On the Algorithmic Solvability of the Spectral Factorization and the Calculation of the Wiener Filter on Turing Machines

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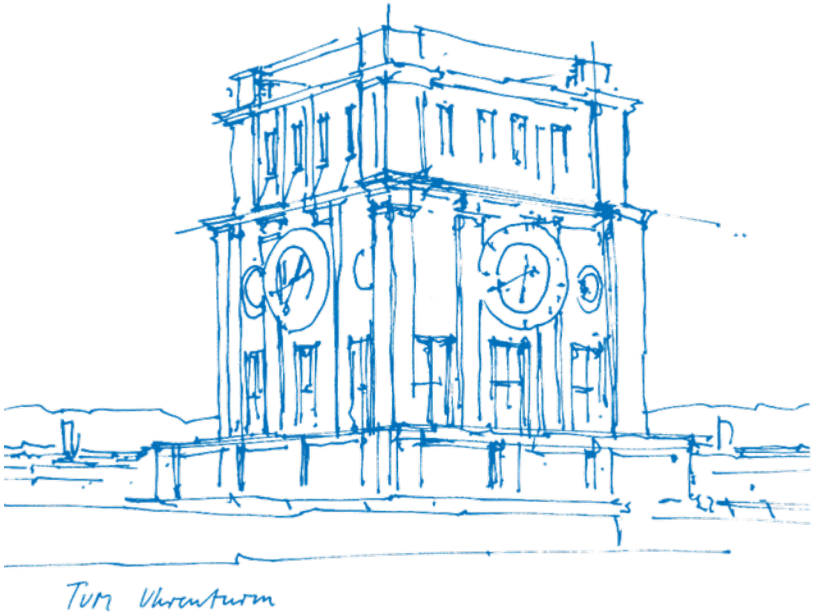


Subject and Outline of the Talk

Is it possible to calculate the spectral factorization on a digital computer?

Is the spectral factor of a computable spectral density computable?

Outline

1. Spectral Factorization – A very short Introduction
2. Review of Computability Theory
   - computable numbers, computable functions, Turing machines, etc.
3. Non-Computability of the Spectral Factorization and a Decision Problem
4. Consequences for Calculating the Causal Wiener filter

Spectral Factorization

Spectral Factorization

- ▷ Let ϕ be a spectral density. That is
  - a non-negative real function on the unit circle $\partial\mathbb{D} = \{z \in \mathbb{C} : |z| = 1\}$
  - satisfying the *Paley–Wiener (Szegö) condition* $\log\phi \in L^1(\partial\mathbb{D})$
- ▷ Spectral factorization is the operation of writing ϕ as

$$\phi(\mathrm{e}^{\mathrm{i}\omega}) = \phi_+(\mathrm{e}^{\mathrm{i}\omega})\,\phi_-(\mathrm{e}^{\mathrm{i}\omega}) = \left|\phi_+(\mathrm{e}^{\mathrm{i}\omega})\right|^2 , \qquad \omega \in [-\pi, \pi) .$$

  with the spectral factor ϕ_+ and its *para-Hermitian conjugate* $\phi_-(z) = \overline{\phi_+(1/\overline{z})}$ for $z \in \mathbb{C}$.
- ▷ The spectral factor ϕ_+ is an *outer function* (a "minimum-phase system"), i.e.
  - $\phi_+(z)$ is analytic for every $z \in \mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$
  - $\phi(z) \neq 0$ for all $z \in \mathbb{D}$.

  It can be written as

$$\phi_+(z) = (\mathrm{S}\phi)(z) = \exp\left(\frac{1}{4\pi}\int_{-\pi}^{\pi} \log\phi(\mathrm{e}^{\mathrm{i}\omega})\,\frac{\mathrm{e}^{\mathrm{i}\omega}+z}{\mathrm{e}^{\mathrm{i}\omega}-z}\,\mathrm{d}\omega\right) \qquad z \in \mathbb{D} .$$

- ▷ We call $\mathrm{S} : \phi \mapsto \phi_+$ the spectral factorization mapping.

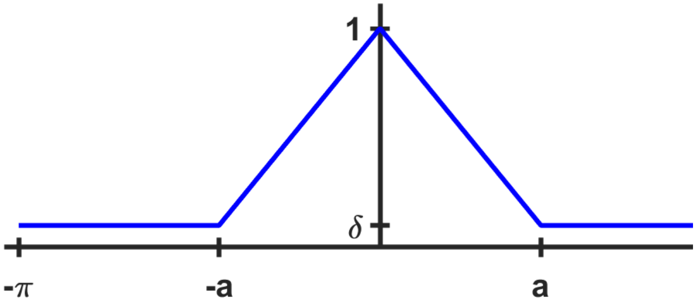
Applications

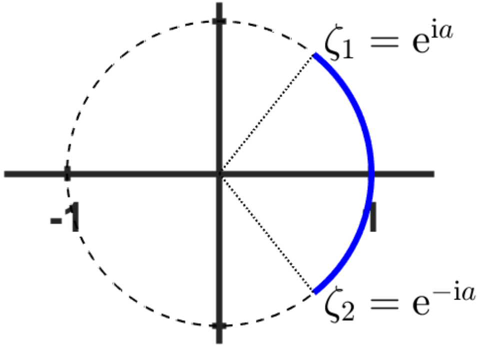
- Wiener–Kolmogorov theory of smoothing and prediction of stationary time series
- causal Wiener filter: Communications, signal processing, control theory, $\cdots$

Spectral Factorization Mapping – Properties

- $S : \phi \mapsto \phi_+$ has very complicated behavior (non-linear mapping, singular integral kernel)
- Even for very simple spectral densities ϕ, the spectral factor can not be written as a closed form expression.

Example (Piecewise linear spectral density):





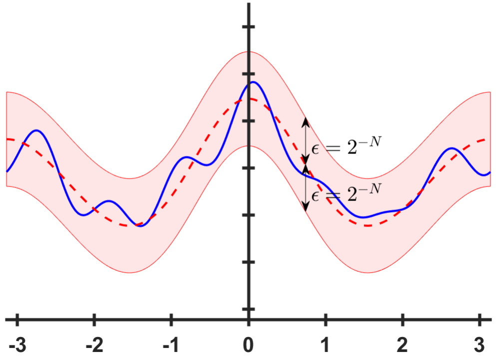
- Left side: a piecewise linear spectral density ϕ
- Right side: The arc on which the spectral factor ϕ_+ of ϕ is only given by a Cauchy principal value.

$$\phi_+(z) = \sqrt{\delta}\exp\left(\frac{1}{4\pi}\int_{-a}^{a}\log\left(\frac{\phi(e^{i\omega})}{\delta}\right)\frac{e^{i\omega}+z}{e^{i\omega}-z}\,d\omega\right), \qquad z \in \mathbb{C}.$$

Computability

Computability – Intuition

- The true spectral factor ϕ_+ is usually not known explicitly.
- A function ϕ_+ is computable if it can be *approximated effectively* by a function p_M which can *perfectly be calculated* on a digital computer.
  - p_M might be a rational polynomial of a certain degree M
  - effective approximation $\Rightarrow$ one can control the approximation error



Computability (an informal definition)

The spectral factor ϕ_+ is computable if there exists an algorithm with the following properties

- It can be implemented on a digital computer (a Turing machine).
- It has two inputs: 1. the spectral density ϕ 2. an error bound $\varepsilon > 0$.
- It is able to determine in finitely many steps an approximation p_M of ϕ_+ such that the true ϕ_+ is guaranteed to be close to p_M, i.e. such that

$$\phi_+ \in \{\psi \in \mathscr{X} \; : \; \|\psi - p_M\|_{\mathscr{X}} < \varepsilon\}$$

where $\mathscr{X}$ is an appropriate Banach space with a corresponding norm $\|\cdot\|_{\mathscr{X}}$.

Computable Rational Numbers

Definition: A sequence $\{r_n\}_{n\in\mathbb{N}} \subset \mathbb{Q}$ of rational numbers is said to be computable if there exist recursive functions $a, b, s : \mathbb{N} \to \mathbb{N}$ with $b(n) \neq 0$ and such that

$$r_n = (-1)^{s(n)} \frac{a(n)}{b(n)}, \qquad n \in \mathbb{N}.$$

A recursive function $a : \mathbb{N} \to \mathbb{N}$ is a mapping that is build form elementary computable functions and recursion and can be calculated on a *Turing machine*.

Turing machine

- can simulate any given algorithm and therewith provide a simple but very powerful model of computation.
- is a theoretical model describing the fundamental limits of any realizable digital computer.
- Most powerful programming languages are called Turing-complete (such as C, C++, Java, etc.).

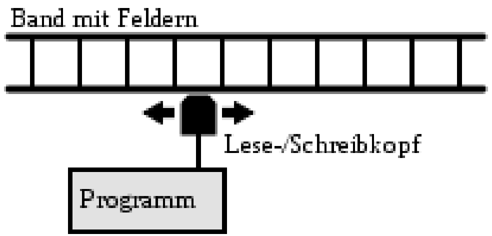


Figure taken from *Wikipedia*

A. M. Turing, "On computable numbers, with an application to the Entscheidungsproblem," *Proc. London Math. Soc.*, vol. s2-42, no. 1, 1937.

Computable Real Numbers

- Any real number $x \in \mathbb{R}$ is the limit of a sequence of rational numbers.
- For $x \in \mathbb{R}$ to be computable, the convergence has to be effective.

Definition (Computable number): A real number $x \in \mathbb{R}$ is said to be *computable* if there exists a computable sequence $\{r_n\}_{n\in\mathbb{N}} \subset \mathbb{Q}$ of rational numbers which *converges effectively* to x,
i.e. if there exists a recursive function $e : \mathbb{N} \to \mathbb{N}$ such that for all $N \in \mathbb{N}$

$$|x - r_n| \leq 2^{-N} \qquad \text{whenever} \quad n \geq e(N) .$$

$\Rightarrow$ $x \in \mathbb{R}$ is computable if a Turing machine can approximate it with exponentially vanishing error.

- $\mathbb{R}_c$ stand for the set of all *computable real numbers*.
- $\mathbb{C}_c = \{x + \mathrm{i}y : x, y \in \mathbb{R}_c\}$ stands for the set of all *computable complex numbers*.
- Note that the set of computable numbers $\mathbb{R}_c \subsetneq \mathbb{R}$ is only countable.

Computable Functions

Definition: A function $f : \partial\mathbb{D} \to \mathbb{R}$ on the unit circle is said to be computable if

(a) f is Banach–Mazur computable, i.e. if f maps computable sequences $\{x_n\}_{n\in\mathbb{N}} \subset \mathbb{R}_c$ onto computable sequences $\{f(x_n)\}_{n\in\mathbb{N}} \subset \mathbb{R}_c$.

(b) f is effective uniformly continuous, i.e. if there is a recursive function $d : \mathbb{N} \to \mathbb{N}$ such that for every $N \in \mathbb{N}$ and all $\zeta_1, \zeta_2 \in \partial\mathbb{D}$ with $|\zeta_1 - \zeta_2| \leq 1/d(N)$ always $|f(\zeta_1) - f(\zeta_2)| \leq 2^{-N}$ is satisfied.

Lemma (equivalent definition of computability):

A function $f : \partial\mathbb{D} \to \mathbb{R}$ on the unit circle is computable if and only if there exists a computable sequence of real trigonometric polynomials $\{p_m\}_{m\in\mathbb{N}}$ which *converges effectively* to f in the uniform norm, i.e. if there exists a recursive function $e : \mathbb{N} \to \mathbb{N}$ such that for all $\theta \in [-\pi, \pi)$ and every $N \in \mathbb{N}$

$$m \geq e(N) \qquad \text{implies} \qquad \left|f(\mathrm{e}^{\mathrm{i}\theta}) - p_m(\mathrm{e}^{\mathrm{i}\theta})\right| \leq 2^{-N} .$$

Remark:

- There exist various notions of computability e.g. *Borel- or Markov computability*.
- Banach–Mazur computability is the weakest form of computability.
  $\Rightarrow$ If a function is not Banach–Mazur computable then it is not computable with respect to any other notion of computability.

Computable Functions in Banach Spaces

We consider functions in a Banach space $\mathscr{X}$ of functions on $\partial\mathbb{D}$ with norm $\|f\|_{\mathscr{X}}$.

Definition: A function $f \in \mathscr{X}$ is said to be $\mathscr{X}$*–computable* if

(a) f is computable (i.e. effectively approximable by rational polynomials p_m).

(b) its norm $\|f\|_{\mathscr{X}}$ is computable $\Rightarrow \|f - p_m\|_{\mathscr{X}}$ converges to zero effectively as $m \to \infty$.

The set of all $\mathscr{X}$-computable functions is denoted by $\mathscr{X}_{\mathrm{c}}$.

For continuous functions $\mathscr{C}(\partial\mathbb{D})$, computability implies $\mathscr{C}(\partial\mathbb{D})$-computability.

Lemma:

Let $f : \partial\mathbb{D} \to \mathbb{R}$ be a computable function on the unit circle. Then f is computable as a continuous function, i.e. $f \in \mathscr{C}_{\mathrm{c}}(\partial\mathbb{D})$.

- J. Avigad and V. Brattka, "Computability and analysis: The legacy of Alan Turing," in *Turing's legacy: developments from Turing's ideas in logic*, ser. Lecture Notes in Logic, Bd. 42. New York: Cambridge University Press, 2014, pp. 1–47.
- K. Weihrauch, *Computable Analysis*. Berlin: Springer-Verlag, 2000.

Non-Computability of the Spectral Factorization

Spectral Densities

We are going to show that the spectral factor

$$\phi_+(z) = (\mathrm{S}\phi)(z) = \exp\left(\frac{1}{4\pi}\int_{-\pi}^{\pi} \log\phi(\mathrm{e}^{\mathrm{i}\theta})\,\frac{\mathrm{e}^{\mathrm{i}\theta}+z}{\mathrm{e}^{\mathrm{i}\theta}-z}\,\mathrm{d}\tau\right), \qquad z \in \mathbb{D}$$

is *not computable*, even for computable spectral densities ϕ with very nice properties.

Definition (Set $\mathscr{D}$ of nice spectral densities):

A spectral density $\phi \in \mathscr{C}(\partial\mathbb{D})$ is said to belong to the set $\mathscr{D}$, if it has the following properties:

- ▷ ϕ is strictly positive on $\partial\mathbb{D}$, i.e. $\min_{\zeta\in\partial\mathbb{D}} \phi(\zeta) = s > 0$.
- ▷ ϕ is absolute continuous.
- ▷ ϕ belongs to the Wiener algebra $\mathscr{W}$, i.e. ϕ possess an absolutely converging Fourier series

$$\phi(\mathrm{e}^{\mathrm{i}\omega}) = \sum_{n=-\infty}^{\infty} c_n(\phi)\,\mathrm{e}^{\mathrm{i}n\omega} \qquad \text{with} \qquad c_n(\phi) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \phi(\mathrm{e}^{\mathrm{i}\omega})\,\mathrm{e}^{-\mathrm{i}n\omega}\,\mathrm{d}\omega\,.$$

- ▷ ϕ has finite Dirichlet energy, i.e.

$$\|\phi\|_{\mathrm{E}}^2 = \sum_{n\in\mathbb{Z}} |n|\,|c_n(\phi)|^2 < \infty\,.$$

- ▷ The spectral factor ϕ_+ has the same nice properties as ϕ, i.e.
  ϕ_+ is absolute continuous, in the Wiener algebra $\mathscr{W}$, and has finite Dirichlet energy.

The Non-Computability of the Spectral Factorization

Theorem:
To every computable point $\zeta \in \partial\mathbb{D}$ on the unit circle, there exists a *computable* spectral density $\phi \in \mathscr{D}$ such that $\phi_+(\zeta)$ is not a computable number, i.e. such that $\phi_+(\zeta) \notin \mathbb{C}_c$.

Remark:

- $\phi_+(\zeta)$ is not a computable number $\Rightarrow$ ϕ_+ is not Banach-Mazur computable.
- So ϕ_+ is not computable in any stronger notion of computability.
- Note that the input, i.e the spectral density ϕ is computable. However, the corresponding spectral factor ϕ_+ might not be computable.

A Decision Problem

- ▷ Assume a certain algorithm $\mathrm{A} : \phi \to \phi_+$ for calculating the spectral factor is given.
- ▷ If $\phi \in \mathscr{D}$ is a spectral density for which ϕ_+ is not computable then the algorithm A may not halt because it is not able to achieve the error bound in finite time.
- ? Can we decide algorithmically whether the spectral factor ϕ_+ of a given density ϕ is computable?

Decision Problem:

Does there exist an algorithm on a Turing machine which is able to decide for a given computable density $\phi \in \mathscr{D}$ whether the corresponding ϕ_+ is computable or not?

Theorem:

There exists no algorithmic solution of the above decision problem.

Consequences for Calculating the Causal Wiener Filter

Application: Wiener Filter

Estimation Problem:

- Estimating a wide-sense stationary (wss) stochastic process $\boldsymbol{x} = \{x_n\}_{n\in\mathbb{Z}}$ from the observation of a correlated wss stochastic process $\boldsymbol{y} = \{y_n\}_{n\in\mathbb{Z}}$.
- Let $r_{\boldsymbol{x}}(n) = \mathbb{E}[x_k \overline{x_{k+n}}]$ and $r_{\boldsymbol{x},\boldsymbol{y}}(n) = \mathbb{E}[x_k \overline{y_{k+n}}]$ be the corresponding *correlation functions* given by

$$r_{\boldsymbol{x}}(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \Phi(\theta)\mathrm{e}^{in\theta}\mathrm{d}\theta \qquad \text{and} \qquad r_{\boldsymbol{x},\boldsymbol{y}}(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \Psi(\theta)\mathrm{e}^{in\theta}\mathrm{d}\theta$$

  with densities $\Phi, \Psi \in L^1(\partial\mathbb{D})$.
- Find a causal linear filter of the form $\widehat{x}_n = \sum_{k=0}^{\infty} \gamma_k\, y_{n-k}$ such that the minimum mean square error $\mathbb{E}\left[|x_n - \widehat{x}_n|^2\right]$ is minimized.

Solution: Causal Wiener Filter

The transfer function $\Gamma(\mathrm{e}^{\mathrm{i}\omega}) = \sum_{k=0}^{\infty} \gamma_k\, \mathrm{e}^{\mathrm{i}k\omega}$ of this optimal filter (Wiener filter) is given by

$$\Gamma(\mathrm{e}^{\mathrm{i}\omega}) = \frac{1}{\Phi_+(\mathrm{e}^{\mathrm{i}\omega})}\left(\mathrm{P}_+\left[\frac{\Psi}{\Phi_-}\right]\right)(\mathrm{e}^{\mathrm{i}\omega})\,, \quad \omega \in [-\pi, \pi)\,.$$

Therein $\mathrm{P}_+ : L^2(\partial\mathbb{D}) \to H^2$ stands for the natural projection $\mathrm{P}_+ : \sum_{n=-\infty}^{\infty} c_n(f)\mathrm{e}^{\mathrm{i}\omega} \mapsto \sum_{n=0}^{\infty} c_n(f) z^n$ from $L^2(\partial\mathbb{D})$ onto the Hardy space $H^2 = \left\{f \in L^2(\partial\mathbb{D}) : c_n(f) = 0 \;\forall\; n < 0\right\}$.

Non-Computability of the Wiener Filter

Theorem:

There exist *computable spectral densities* $\Phi \in \mathscr{D}$ and *computable cross-correlations* $\Psi \in \mathscr{C}(\partial\mathbb{D})$ with Ψ is absolute continuous, $\Psi \in \mathscr{W}$, and Ψ has finite Dirichlet energy, such that the corresponding causal Wiener filter Γ satisfies

- ▷ Γ is absolute continuous
- ▷ Γ belongs to the Wiener algebra $\mathscr{W}$
- ▷ Γ has finite Dirichlet energy $\|\Gamma\|_{\mathrm{E}} < \infty$

but such that $\Gamma(1)$ *is not computable*.

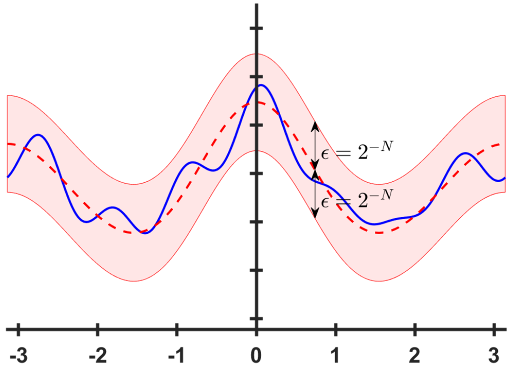
Proof:

- Choose Φ such that Φ_+ is not computable (see Theorem above).
- Choose the cross-correlation density Ψ as $\Psi(\zeta) = 1$ for all $\zeta \in \partial\mathbb{D}$.
- Verify the properties claimed by the theorem.

Summary

- There is no closed form expression for the spectral factor ϕ_+ of a spectral density, in general.
  $\Longrightarrow$ Numerically approximation methods (on digital computers) are applied to determine ϕ_+.

- Numerically approximation:
  Given ϕ and $\varepsilon > 0$, determine (in finite time) a confidence interval of width 2ε in which the (unknown) spectral factor ϕ_+ lies.
  $\Rightarrow$ ϕ_+ is computable.



- Main result: There exist computable spectral densities ϕ with very decent analytic properties (finite energy, absolute continuous, etc.) for which the spectral factor ϕ_+ is not computable.

- Decision problem: It is impossible to decide algorithmically (i.e. by an algorithm on an abstract Turing machine) whether a given computable spectral density ϕ possesses a computable spectral factor ϕ_+.

- Application: The causal Wiener filter is not computable even for very simple densities with decent analytic properties.