{\mathrm{N}\in\mathbb{N}}$ *be a sequence of bounded linear operators, mapping from* B_1 *into* B_2*, with:*

1. $\limsup_{\mathrm{N}\to\infty} \|\mathrm{T}_\mathrm{N}\|_{\mathrm{B}_1\to\mathrm{B}_2} = \infty$,
2. *there exists a dense subset* $\mathcal{K}$ *of* B_1 *such that* $\lim_{\mathrm{N}\to\infty} \|\mathrm{T}f - \mathrm{T}_\mathrm{N} f\|_{\mathrm{B}_2} = 0$ *for all* $f \in \mathcal{K}$*, and*
3. *there exists an infinite dimensional closed subspace* $\underline{\mathrm{B}}_1$ *of* B_1 *such that* $\sup_{\mathrm{N}\in\mathbb{N}} \|\mathrm{T}_\mathrm{N} f\|_{\mathrm{B}_2} \leqslant C_4 \|f\|_{\underline{\mathrm{B}}_1}$ *for all* $f \in \underline{\mathrm{B}}_1$.

Then, the set

$$\mathrm{D}_{UB} = \left\{ f \in \mathrm{B}_1\colon \limsup_{\mathrm{N}\to\infty} \|\mathrm{T}_\mathrm{N} f\|_{\mathrm{B}_2} = \infty \right\}$$

is spaceable.

Motivation from a Mathematical Point of View

- The Banach–Mazur theorem shows that every Banach space is isometrically embeddable in $\mathcal{C}[0,1]$.
- In general: For a given infinite dimensional Banach space $\mathcal{B}$, all isometrically isomorphic subspaces of $\mathcal{C}[0,1]$ contain non-smooth functions.
- Let $\mathcal{N}_{\mathsf{D}}$ be the set of nowhere differentiable functions in $\mathcal{C}[0,1]$. It is not difficult to see that every Banach space $\mathcal{B}$ is isometrically embeddable in $\mathcal{N}_{\mathsf{D}} \cup \{0\}$.
  $\Rightarrow \mathcal{N}_{\mathsf{D}} \cup \{0\}$ is spaceable and has a very rich structure.

General question:
$T_N : \mathcal{C}[0,1] \to \mathcal{C}[0,1]$ with the above properties according to the key lemma.
Can we have the same behavior for $\mathcal{D}_{\mathsf{UB}} \cup \{0\}$?

Sketch of Proof I

We use the basic fact that every infinite dimensional Banach space contains an infinite dimensional closed subspace with a Schauder basis.

$\Rightarrow$ There exists an infinite dimensional closed subspace $\underline{\underline{B}}_1$ of $\underline{B}_1$, and functions $\{\phi_n\}_{n\in\mathbb{N}} \subset \underline{\underline{B}}_1$, as well as continuous linear functionals $\{\Phi_n\}_{n\in\mathbb{N}} \subset \underline{\underline{B}}_1^*$, such that

$$f = \sum_{n=1}^{\infty} \Phi_n(f)\phi_n$$

for all $f \in \underline{\underline{B}}_1$, where the series converges in the B_1-norm.

The coefficient functionals $\{\Phi_n\}_{n\in\mathbb{N}}$ can be extended to continuous linear functionals $\{\Phi_n^{\text{ex}}\}_{n\in\mathbb{N}}$ defined on B_1 which satisfy $\|\Phi_n\|_{\underline{\underline{B}}_1^*} = \|\Phi_n^{\text{ex}}\|_{B_1^*}$, $n \in \mathbb{N}$.

Let $q_n = \max\{1, \|\Phi_n^{\text{ex}}\|_{B_1^*}\}$, and consider the functions

$$h_n = \phi_n + \frac{1}{2^{n+1} q_n}\varphi_n, \quad n \in \mathbb{N},$$

Sketch of Proof II

where $\{\varphi_n\}_{n\in\mathbb{N}}$ are the functions from the key lemma. We have

$$\sum_{n=1}^{\infty} \|\Phi_n^{\text{ex}}\|_{B_1^*} \|\phi_n - h_n\|_{B_1} = \sum_{n=1}^{\infty} \|\Phi_n^{\text{ex}}\|_{B_1^*} \left\| \frac{1}{2^{n+1} q_n} \varphi_n \right\|_{B_1}$$
$$= \sum_{n=1}^{\infty} \|\Phi_n^{\text{ex}}\|_{B_1^*} \frac{1}{2^{n+1} q_n} \leqslant \sum_{n=1}^{\infty} \frac{1}{2^{n+1}} = \frac{1}{2} < 1.$$

$\Rightarrow \{h_n\}_{n\in\mathbb{N}}$ is a basic sequence in B_1 that is equivalent to $\{\phi_n\}_{n\in\mathbb{N}}$.

Let $\quad D_1 = \left\{ f \in B_1 \colon \exists \{\alpha_n\}_{n\in\mathbb{N}} \subset \mathbb{C} \text{ with } \lim_{N\to\infty} \left\| f - \sum_{n=1}^{N} \alpha_n h_n \right\|_{B_1} = 0 \right\}.$

D_1 is a closed subspace of B_1.

Sketch of Proof III

For $f \in D_1$ we have

$$f = \sum_{n=1}^{\infty} \alpha_n h_n = \underbrace{\sum_{n=1}^{\infty} \alpha_n \phi_n}_{=g} + \underbrace{\sum_{n=1}^{\infty} \alpha_n \frac{1}{2^{n+1} q_n} \varphi_n}_{=v}. \tag{1}$$

Note that $g \in \underline{\underline{B}}_1$. Using (1) we obtain for $N \in \mathbb{N}$ that

$$T_N f = T_N g + T_N v$$

and

$$\|T_N f - T_N v\|_{B_2} = \|T_N g\|_{B_2} \leqslant C_4 \|g\|_{\underline{\underline{B}}_1}, \tag{2}$$

where the last inequality follows from assumption 3 and $\underline{\underline{B}}_1 \subset \underline{B}_1$. Let n_0 be the smallest index such that $\alpha_{n_0} \neq 0$. Then we have

$$T_N v = \frac{\alpha_{n_0}}{2^{n_0+1} q_{n_0}} T_N \varphi_{n_0} + \sum_{n=n_0+1}^{\infty} \frac{\alpha_n}{2^{n+1} q_n} T_N \varphi_n. \tag{3}$$

Sketch of Proof IV

We have

$$\left\| \sum_{n=n_0+1}^{\infty} \frac{\alpha_n}{2^{n+1} q_n} T_{N_k^{n_0}} \varphi_n \right\|_{B_2} \leqslant \frac{C_3}{2} \|g\|_{\underline{\underline{B}}_1} \tag{4}$$

for all $k \in \mathbb{N}$. From (3) and (4) we see that

$$\left\| T_{N_k^{n_0}} v - \frac{\alpha_{n_0}}{2^{n_0+1} q_{n_0}} T_{N_k^{n_0}} \varphi_{n_0} \right\|_{B_2} \leqslant \frac{C_3}{2} \|g\|_{\underline{\underline{B}}_1} \tag{5}$$

for all $k \in \mathbb{N}$. Combining (2) and (5), it follows that

$$\|T_{N_k^{n_0}} f\|_{B_2} \geqslant \frac{|\alpha_{n_0}|}{2^{n_0+1} q_{n_0}} \|T_{N_k^{n_0}} \varphi_{n_0}\|_{B_2} - C_4 \|g\|_{\underline{\underline{B}}_1} - \frac{C_3}{2} \|g\|_{\underline{\underline{B}}_1}$$

for all $k \in \mathbb{N}$.

General Remarks on Spaceability

- It is not clear whether the spaceability extension of the Banach–Steinhaus theory is also valid in a general setting, i.e., without assumption 3.
- The key lemma seems to be to weak to prove spaceability in the general setting.
- A better understanding of the geometry of the spaces is necessary. Most constructions according to the proof of Theorem 2 result in a Banach space which is decomposable into two infinite dimensional complemented subspaces $\mathrm{B} = W_1 \oplus W_2$.
- However, according to Gowers' dichotomy theorem, every Banach space B has a subspace which either has an unconditional basis or is hereditarily indecomposable.

T. W. Gowers, **An Infinite Ramsey Theorem and Some Banach-Space Dichotomies,** Annals of Mathematics, 156, pp. 797–833, 2002.

Outline

Signal Spaces

Definition (Paley–Wiener Space)

For $1 \leqslant p \leqslant \infty$ we denote by $\mathcal{PW}_\sigma^p$ the Paley-Wiener space of functions f with a representation $f(z) = \frac{1}{2\pi} \int_{-\sigma}^{\sigma} g(\omega)\, e^{iz\omega}\, d\omega$, $z \in \mathbb{C}$, for some $g \in L^p[-\sigma, \sigma]$.

The norm for $\mathcal{PW}_\sigma^p$ is given by $\|f\|_{\mathcal{PW}_\sigma^p} = \left(\frac{1}{2\pi} \int_{-\sigma}^{\sigma} |\hat{f}(\omega)|^p \, d\omega \right)^{1/p}$.

Properties:

- $\mathcal{PW}_\sigma^p \supset \mathcal{PW}_\sigma^s$ for $1 \leqslant p < s \leqslant \infty$
- $\|f\|_\infty \leqslant \|f\|_{\mathcal{PW}_\sigma^1}$
- $\mathcal{PW}_\sigma^2$ is the space of bandlimited functions with finite $L^2(\mathbb{R})$-norm (finite energy).

Without loss of generality, we can restrict to $\sigma = \pi$.

Stable Linear Time Invariant Systems

A linear system $T : \mathcal{PW}_\pi^p \to \mathcal{PW}_\pi^p$ is called stable linear time invariant (LTI) system if:

- T is bounded, i.e., $\|T\| = \sup_{\|f\|_{\mathcal{PW}_\pi^p} \leqslant 1} \|Tf\|_{\mathcal{PW}_\pi^p} < \infty$ and
- T is time invariant, i.e., $\big(Tf(\cdot - a)\big)(t) = (Tf)(t-a)$ for all $f \in \mathcal{PW}_\pi^p$ and $t, a \in \mathbb{R}$.

By $\mathcal{T}$ we denote the space of all stable LTI systems.

The Hilbert transform H and the low-pass filter are stable LTI systems.

Example (Hilbert transform)

The Hilbert transform Hf of a signal $f \in \mathcal{PW}_\pi^1$ is defined by

$$(Hf)(t) = \frac{1}{2\pi} \int_{-\pi}^{\pi} -i \operatorname{sgn}(\omega) \hat{f}(\omega)\, e^{i\omega t} \, d\omega,$$

where sgn denotes the signum function.

Finite Energy and Mixed Signal Representation

Approximate the system output Tf from the samples of f:

$$(Tf)(t) = \sum_{k=-\infty}^{\infty} f(k) h_T(t-k).$$

Mixed signal representation:

$$(Tf)(t) \stackrel{?}{=} \sum_{k=-\infty}^{\infty} f(t-k) h_T(k).$$

Finite Energy and Mixed Signal Representation

For all $f \in \mathcal{PW}_{\pi}^{2}$

$$\sum_{k=-\infty}^{\infty} f(k) h_T(t-k)$$

converges in $L^2(\mathbb{R})$ and consequently globally uniformly,

and we have

$$\lim_{N\to\infty} \max_{t\in\mathbb{R}} \left| (Tf)(t) - \sum_{k=-\infty}^{\infty} f(t-k) h_T(k) \right| = 0.$$

for the mixed signal representation.

Finite Energy and Mixed Signal Representation

Theorem 3

There exist an infinite dimensional closed subspace $\mathrm{D}_{sig}^{(2)} \subset \mathcal{PW}_\pi^2$ *and an infinite dimensional closed subspace* $\mathrm{D}_{sys}^{(2)} \subset \mathcal{T}$ *such that for all* $\mathrm{f} \in \mathrm{D}_{sig}^{(2)}$, $\mathrm{f} \not\equiv 0$, *and all* $\mathrm{T} \in \mathrm{D}_{sys}^{(2)}$, $\mathrm{T} \not\equiv 0$, *we have*

$$\limsup_{\mathrm{N} \to \infty} \int_{-\infty}^{\infty} \left| \sum_{\mathrm{k}=-\mathrm{N}}^{\mathrm{N}} \mathrm{f}(\mathrm{t}-\mathrm{k}) \mathrm{h}_{\mathrm{T}}(\mathrm{k}) \right|^2 \mathrm{d}t = \infty.$$

All $\mathrm{T} \in \mathrm{D}_{sys}^{(2)}$ *are such that* $\widehat{\mathrm{h}}_{\mathrm{T}}$ *is continuous.*

Representation of Stable LTI Systems for $\mathcal{PW}_\pi^1$

- For every stable LTI system $T : \mathcal{PW}_\pi^1 \to \mathcal{PW}_\pi^1$ there is exactly one function $\hat{h}_T \in L^\infty[-\pi, \pi]$ such that

$$(Tf)(t) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \hat{h}_T(\omega) \hat{f}(\omega) \, e^{i\omega t} \, d\omega$$

  for all $f \in \mathcal{PW}_\pi^1$, and the integral is absolutely convergent.
- Every $\hat{h}_T \in L^\infty[-\pi, \pi]$ defines a stable LTI system $T : \mathcal{PW}_\pi^1 \to \mathcal{PW}_\pi^1$.
- $h_T = T \,\text{sinc}$

The operator norm $\|T\| := \sup_{\|f\|_{\mathcal{PW}_\pi^1} \leqslant 1} \|Tf\|_{\mathcal{PW}_\pi^1}$ is given by $\|T\| = \|\hat{h}_T\|_\infty$.

System Approximation

Convergence of $\sum_{k=-\infty}^{\infty} f(k) h_T(t-k)$ is problematic.

Theorem 4

For all $t \in \mathbb{R}$ *there exist a stable LTI system* $T \in \mathcal{T}$ *and a signal* $f \in \mathcal{PW}_\pi^1$ *such that*

$$\limsup_{N \to \infty} \left| \sum_{k=-N}^{N} f(k) h_T(t-k) \right| = \infty.$$

Even true for oversampling and arbitrary choice of the reconstruction kernel.

System Approximation

We know:

There exist a stable LTI system $\mathrm{T} \in \mathcal{T}$ and a signal $f \in \mathcal{PW}_\pi^1$ such that

$$\limsup_{N\to\infty} \left| \sum_{k=-N}^{N} f(k) h_T(t-k) \right| = \infty. \tag{*}$$

Questions:

- What can we say about the structure of the set of systems $\mathrm{T} \in \mathcal{T}$ and the set of signals $f \in \mathcal{PW}_\pi^1$ such that (*) is true.
- Do both sets contain an infinite dimensional closed linear subspace?

Joint Spaceability for System Approximation

Theorem 5

Let $t \in \mathbb{R}$ *be arbitrary but fixed. There exist an infinite dimensional closed subspace* $\mathcal{D}_{sig} \subset \mathcal{PW}_\pi^1$ *and an infinite dimensional closed subspace* $\mathcal{D}_{sys} \subset \mathcal{T}$ *such that for all* $f \in \mathcal{D}_{sig}$, $f \not\equiv 0$, *and all* $T \in \mathcal{D}_{sys}$, $T \not\equiv 0$, *we have*

$$\limsup_{N\to\infty} \left| \sum_{k=-N}^{N} f(k) h_T(t-k) \right| = \infty.$$

All $T \in \mathcal{D}_{sys}$ *are such that* $\widehat{h}_T$ *is continuous.*

Joint spaceability: For any pair of signal and system $(f, T) \in \mathcal{D}_{\text{sig}} \times \mathcal{D}_{\text{sys}}$, $f \not\equiv 0$, $T \not\equiv 0$, we have divergence.

Joint Spaceability for Mixed-Signal Representation

The previous result implies joint spaceability for the mixed signal representation.

Corollary

Let $t \in \mathbb{R}$ be arbitrary but fixed. There exist an infinite dimensional closed subspace $D_{sig} \subset \mathcal{PW}_\pi^1$ and an infinite dimensional closed subspace $D_{sys} \subset \mathcal{T}$ such that for all $f \in D_{sig}$, $f \not\equiv 0$, and all $T \in D_{sys}$, $T \not\equiv 0$, we have

$$\limsup_{N\to\infty} \left| \sum_{k=-N}^{N} f(t-k) h_T(k) \right| = \infty.$$

All $T \in D_{sys}$ are such that $\widehat{h}_T$ is continuous.

Outline

Hilbert Transformation and Causality

Let $\{a_n\}_{n=0}^{\infty}$ be a *causal sequence* in $\ell^2(\mathbb{Z}_+)$ then

$$f(z) = \sum_{n=0}^{\infty} a_n z^n = u(z) + \mathrm{i}\, v(z)\,, \qquad |z| < 1$$

belongs to the *Hardy space* $\mathrm{H}^2(\mathbb{D})$ of analytic functions in $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ with

$$\|f\|_2 = \left(\sup_{0<r<1} \frac{1}{2\pi} \int_{-\pi}^{\pi} \left| f(r e^{\mathrm{i}\theta}) \right|^2 d\theta \right)^{1/2} < \infty\,.$$

- u, v are real and harmonic in $\mathbb{D}$
- the boundary functions $f, u, v \in L^2(\partial\mathbb{D})$ exist

$$f(e^{\mathrm{i}\theta}) = \lim_{r\to 1} f(r\, e^{\mathrm{i}\theta}) \quad , \quad u(e^{\mathrm{i}\theta}) = \lim_{r\to 1} u(r\, e^{\mathrm{i}\theta}) \quad , \quad v(e^{\mathrm{i}\theta}) = \lim_{r\to 1} v(r\, e^{\mathrm{i}\theta})$$

- v is the *harmonic conjugate* of u: $v = \widetilde{u}$ (*Kramers-Kronig-Relation*)
- $\widetilde{u}$ is given as the *Hilbert transform* of u:

$$\widetilde{u}(e^{\mathrm{i}\theta}) = \big(\mathrm{H}u\big)(e^{\mathrm{i}\theta}) = \lim_{\epsilon\to 0} \frac{1}{2\pi} \int_{\epsilon \leqslant |\tau| \leqslant \pi} \frac{f(e^{\mathrm{i}(\theta-\tau)})}{\tan(\tau/2)}\, d\tau$$

Phase Retrieval using Hilbert Transform

Problem

Let $f \in H^2(\mathbb{D})$ *with* $f(z) \neq 0$ *for all* $z \in \mathbb{D}$.

Assume that only magnitude measurements $\left|f(e^{i\theta})\right|$, $\theta \in [-\pi, \pi)$ *are known.*

Goal: Recover the phase information of f.

- write $f(e^{i\theta}) = \left|f(e^{i\theta})\right| e^{i\phi(\theta)}$
- then $\log f(e^{i\theta}) = \log\left|f(e^{i\theta})\right| + i\,\phi(\theta)$ is analytic in $\mathbb{D}$
- recover the phase ϕ as the Hilbert transform of $\log|f|$:

$$\phi(\theta) = \big(H[\log|f|]\big)(e^{i\theta}) = \lim_{\epsilon \to 0} \frac{1}{2\pi} \int_{\epsilon \leqslant |\tau| \leqslant \pi} \frac{\log\left|f(e^{i(\theta-\tau)})\right|}{\tan(\tau/2)}\, d\tau$$

In principle, the phase retrieval can be solved using the Hilbert transform!
Is this approach also practically realizable?

Hilbert Transform – Basic Properties

- L^p-Theory
  - ▷ $H : L^1(\partial\mathbb{D}) \to$ weak $L^1(\partial\mathbb{D})$ (*Kolmogoroff*)
  - ▷ $H : L^p(\partial\mathbb{D}) \to L^p(\partial\mathbb{D})$, $1 < p < \infty$
  - ▷ $H : L^\infty(\partial\mathbb{D}) \to \mathrm{BMO}$ (*Ch. Fefferman* & *E. M. Stein*)
  - ▷ $H : H^1 \to H^1$ (*L. Carleson* & *E. M. Stein*)
  - ▷ H^1–BMO Duality (*Ch. Fefferman*)

- Hilbert transform on $\mathcal{C}(\partial\mathbb{D})$
  - ▷ $H : \mathcal{C}(\partial\mathbb{D}) \to L^p(\partial\mathbb{D}),\ 1 \leqslant p < \infty$
  - ▷ $H : \mathcal{C}(\partial\mathbb{D}) \nrightarrow \mathcal{C}(\partial\mathbb{D})$
  - ▷ $H : \mathcal{C}(\partial\mathbb{D}) \to \mathrm{VMO}$ (*Ch. Fefferman*)

J.B. Garnett **Bounded analytic functions** Academic Press, New York, 1981.

Signal Space for Hilbert Transform Approximations

We consider the Hilbert transform on the Banach space $\mathcal{B}$ of all *continuous functions* on the unit circle $\partial\mathbb{D} := \{z \in \mathbb{C} \;:\; |z| = 1\}$ *with continuous conjugate*

$$\mathcal{B} := \left\{ f \in \mathcal{C}(\partial\mathbb{D}) : \widetilde{f} = Hf \in \mathcal{C}(\partial\mathbb{D}) \right\}$$

equipped with the norm

$$\left\|f\right\|_{\mathcal{B}} := \max\left\{\|f\|_\infty, \|Hf\|_\infty\right\} \qquad \text{with} \qquad \left\|f\right\|_\infty = \max_{\theta\in[-\pi,\pi)} \left|f(e^{i\theta})\right| .$$

Dirichlet Spaces

- Let f be an analytic function in $\mathbb{D}$ with finite Dirichlet integral

$$D(f) = \frac{1}{\pi} \iint\limits_{|z|<1} \left| f'(z) \right|^2 \mathrm{d}z = \sum_{n=1}^{\infty} n \left| \widehat{f}_n \right|^2 < \infty \,.$$

- Write $f = u + i\widetilde{u}$ with the real harmonic function u and $\widetilde{u} = \mathrm{H}u$. Then

$$D(u) = \frac{1}{2\pi} \iint\limits_{|z|<1} \left\| (\operatorname{grad} u)(z) \right\|_{\mathbb{R}^2}^2 \mathrm{d}z = 2 \sum_{n=1}^{\infty} n \left| \widehat{u}_n \right|^2 < \infty \,.$$

- We define the analytic and harmonic Dirichlet spaces by

$$\begin{aligned} \mathcal{D}_a &= \big\{ f \, : \, f \text{ analytic in } \mathbb{D}, & \text{with } D(f) < \infty \big\} \\ \mathcal{D}_h &= \big\{ u \, : \, u \text{ harmonic in } \mathbb{D}, & \text{with } D(u) < \infty \big\} \end{aligned}$$

with the Sobolev-$\mathcal{H}^{\frac{1}{2}}(\partial\mathbb{D})$ norms

$$\|f\|_{\mathcal{H}^{1/2}} = \left(\widehat{f}_0^2 + D(f)^2 \right)^{\frac{1}{2}} \quad \text{and} \quad \|u\|_{\mathcal{H}^{1/2}} = \left(\widehat{u}_0^2 + D(u)^2 \right)^{\frac{1}{2}}$$

Functions with Finite Dirichlet Energy

Problem (Dirichlet problem)

For $1 < p \leqslant \infty$*, assume that* $u_0 \in L^p(\partial\mathbb{D})$ *is given. Find a potential* u *such that*

$$(\Delta u)(z) = 0 \quad \textit{for all } z \in \mathbb{D}$$

$$u(\zeta) = u_0(\zeta) \quad \textit{for almost all } \zeta \in \partial\mathbb{D}$$

$$\sup_{0<r<1} \left(\frac{1}{2\pi}\int_{-\pi}^{\pi} \left|u(re^{i\theta})\right|^p \, d\theta\right)^{1/p} < \infty \, .$$

Dirichlet's Principle – Solution minimizes the field energy

$$\min D(u) \quad \text{s.t.} \quad u \text{ is harmonic in } \mathbb{D}$$
$$u(\zeta) = u_0(\zeta) \text{ for all } \zeta \in \partial\mathbb{D}$$

The Dirichlet Problem has a solution only if $\|u_0\|_{\mathcal{H}^{1/2}} < \infty$.
$\mathcal{D}_h$ is the set of all finite energy solutions of a Dirichlet problem.

Approximation Problem

Continuous functions of finite energy

$$\mathcal{B}^{1/2} = \mathcal{B} \cap \mathcal{D}_h \qquad \text{with} \qquad \|f\|_{\mathcal{B}^{1/2}} = \max\left(\|f\|_{\mathcal{B}}, \|f\|_{\mathcal{H}^{1/2}}\right)$$

The set of all functions f which are harmonic in $\mathbb{D}$ and such that

- f is continuous on $\partial\mathbb{D}$
- f has a continuous conjugate $\widetilde{f}$
- the Dirichlet energy $D(f)$ is finite

Goal

Find a (realizable) sequence $\{H_N\}_{N\in\mathbb{N}}$ of bounded linear operators $H_N : \mathcal{B} \to \mathcal{B}$ such that

$$\lim_{N\to\infty} \left\|H_N f - \widetilde{f}\right\|_\infty = \lim_{N\to\infty} \left\|H_N f - Hf\right\|_\infty = 0 \qquad \text{for all } f \in \mathcal{B}^{1/2} .$$

Example: Approximation from Frequency Samples

Given $\mathrm{f} \in \mathcal{B}$ be arbitrary, and let $\{\widehat{f}_n\}_{n\in\mathbb{Z}}$ be its *Fourier coefficients*

$$\widehat{f}_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(e^{i\theta})\, e^{in\theta}\ d\theta\,, \qquad n \in \mathbb{Z}\,.$$

Consider the Nth-order *Fejér mean*

$$(F_N f)(e^{i\theta}) = \sum_{n=-N}^{N}\left(1 - \frac{|n|}{N}\right)\widehat{f}_n\, e^{in\theta} = \frac{N}{2\pi}\int_{-\pi}^{\pi} f(e^{i\theta})\,\mathcal{F}_N(t-\theta)\ d\theta$$

and define $\widetilde{F}_N := HF_N$.

Theorem 6

$$\lim_{N\to\infty}\left\|\widetilde{F}_N f - \widetilde{f}\right\|_\infty = 0 \qquad \textit{for all} \qquad f \in \mathcal{B}\,.$$

Proof:

$$\left\|\widetilde{F}_N f - \widetilde{f}\right\|_\infty = \left\|HF_N f - \widetilde{f}\right\|_\infty = \left\|\widetilde{F_N f} - \widetilde{f}\right\|_\infty = \left\|F_N \widetilde{f} - \widetilde{f}\right\|_\infty\,.$$

Practical Constraints on Approximation Sequences

- ▷ Previously constructed operators $\{\widetilde{F}_N\}_{N\in\mathbb{N}}$ are based on the *exact knowledge* of the Fourier coefficients $\{\widehat{f}_n\}_{n=-N}^{N}$.
- ▷ Equivalently, these operators are based on the knowledge of $f \in \mathcal{B}$ on the whole unit circle $\partial\mathbb{D}$.

$\Rightarrow$ Analog computers/devices are needed for implementation.

- ↯ Practical applications $\Rightarrow$ digital signal processing.
- ↯ Signals f are only known on finite number of sampling points $\{f(\zeta_m)\}_{m=1}^{M}$.
- ↯ Previous approximation sequence $\{\widetilde{F}_N\}_{N\in\mathbb{N}}$ can not be implemented.

$\Rightarrow$ Consider approximation sequences $\{H_N\}_{N\in\mathbb{N}}$ which are based on sampled data.

Axiomatic Properties of Approximation Sequences

The properties of the investigated approximation sequences $\{H_N\}_{N\in\mathbb{N}}$ of bounded linear operators are described by two axioms:

(A) Concentration on a finite sampling set: For every $N \in \mathbb{N}$ there exists a finite sampling set $Z_N = \{\zeta_n : n = 1, \dots, M_N\} \subset \partial\mathbb{D}$ such that

$$f(\zeta_n) = g(\zeta_n) \qquad \text{for all } \zeta_n \in Z_N$$

implies

$$\big(H_N f\big)(\zeta) = \big(H_N g\big)(\zeta) \qquad \text{for all } \zeta \in \partial\mathbb{D}\,.$$

(B) Weak convergence on $\mathcal{B}$: For every $f \in \mathcal{B}$, the sequence $\{H_N f\}_{N\in\mathbb{N}}$ converges weakly to Hf, i.e.

$$\lim_{N\to\infty} \big\langle H_N f, \varphi \big\rangle_2 = \big\langle Hf, \varphi \big\rangle_2 \qquad \text{for all } \varphi \in \mathcal{C}^\infty(\partial\mathbb{D})\,.$$

Consequences & Properties

Lemma

A sequence $\{H_N\}_{N\in\mathbb{N}}$ *satisfies Axiom (A) if and only if to every* $N \in \mathbb{N}$ *there exists a finite sampling set*

$$Z_N = \left\{\zeta_{1,N}, \lambda_{2,N}, \dots, \lambda_{M_N,N}\right\} \subset \partial\mathbb{D} \qquad \textit{with} \qquad M_N \in \mathbb{N}$$

and functions $\{h_{n,N} \; : \; n = 1, \dots, M_N\} \subset \mathcal{B}$ *such that*

$$\big(H_N f\big)(\zeta) = \sum_{n=1}^{M_N} f(\zeta_{n,N})\, h_{n,N}(\zeta) \qquad \textit{for all } f \in \mathcal{B}\,.$$

Lemma

Let $\{H_N\}_{N\in\mathbb{N}}$ *be a sequence which satisfies Axioms (A) and (B). Then every* $f \in \mathcal{C}_0(\partial\mathbb{D}) = \{f \in \mathcal{C}(\partial\mathbb{D}) \; : \; \widehat{f}_0 = 0\}$ *satisfies*

$$\lim_{N\to\infty} \big\langle H_N f, \varphi \big\rangle_2 = \big\langle Hf, \varphi \big\rangle_2 \qquad \textit{for all } \varphi \in \mathcal{C}^\infty(\partial\mathbb{D})\,.$$

Example – Sampled (Conjugate) Fejér Mean

- Consider again the N-th order *Fejér mean* of $f \in \mathcal{B}$

$$\big(F_N f\big)(e^{i\theta}) = \frac{N}{2\pi} \int_{-\pi}^{\pi} f(e^{i\theta})\, \mathcal{F}_N(\theta - \tau)\, d\tau \tag{Δ}$$

  with the so-called *Fejér kernel*

$$\mathcal{F}_N(\tau) = \left(\frac{\sin(N\tau/2)}{N\, \sin(\tau/2)} \right)^2 .$$

- Approximate the integral in (Δ) by its Riemann sum based on the rectangular integration rule yields the *sampled Fejér mean*

$$\big(S_N f\big)(e^{i\theta}) = \sum_{n=0}^{N-1} f\big(e^{i\, n\, 2\pi/N}\big)\, \mathcal{F}_N\big(\theta - n\tfrac{2\pi}{N}\big) \approx (F_N f)(e^{i\theta}) .$$

  It show the same approximation behavior as (Δ):

$$\lim_{N\to\infty} \big\| S_N f - f \big\|_\infty = \lim_{N\to\infty} \big\| F_N f - f \big\|_\infty = 0 \qquad \text{for all } f \in \mathcal{C}(\partial\mathbb{D}) .$$

Example – Sampled Conjugate Fejér Mean

- Now we define the approximation operators $H_N^{\mathcal{F}} := HS_N$. This yields

$$\left(H_N^{\mathcal{F}} f\right)(e^{i\theta}) = \left(HS_N f\right)(e^{i\theta}) = \sum_{n=0}^{N-1} f\left(e^{i\, n\, 2\pi/N}\right) \widetilde{\mathcal{F}}_N\left(\theta - n\,\tfrac{2\pi}{N}\right)$$

with the *conjugate Fejér kernel* $\widetilde{\mathcal{F}}_N = H\mathcal{F}_N$ given by

$$\widetilde{\mathcal{F}}_N(\tau) = \frac{N \sin\tau - \sin(N\tau)}{2\left[N \sin(\tau/2)\right]^2} = \frac{1}{N}\left(\frac{1}{\tan(\tau/2)} - \frac{\sin(N\tau)}{2N\sin^2(\tau/2)}\right) .$$

▷ $\{H_N^{\mathcal{F}}\}_{N\in\mathbb{N}}$ is an approximation sequence satisfying Axioms (A) and (B).

▷ Replace the rectangular integration rule by any *other integration method* gives similar operators but with *other kernels*.

Divergence of Hilbert Transform Approximations

Theorem 7

To every sequence $\{\mathrm{H}_N\}_{N\in\mathbb{N}}$ *which satisfies Axioms (A) and (B) there exists a residual set* $\mathrm{D}_{UB} \subset \mathcal{B}^{1/2}$ *such that*

$$\limsup_{N\to\infty} \left\|\mathrm{H}_N f\right\|_\infty = \infty \qquad \text{for all } f \in \mathrm{D}_{UB}\,,$$

and such that D_{UB} *is spaceable.*

- Since $\mathcal{B}^{1/2} \subset \mathcal{B}$, the same results holds for $\mathcal{B}$.

Sketch of Proof – Interpolation Lemmas

$\mathcal{A}(\mathbb{D})$ is the *disk algebra* of functions analytic in $\mathbb{D}$ and continuous in $\mathbb{D} \cup \partial\mathbb{D}$.

$$\mathcal{A}^{1/2} = \mathcal{A}(\mathbb{D}) \cap \mathcal{D}_a \qquad \text{with} \qquad \|f\|_{\mathcal{A}^{1/2}} = \max\left(\|f\|_\infty, \|f\|_{\mathcal{H}^{1/2}}\right)$$

Lemma (Interpolation by functions from $\mathcal{A}^{1/2}$)

There exists a universal constant $C_a \geqslant 1$ *such that for every finite sampling set* $Z_N = \{\zeta_1, \ldots, \zeta_N\} \subset \partial\mathbb{D}$ *the following statement is true:*
To every complex-valued $f \in \mathcal{C}(\partial\mathbb{D})$ *there exists a* $g \in \mathcal{A}^{1/2}$ *such that*

$$g(\zeta_n) = f(\zeta_n) \quad \textit{for all } \zeta_n \in Z_N \qquad \textit{and} \qquad \|g\|_{\mathcal{A}^{1/2}} \leqslant C_a \|f\|_\infty .$$

Corollary (Interpolation by functions from $\mathcal{B}^{1/2}$)

There exists a universal constant $C_a \geqslant 1$ *such that for every finite sampling set* $Z_N = \{\zeta_1, \ldots, \zeta_N\} \subset \partial\mathbb{D}$ *the following statement is true:*
To every real-valued $f \in \mathcal{C}(\partial\mathbb{D})$ *there exists a* $u \in \mathcal{B}^{1/2}$ *such that*

$$u(\zeta_n) = f(\zeta_n) \quad \textit{for all } \zeta_n \in Z_N \qquad \textit{and} \qquad \|u\|_{\mathcal{B}^{1/2}} \leqslant C_a \|f\|_\infty .$$

Sketch of Proof – Divergence Theorem

(1) Using

- Corollary – Interpolation of continuous functions by functions from $\mathcal{B}^{1/2}$
- Axiom (A) – Operators $\{H_N\}_{N\in\mathbb{N},}$ are concentrated on sampling sets

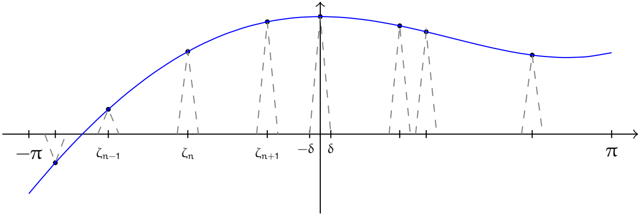
$$\|H_N\|_{\mathcal{C}(\partial\mathbb{D})\to\mathcal{C}(\partial\mathbb{D})} = \sup_{f\in\mathcal{C}(\partial\mathbb{D})} \frac{\|H_N f\|_\infty}{\|f\|_\infty}$$
$$\leqslant C_a \sup_{u\in\mathcal{B}^{1/2}} \frac{\|H_N u\|_\infty}{\|u\|_{\mathcal{B}^{1/2}}} = C_a \|H_N\|_{\mathcal{B}^{1/2}\to\mathcal{C}(\partial\mathbb{D})} .$$

(2) Show that

$$\sup_{N\in\mathbb{N}} \|H_N\|_{\mathcal{C}(\partial\mathbb{D})\to\mathcal{C}(\partial\mathbb{D})} = +\infty$$

Main technical challenge: Interpolation Lemma

Proof Interpolation Lemma – Previous Approach



- Interpolation with triangle function (B-spline of order 1)

$$g(t) = \sum_{n=1}^{N} f(\zeta_n)\, h_\delta(t-\zeta_n) \qquad \text{with} \qquad h_\delta(t) = \max\left(1 - \frac{|t|}{\delta}, 0\right)$$

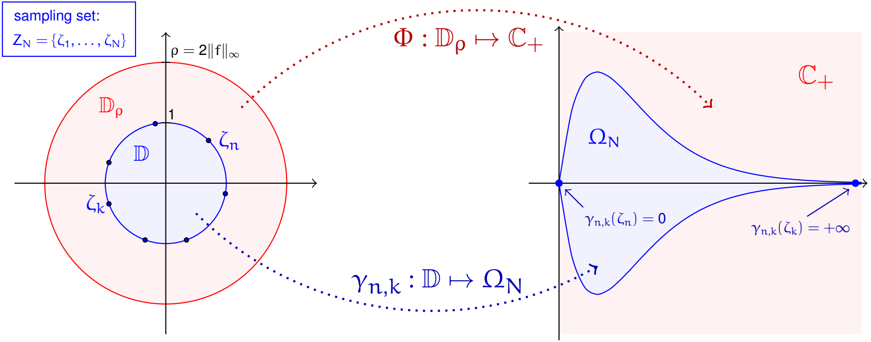
√ $\|h_\delta\|_\infty = 1$, $\quad \|\widetilde{h}_\delta\|_\infty \leqslant 1-\delta$, $\quad |\widetilde{h}_\delta(t)| \leqslant \epsilon$ for $|t| \gtrsim \delta$.

ϟ **Difficulty**:

$$\text{Behavior of} \quad \|h_\delta\|_{\mathcal{H}^s} \quad \text{for all } s \geqslant \frac{1}{2} \text{ and } \delta \text{ small .}$$

Therefore $\|g\|_{\mathcal{H}^s}$ cannot be controlled for N large and $s \geqslant 1/2$.

Mapping Theorem of Riemann & Carathéodory



- Ω_N: open region in $\mathbb{C}$, $\partial\Omega_N$: Jordan curve,

$$A(\Omega_N) = \pi\, D(\gamma_{n,k}) = \iint_{\mathbb{D}} |\gamma'_{n,k}(z)|^2 \, dz = \iint_{|x+iy|<1} |\gamma'_{n,k}(x+iy)|^2 \, dx\, dy$$

- Riemann mapping theorem: Conformal mappings $\gamma : \mathbb{D} \mapsto \Omega_N$
- Carathéodory: γ has an extension $\gamma : \partial\mathbb{D} \mapsto \partial\Omega_N$
- Uniqueness of γ: Fix the value of γ at two points on $\partial\mathbb{D}$: $\Rightarrow \gamma_{n,k}$

Proof of the Interpolation Lemma – Continued

- Based on the the conformal mappings $\gamma_{n,k}$ and Φ, one defines the interpolating function g as follows

$$q_n(z) := \sum_{k=1\,,\,k\neq n}^{N} \gamma_{n,k}(z) + \frac{1}{\Phi(f(\zeta_n))}\,, \qquad n = 1, \dots, N$$

$$g(z) := \Phi^{-1}\left(\sum_{n=1}^{N} 1/q_n(z)\right)$$

- By the above construction, g has the desired properties:
  - g is analytic in $\mathbb{D}$
  - Interpolation property: $g(\zeta_n) = f(\zeta_n)$ for all $n = 1, \dots, N$.
  - Norm and energy bounded: $\|g\|_\infty \leqslant 2\|f\|_\infty$ and $D(g) \leqslant \|f\|_\infty$

Further Questions, Extensions

- The previous theorem showed the weak divergence of sampling based Hilbert transform approximations $\{H_N\}_{N\in\mathbb{N}}$ satisfying Axioms (A) and (B):

$$\limsup_{N\to\infty} \left\|H_N f\right\|_\infty = \infty \qquad \text{for all } f \in D_{\mathsf{UB}} \subset \mathcal{B}^{1/2}$$

  where $D_{\mathsf{UB}} \cup \{0\}$ is a linear subspace of $\mathcal{B}^{1/2}$ (spaceability).
- For $\mathcal{H}^s$, $s > 1/2$, it is a simple exercise to construct a convergent approximation process for the Hilbert transform, satisfying Axioms (A) and (B).

  ? Do these approximation methods also *diverge strongly* on $\mathcal{B}^{1/2}$?

$$\lim_{N\to\infty} \left\|H_N f\right\|_\infty = \infty \qquad \text{for some } f \in \mathcal{B}^{1/2}\,.$$

  ? Has the corresponding divergence set any (linear) structure?

Remarks

- In general, it is extremely easy to approximate continuous functions by sampling series which satisfy the two Axioms (A) and (B).
- Our connection with the Hilbert transform, implies a relation to *causality*.
- Causality $\Rightarrow$ we have to work in the framework of *analytic functions* to satisfy the causality restriction.
- In *Curt McMullen's* solution of *Smale's conjecture*, he was restricted to *analytic* rational functions.
- If one allows general rational functions, then there are always convergent algorithms.

Conclusions

- Lineability of divergence set in the Banach–Steinhaus theorem without any further assumptions.
- Spaceability of divergence set in the Banach–Steinhaus theorem under the condition that the boundedness set is spaceable.
- Joint spaceability of divergence set for mixed signal representation and sampling type representation in different Paley–Wiener spaces.
- Convergence behavior of a classical phase retrieval problem for Sobolev spaces.

Thank you!