The Solvability Complexity Index of Sampling-based Hilbert Transform Approximations

Holger Boche   Volker Pohl

Technical University of Munich
Department of Electrical and Computer Engineering
Chair of Theoretical Information Technology

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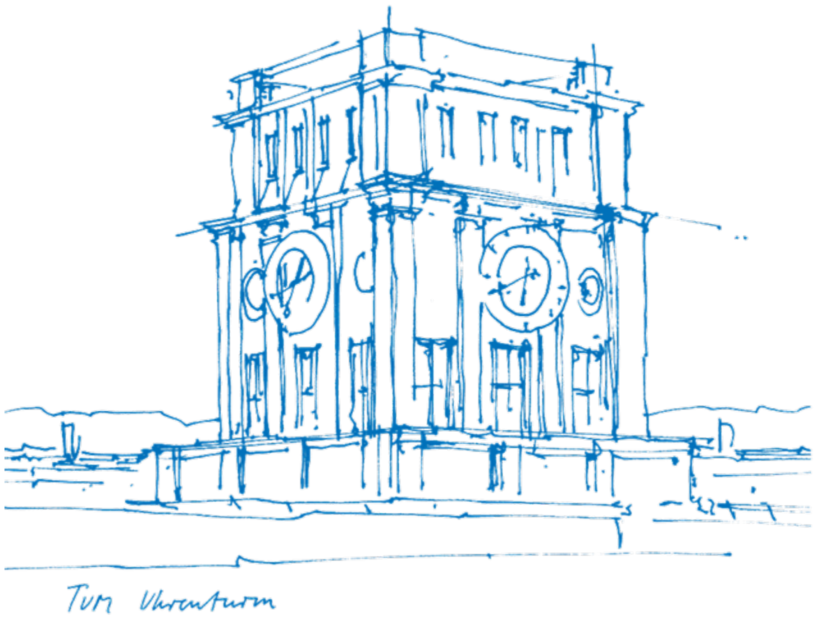

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Outline of this Talk

1. The Solvability Complexity Index and Towers of Algorithms
2. The Hilbert Transform and Sampling-based Hilbert Transform Approximations
3. Apply the the SCI Framework to the Hilbert Transform
4. Conlusions

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The Solvability Complexity Index

The Solvability Complexity Index

- The Solvability Complexity Index (SCI) is a new measure to characterize the complexity of certain computational problems.
  - J. Ben-Artzi, A. C. Hansen, O. Nevanlinna, and M. Seidel, "Can everything be computed? - On the Solvability Complexity Index and towers of algorithms." *preprint: arXiv:1508.03280v1*, Aug. 2015.
  - J. Ben-Artzi, A. C. Hansen, O. Nevanlinna, and M. Seidel, "New barriers in complexity theory: On the solvability complexity index and the towers of algorithms," *Comptes Rendus Mathematique*, vol. 353, no. 10, pp. 931–936, 2015.
- Solving computational problems by towers of algorithms.
- The SCI is the minimal height of such a tower.
- The SCI counts the minimum number of limits necessary for computing a desired quantity.

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The Solvability Complexity Index – Polynomials

- Let $\mathscr{P}_N$ set of polynomials of degree $N \in \mathbb{N}$.
- Can we find an algorithm to compute the roots of every $p \in \mathscr{P}_N$ using only finitely many arithmetic operations?
  - Yes! for $N \leq 4$
  - No! for $N \geq 5$
- What if we allow for a limiting process?
  Does there exists a sequence of operations $\{\Gamma_n : \mathscr{P}_N \to \mathbb{C}\}$ where each Γ_n involves only finitely many arithmetic operations such that
  $$\lim_{n\to\infty} \Gamma_n(p) = z_0$$
  is a zero of p?
  - Yes! for all $N \in \mathbb{N}$
  - Note: Newton's method will not work.
    An there is no „purely iterative generally convergent algorithm“ for root finding.

S. Smale, “The fundamental theorem of algebra and complexity theory,” *Bull. Amer. Math. Soc.*, vol. 4, 1981.
P. Doyle and C. McMullen, “Solving the quintic by iteration,” *Acta Math.*, vol. 163, pp. 151–180, 1989.

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The Solvability Complexity Index

- **There are important computational problems which can not be solved by one limiting process!**
  - „purely iterative generally convergent algorithm“ for root finding.
  - ...
  - calculation of the Hilbert transform from samples.
- What if we allow for a several limiting processes?

  Does there exist a sequence $\{\Gamma_{n_k,\dots,n_1}\}$ of operations depending on several indices $n_k,\dots,n_1 \in \mathbb{N}$ such that

$$\lim_{n_1\to\infty} \cdots \lim_{n_k\to\infty} \Gamma_{n_k,\dots,n_1}(p) = \Xi(p)$$

  converges to the desired result $\Xi(p)$?

H. Boche and V. Pohl, "On the calculation of the Hilbert transform from interpolated data," *IEEE Trans. Inf. Theory*, vol. 54, no. 5, pp. 2358–2366, May 2008.

H. Boche and V. Pohl, "Calculating the Hilbert transform on spaces with energy concentration: Convergence and divergence regions," *IEEE Trans. Inf. Theory*, vol. 65, no. 1, pp. 586–603, Jan. 2019.

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The SCI Framework

The SCI framework, is based on the following basic objects

- Ω a primary set, is the set of objects on which we perform the computations
- $\mathscr{M}$ a (pseudo) metric space
- Ξ a problem function, is the mapping $\Xi : \Omega \to \mathscr{M}$ which we want to compute for our objects in Ω
- Λ is the evaluation set, that is a set of (measurement) functionals $\lambda : \Omega \to \mathbb{C}$

Definition (Computational problem)

A collection $\{\Xi, \Omega, \mathscr{M}, \Lambda\}$ is said to be a *computational problem*.

Example ($\{\mathscr{C}(\mathbb{T}), \mathbb{R}, \|\cdot\|_\infty, \mathbb{T}\}$ – „Sampling-based computational problem“)

- $\Omega = \mathscr{C}(\mathbb{T})$, the Banach space of continuous functions on a compact interval $\mathbb{T} \subset \mathbb{R}$
- $\mathscr{M} = \mathbb{R}$ the real numbers.
- $\Xi : \mathscr{C}(\mathbb{T}) \to \mathbb{R}$ is given by $\Xi(f) = \|f\|_\infty = \max_{t\in\mathbb{T}} |f(t)|$
- $\Lambda = \{\lambda_\tau : \tau \in \mathbb{T}\}$ consists of all point evaluations

$$\lambda_\tau(f) = f(\tau) \qquad \text{for all } f \in \mathscr{C}(\mathbb{T})\,.$$

Fundamental Algorithm

- Let $\{\Xi, \Omega, \mathscr{M}, \Lambda\}$ be a given computational problem.
- The problem of calculating $\Xi(f)$ will be done by a so called tower of algorithms.
- The next definition characterizes the algorithms at the lowest (the fundamental) level of the tower.

Definition (Fundamental Algorithm)

Let $\{\Xi, \Omega, \mathscr{M}, \Lambda\}$ be a computational problem. A mapping $\Gamma : \Omega \to \mathscr{M}$ is said to be a *fundamental algorithm* if for every $f \in \Omega$ the following conditions are satisfied:

(i) There exists a finite subset $\Lambda_\Gamma(f) \subset \Lambda$ of evaluations.
(ii) The action of Γ on f depends only on the values $\{\lambda(f) : \lambda \in \Lambda_\Gamma(f)\}$.
(iii) For every $g \in \Omega$, with $\lambda(g) = \lambda(f)$ for all $\lambda \in \Lambda_\Gamma(f)$, holds $\Lambda_\Gamma(g) = \Lambda_\Gamma(f)$.

We say that Γ is a fundamental algorithm of type

L for "*linear*", if Γ is linear.
G for "*general*", if there is no further restriction on Γ.

Remark:

- The calculation of $\Gamma(f)$ is based on finitely many measurements of f.
- The evaluation set $\lambda_\Gamma(f)$ may be chosen adaptively, dependent on the actual object f.
- Condition (iii): well definiteness and consistence. It limits adaptivity.

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Tower of Algorithms

Definition (Tower of Algorithms)

A *tower of algorithms* of type α and of height k for $\{\Xi, \Omega, \mathscr{M}, \Lambda\}$ is a family of sequences of operators

$$\begin{aligned} \Gamma_{n_k} &: \Omega \to \mathscr{M}, \\ \Gamma_{n_k, n_{k-1}} &: \Omega \to \mathscr{M}, \\ &\vdots \\ \Gamma_{n_k, n_{k-1}, \dots, n_1} &: \Omega \to \mathscr{M}, \end{aligned}$$

where $n_k, \dots, n_1 \in \mathbb{N}$ and where the operators $\Gamma_{n_k, n_{k-1}, \dots, n_1}$ at the lowest level of the tower are fundamental algorithms of type α. Moreover, for every $f \in \Omega$ holds

$$\begin{aligned} \Xi(f) &= \lim_{n_k \to \infty} \Gamma_{n_k}(f) \\ \Gamma_{n_k}(f) &= \lim_{n_{k-1} \to \infty} \Gamma_{n_k, n_{k-1}}(f) \\ &\vdots \\ \Gamma_{n_k, \dots, n_2}(f) &= \lim_{n_1 \to \infty} \Gamma_{n_k, \dots, n_2, n_1}(f) \end{aligned}$$

where the convergence is always in the metric of $\mathscr{M}$.

Solvability Complexity Index

Definition (Solvability Complexity Index)

We say that a computational problem $\{\Xi, \Omega, \mathscr{M}, \Lambda\}$ has a *Solvability Complexity Index*

$$\mathrm{SCI}\,(\Xi, \Omega, \mathscr{M}, \Lambda)_\alpha = k$$

with respect to towers of algorithms of type α if k is the smallest integer for which there exists a tower of algorithms of type α of height k. If no such tower exists then $\mathrm{SCI}\,(\Xi, \Omega, \mathscr{M}, \Lambda)_\alpha = \infty$.

- ▷ We say that $\mathrm{SCI}\,(\Xi, \Omega, \mathscr{M}, \Lambda)_\alpha = 0$, if $\Xi(f)$ can be calculate by finitely many algebraic operations.

Hilbert Transformation

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The Hilbert Transform

- We consider continuous functions $f \in \mathscr{C}(\mathbb{T})$ on the interval $\mathbb{T} = [-\pi, \pi]$ with $f(-\pi) = f(\pi)$
- The Fourier coefficients of $f \in \mathscr{C}(\mathbb{T})$ are given by

$$c_k(f) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) \mathrm{e}^{-\mathrm{i}kt} \, \mathrm{d}t \,, \qquad k = 0, \pm 1, \pm 2, \ldots \,.$$

- Assume that the Fourier series of f converges to f (in some sense)

$$f(t) = \sum_{k=-\infty}^{\infty} c_k(f) \mathrm{e}^{\mathrm{i}kt} \,, \qquad t \in \mathbb{T} \,.$$

- The conjugate function associated with f is

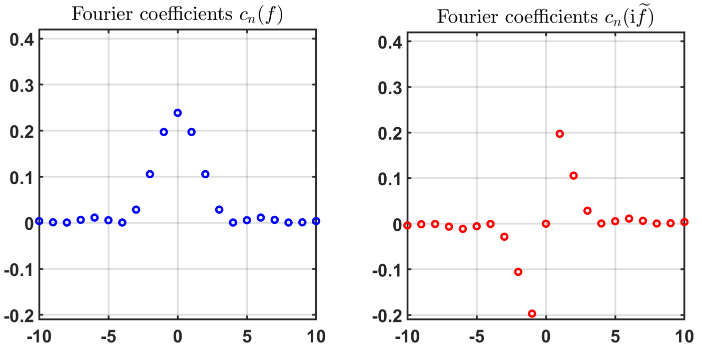
$$\widetilde{f}(t) = (\mathrm{H}f)(t) = -\mathrm{i} \sum_{k=-\infty}^{\infty} \mathrm{sgn}(k) \, c_k(f) \mathrm{e}^{\mathrm{i}kt} \,, \quad t \in \mathbb{T} \qquad \text{with} \qquad \mathrm{sgn}(k) = \begin{cases} 0 & : \; k = 0 \\ \frac{k}{|k|} & : \; k \neq 0 \end{cases}$$

- The Hilbert transform is the mapping

$$\mathrm{H} : f \mapsto \widetilde{f}$$

.

- Applications: Causality, Kramers–Kronig relation, ...



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Signal Spaces for the Hilbert Transform

- ▷ Let $f \in \mathscr{C}(\mathbb{T})$ with Fourier coefficients $\{c_k(f)\}_{k\in\mathbb{Z}}$.
- ▷ For $\beta \geq 0$, we define the seminorm

$$\|f\|_\beta = \left(\sum_{k\in\mathbb{Z},\ k\neq 0} |k| \, (1 + \log|k|)^\beta \, |c_k(f)|^2 \right)^{1/2} .$$

- ▷ Therewith, we define for any $\beta \geq 0$ the Banach space

$$\mathscr{B}_\beta = \left\{ f \in \mathscr{C}(\mathbb{T}) \, : \, \widetilde{f} \in \mathscr{C}(\mathbb{T}) \text{ and } \|f\|_\beta < \infty \right\}$$

with the norm

$$\|f\|_{\mathscr{B}_\beta} = \max\left(\|f\|_{\mathscr{C}(\mathbb{T})}, \|\widetilde{f}\|_{\mathscr{C}(\mathbb{T})}, \|f\|_\beta \right) .$$

Remark

- For every $f \in \mathscr{B}_\beta$ the Fourier series and the conjugate Fourier series converge uniformly to f and $\widetilde{f}$
- The space $\mathscr{B}_{\beta=0}$ corresponds to the usual Sobolev space $H^{1/2}(\mathbb{T}) = W^{1/2,2}(\mathbb{T})$.
- The value $\|f\|^2_{\beta=0}$ corresponds to the Dirichlet energy of f.
- $\mathscr{B}_{\beta_2} \subset \mathscr{B}_{\beta_1} \subset \mathscr{B}_{\beta=0}$ for all $\beta_2 \geq \beta_1 \geq 0$.

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No Sampling-based Hilbert Transform Approximation

Theorem

Let $\{\Gamma_n\}_{n\in\mathbb{N}}$ *be a sequence of lower semicontinuous mappings* $\Gamma_n : \mathscr{B}_\beta \to \mathscr{C}(\mathbb{T})$ *such that to every* $n \in \mathbb{N}$ *there exists a finite sampling set* $\mathscr{T}_n \subset \mathbb{T}$ *such that*

$$f_1(\tau) = f_2(\tau) \quad \text{for all } \tau \in \mathscr{T}_n \qquad \text{implies} \qquad \Gamma_n(f_1) = \Gamma_n(f_2)\,.$$

Then for every $0 \le \beta \le 1$ *there exist functions* $f \in \mathscr{B}_\beta$ *such that*

$$\limsup_{n\to\infty} \left\| \widetilde{f} - \Gamma_n(f) \right\|_{\mathscr{C}(\mathbb{T})} > 0\,.$$

- The operators $\Gamma_n : \mathscr{B}_\beta \to \mathscr{C}(\mathbb{T})$ can be non-linear.
- So the Hilbert transform on $\mathscr{B}_\beta$ can not be calculated by a single limit (i.e. $\mathrm{SCI} > 1$).
- So what is the SCI for calculating the Hilbert transform?

H. Boche and V. Pohl, “Investigations on the approximability and computability of the Hilbert transform with applications,” *Appl. Comput. Harmon. Anal.*, Sep. 2018, in press.

The Solvability Complexity Index for Calculating the Hilbert Transform

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The Computational Problem

Now we investigate the Hilbert transform calculation using the SCI farmework.

- For $\beta \in [0,1]$, we consider the computational problems

$$\{\mathrm{H}, \mathscr{B}_\beta, \mathscr{C}(\mathbb{T}), \Lambda\} \qquad \text{and} \qquad \{\mathrm{H}, \mathscr{B}_\beta, \mathscr{B}_\beta, \Lambda\}$$

  i.e.
  - The primary set Ω is $\mathscr{B}_\beta$ for some $\beta \in [0,1]$.
  - The metric space $\mathscr{M}$ is either $\mathscr{C}(\mathbb{T})$ or $\mathscr{B}_\beta$.
  - The problem function Ξ is the Hilbert transform $\mathrm{H} : \mathscr{B}_\beta \to \mathscr{M}$ is the Hilbert transform.
  - The evaluations set Λ contains the sampling functionals $\lambda_\tau : \mathscr{B}_\beta \to \mathbb{C}$ with $\tau \in \mathbb{T}$ defined by

$$\lambda_\tau(f) = f(\tau), \quad f \in \mathscr{B}_\beta .$$

- So the evaluation set consist now of point evaluations (i.e. sampling).

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Sampling-based Fundamental Algorithms

Definition (Sampling-based Fundamental Algorithm)

Let $\{\mathrm{H}, \mathscr{B}_\beta, \mathscr{M}, \Lambda\}$ be a sampling-based computational problem for calculating the Hilbert transform. A lower semicontinuous mapping $\Gamma : \mathscr{B}_\beta \to \mathscr{M}$ is said to be a *sampling-based fundamental algorithm* if

(i) There exists a finite subset $T_\Gamma \subset \mathbb{T}$

(ii) For all $f, g \in \mathscr{B}_\beta$

$$\begin{aligned} & f(\tau) = g(\tau) && \text{for all } \tau \in T_\Gamma \\ \text{implies} \quad & [\Gamma(f)](t) = [\Gamma(g)](t) && \text{for all } t \in \mathbb{T}. \end{aligned}$$

Moreover, we say that Γ is of type

1. *linear (L)*, if Γ is linear.
2. *general (G)*, if there is no further restriction on Γ.

Remark

- ▷ Every fundamental algorithm Γ is concentrated on a finite sampling set Λ_Γ.
- ▷ No adaptivity, i.e. the sampling sets Λ_Γ depend only on Γ but not on the actual function.

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The SCI of the Hilbert Transform

Theorem

Let $\beta \in [0,1]$ *be arbitrary, then we have*

$$\mathrm{SCI}\left(\mathrm{H},\mathscr{B}_\beta,\mathscr{B}_\beta,\Lambda\right)_L = \mathrm{SCI}\left(\mathrm{H},\mathscr{B}_\beta,\mathscr{C}(\mathbb{T}),\Lambda\right)_G = 2\,.$$

Remark

- We also get

$$2 = \mathrm{SCI}\left(\mathrm{H},\mathscr{B}_\beta,\mathscr{C}(\mathbb{T}),\Lambda\right)_G \le \mathrm{SCI}\left(\mathrm{H},\mathscr{B}_\beta,\mathscr{C}(\mathbb{T}),\Lambda\right)_L \le \mathrm{SCI}\left(\mathrm{H},\mathscr{B}_\beta,\mathscr{B}_\beta,\Lambda\right)_L = 2$$
$$2 = \mathrm{SCI}\left(\mathrm{H},\mathscr{B}_\beta,\mathscr{C}(\mathbb{T}),\Lambda\right)_G \le \mathrm{SCI}\left(\mathrm{H},\mathscr{B}_\beta,\mathscr{B}_\beta,\Lambda\right)_G \le \mathrm{SCI}\left(\mathrm{H},\mathscr{B}_\beta,\mathscr{B}_\beta,\Lambda\right)_L = 2$$

- *Non-linear methods give no improvement* over linear approximation methods (as far as the SCI is concerned).

Proof

- Our previous already implied that $\mathrm{SCI}\left(\mathrm{H},\mathscr{B}_\beta,\mathscr{B}_\beta,\Lambda\right)_G > 1$.
- It remains to show that $\mathrm{SCI}\left(\mathrm{H},\mathscr{B}_\beta,\mathscr{B}_\beta,\Lambda\right)_L \le 2$.

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Tower of Algorithms for the Hilbert Transform

$$(\mathrm{H}f)(t) = -\mathrm{i}\sum_{k=-\infty}^{\infty} \operatorname{sgn}(k)\, c_k(f)\,\mathrm{e}^{\mathrm{i}kt}\,, \quad t \in \mathbb{T} \qquad \text{with} \qquad c_k(f) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\,\mathrm{e}^{-\mathrm{i}kt}\,\mathrm{d}t\,.$$

- Let $n_1, n_2 \in \mathbb{N}$ be fixed. For $f \in \mathscr{B}_\beta$, we consider

$$c_k(f,n_1) = \tfrac{1}{n_1}\textstyle\sum_{l=0}^{n_1-1} f\left(l\tfrac{2\pi}{n_1}\right)\mathrm{e}^{-\mathrm{i}\frac{2\pi}{n_1}lk}\,, \qquad \text{for all } |k| \leq n_2\,.$$

  $c_k(f,n_1)$ is the approximation of the kth Fourier coefficient of f by a Riemann sum based on n_1 equally spaced sampling points and we have

$$\lim_{n_1\to\infty} c_k(f,n_1) = c_k(f) \qquad \text{for all } |k| \leq n_2\,.$$

- Define the operators in the lower level of the tower by

$$(\Gamma_{n_2,n_1}f)(t) = -\mathrm{i}\textstyle\sum_{k=-n_2}^{n_2} \operatorname{sgn}(k)\, c_k(f,n_1)\,\mathrm{e}^{\mathrm{i}kt}\,, \qquad t \in \mathbb{T}\,,$$

- Define the operators in the upper level of the tower by

$$(\Gamma_{n_2}f)(t) = -\mathrm{i}\textstyle\sum_{k=-n_2}^{n_2} \operatorname{sgn}(k)\, c_k(f)\,\mathrm{e}^{\mathrm{i}kt}\,, \qquad t \in \mathbb{T}\,.$$

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Discussion

- Tower of algorithm

$$(\Gamma_{n_2} f)(t) = -\mathrm{i} \sum_{k=-n_2}^{n_2} \operatorname{sgn}(k)\, c_k(f)\, \mathrm{e}^{\mathrm{i}kt}$$

$$(\Gamma_{n_2,n_1} f)(t) = -\mathrm{i} \sum_{k=-n_2}^{n_2} \operatorname{sgn}(k)\, c_k(f,n_1)\, \mathrm{e}^{\mathrm{i}kt} \quad \text{with} \quad c_k(f,n_1) = \frac{1}{n_1} \sum_{l=0}^{n_1-1} f\left(l\frac{2\pi}{n_1}\right) \mathrm{e}^{-\mathrm{i}\frac{2\pi}{n_1} lk} .$$

- The first layer calculates the Fourier coefficients $c_k(f)$ of f from the samples $\left\{ f\left(l\frac{2\pi}{n_1}\right)\right\}$ of f by the first limit

$$\lim_{n_1\to\infty} c_k(f,n_1) = c_k(f) .$$

- The second layer calculates $\widetilde{f} = \mathrm{H}f$ by the conjugate Fourier series (second limit).

- How to get a tower of algorithm of height 1?
  $\Rightarrow$ Other evaluation set Λ (i.e. other measurement functionals)

$$\lambda_k : f \mapsto c_k(f) , \qquad k = 0, \pm 1, \pm 2, \dots$$

  $\Rightarrow$ Smaller primary set $\Omega \subset \mathscr{B}_\beta$.

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Conclusions

Conclusions and Discussion

- The SCI for calculating the Hilbert transform $\mathrm{H}f$ from samples of $f \in \mathscr{B}_\beta$ is equal to 2.
  - One needs 2 limits to calculate $\widetilde{f} = \mathrm{H}(f)$.
  - The SCI = 2 for linear and general towers of algorithms.
  - This statement holds primary sets $\mathscr{B}_\beta$ with $\beta \in [0,1]$

- For $\beta > 1$ we (probably) get

$$\mathrm{SCI}\left(\mathrm{H}, \mathscr{B}_\beta, \mathscr{B}_\beta, \Lambda\right)_L = 1 \,.$$

- Similar results for other sampling based problems:
  - Recovering functions from samples on the space of uniformly convergent Fourier series.
  - Solving the Dirichlet problem based on samples of the boundary function.
  - Calculating the Spectral factorization.