# The Divergence of all Sampling-based Methods for Calculating the Spectral Factorization

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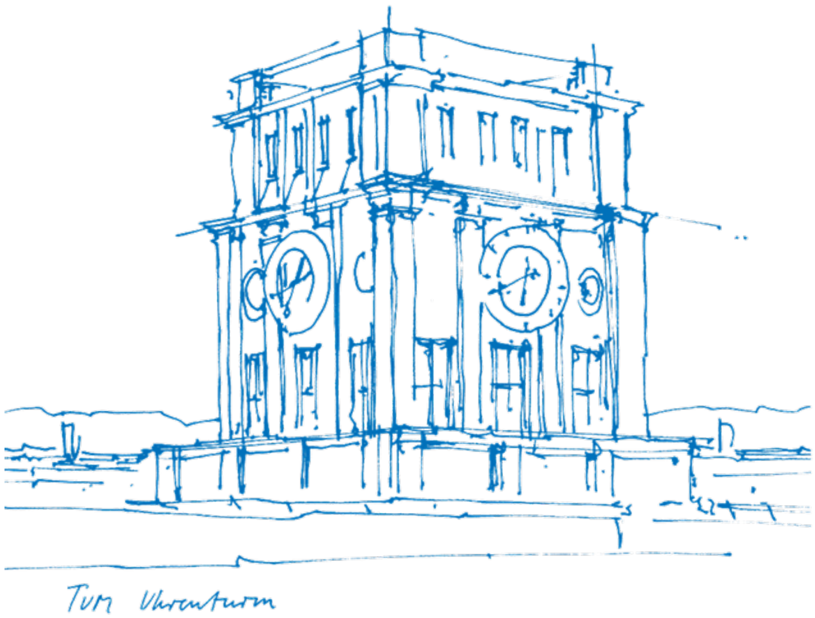
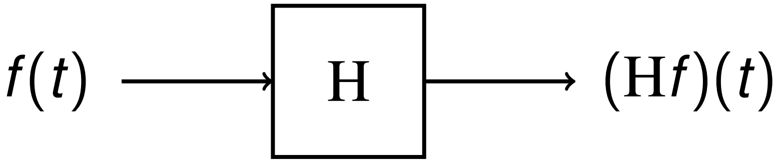
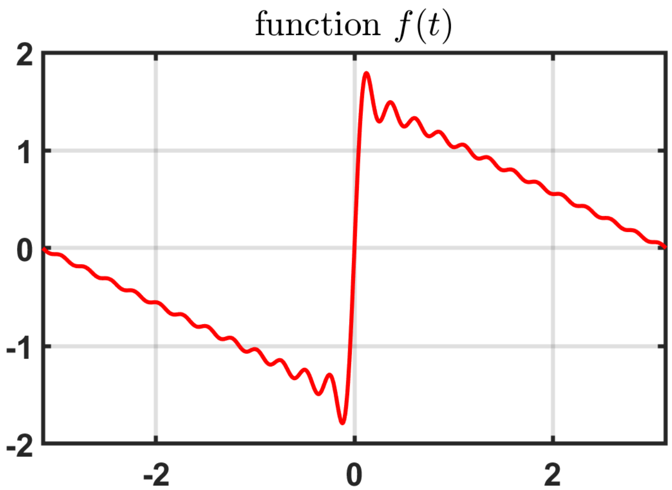


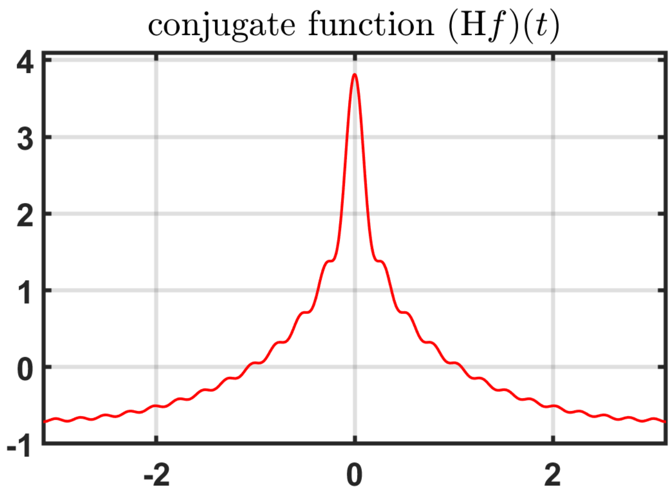
# General Question

- Can every continuous-time system $\mathrm{H}$ be simulated on a digital computer?
- Can every continuous-time system $\mathrm{H}$ be approximated by a time-discrete system?

$$\mathrm{H}: \mathscr{X} \to \mathscr{Y}$$



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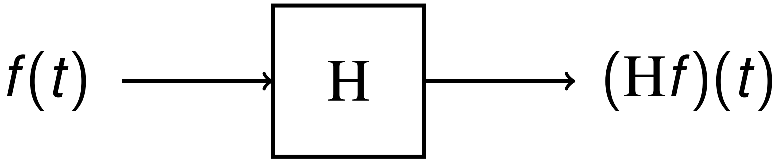


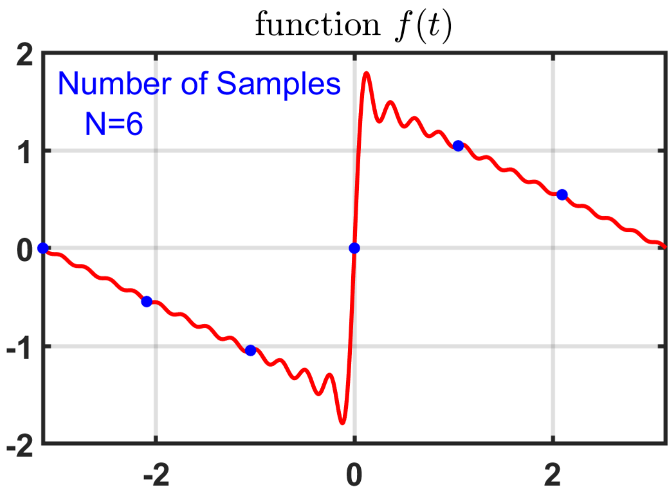


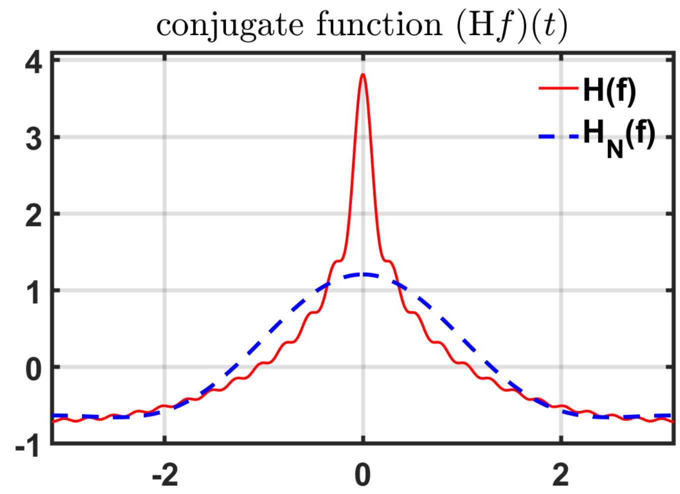

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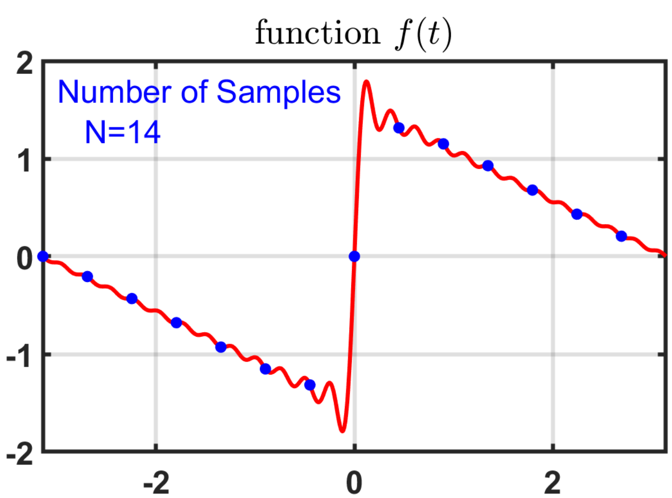


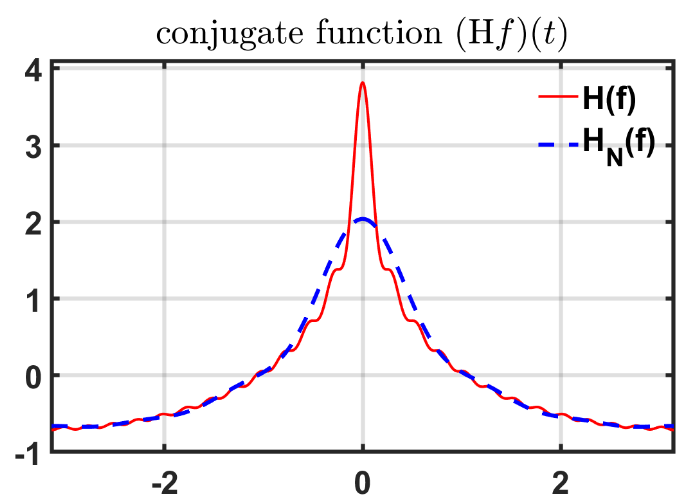

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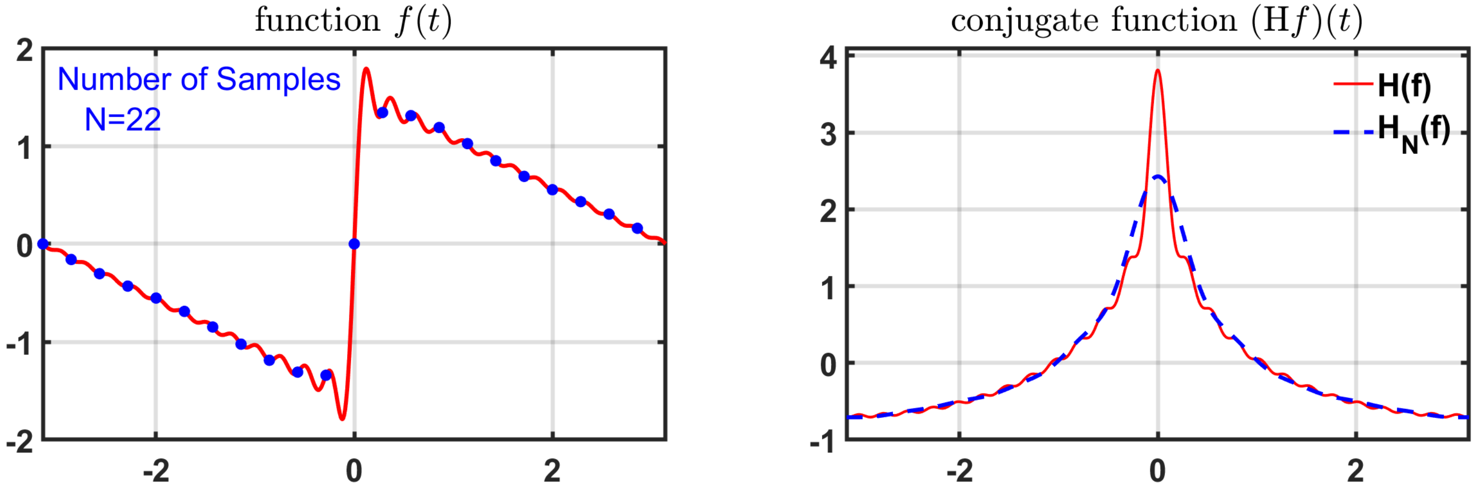

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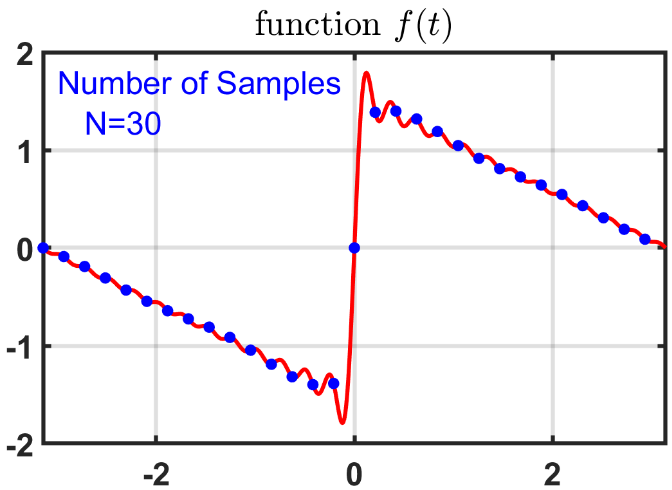


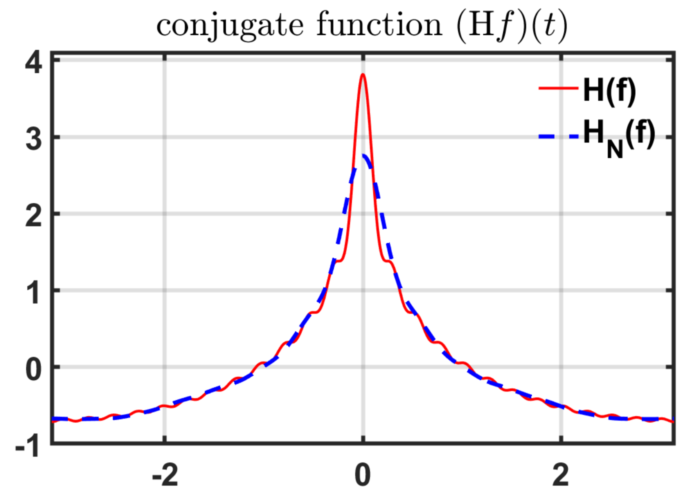

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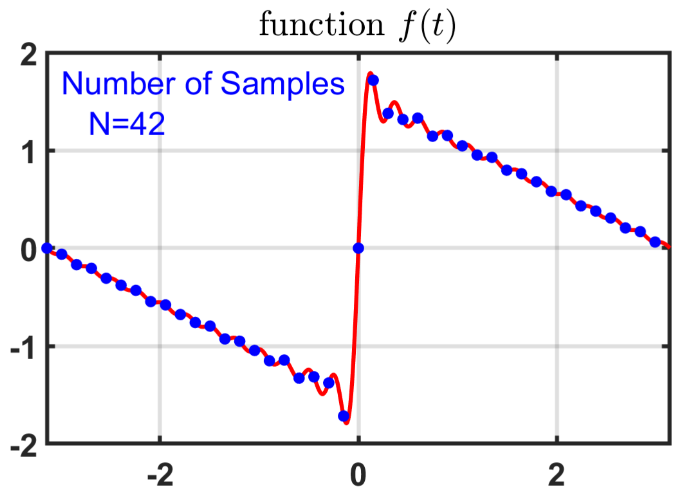


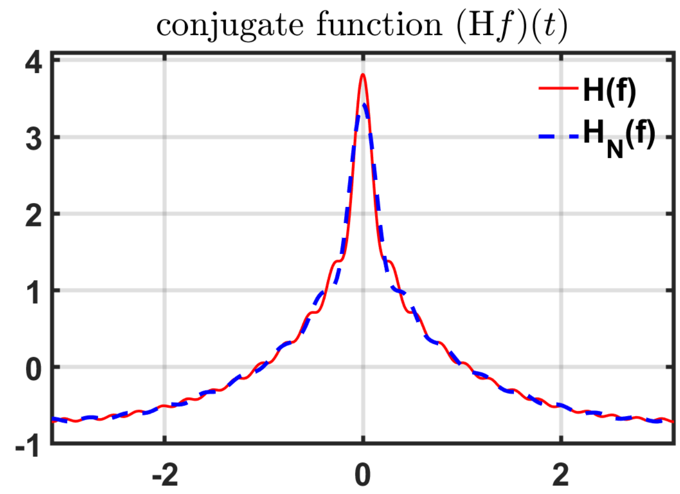

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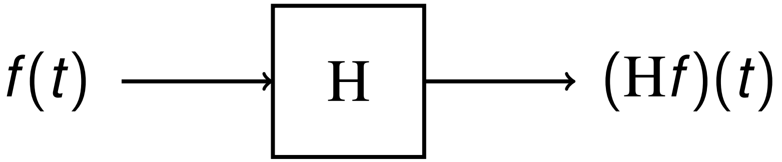


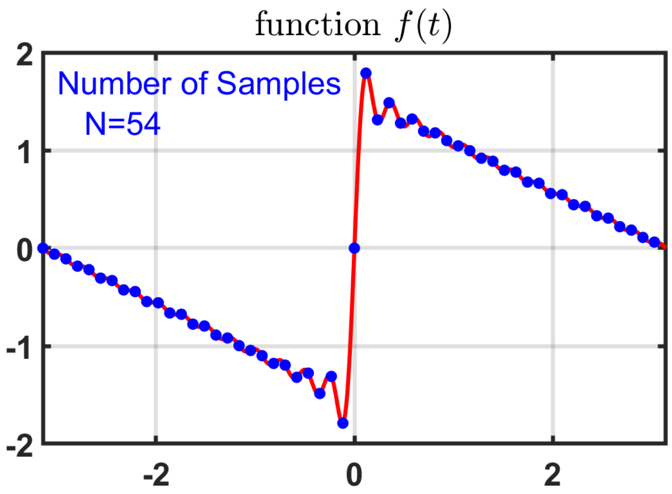


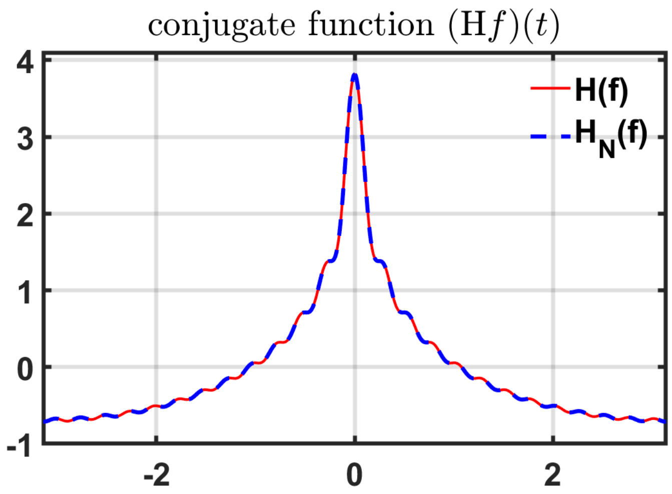

# This Talk – Outline

- ▷ Here we consider the spectral factorization mapping $\mathrm{H} = \mathrm{S}$.
- ▷ Answer depends on the signal space $\mathscr{X}$
  - For which signal space the spectral factorization can not be calculated on digital computers?
  - For which it is possible?

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## Outline

1. Spectral Factorization – A very short Introduction
2. Signal Spaces, Sets of Spectral Densities
3. Sampling-based Algorithms – Axioms
4. Main Result - No Sampling-based Algorithms for Spectral Factorization
5. Extension, Outlook

# Spectral Factorization

# Spectral Factorization

▷ Let $\phi$ be a spectral density. That is

– a non-negative real function on the unit circle $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$

– satisfying the *Paley–Wiener (Szegö) condition* $\log \phi \in L^1(\mathbb{T})$

▷ Spectral factorization is the operation of writing $\phi$ as

$$\phi(e^{i\theta}) = \phi_+(e^{i\theta})\,\phi_-(e^{i\theta}) = \left|\phi_+(e^{i\theta})\right|^2, \qquad \theta \in [-\pi, \pi) .$$

with the spectral factor $\phi_+$ and its *para-Hermitian conjugate* $\phi_-(z) = \overline{\phi_+(1/\overline{z})}$ for $z \in \mathbb{C}$.

▷ The spectral factor $\phi_+$ is an *outer function* (a "minimum-phase system"), i.e.

– $\phi_+(z)$ is analytic for every $z \in \mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$.

– $\phi(z) \neq 0$ for all $z \in \mathbb{D}$.

It can be written as

$$\phi_+(z) = (\mathrm{S}\phi)(z) = \exp\left(\frac{1}{4\pi}\int_{-\pi}^{\pi} \log\phi(e^{i\omega})\,\frac{e^{i\omega}+z}{e^{i\omega}-z}\,d\omega\right), \qquad z \in \mathbb{D} .$$

▷ We call $\mathrm{S} : \phi \mapsto \phi_+$ the spectral factorization mapping.

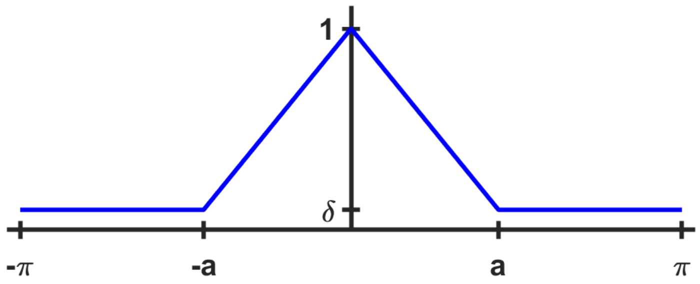
## Applications

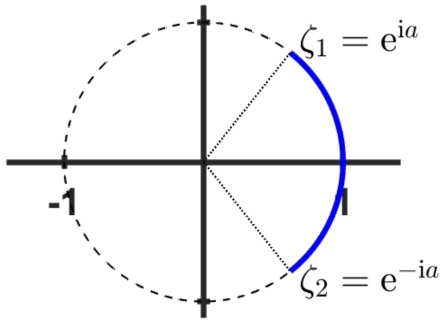
- Wiener–Kolmogorov theory of smoothing and prediction of stationary time series
- causal Wiener filter: Communications, signal processing, control theory, $\cdots$

# Spectral Factorization Mapping – Properties

- $\mathrm{S} : \phi \mapsto \phi_+$ has very complicated behavior (non-linear mapping, singular integral kernel)
- Even for very simple spectral densities $\phi$, the spectral factor can not be written as a closed form expression.

**Example (Piecewise linear spectral density):**




- Left side: a piecewise linear spectral density $\phi$
- Right side: The arc on which the spectral factor $\phi_+$ of $\phi$ is only given by a Cauchy principal value.

$$\phi_+(z) = \sqrt{\delta}\exp\left(\frac{1}{4\pi}\int_{-a}^{a}\log\left(\frac{\phi(e^{i\omega})}{\delta}\right)\frac{e^{i\omega}+z}{e^{i\omega}-z}\,d\omega\right), \qquad z\in\mathbb{C}.$$

# Sets of Spectral Densities

# Basic Notations

- ▷ Continuous functions on the unit circle: $\mathscr{C}(\mathbb{T}) = \{z \in \mathbb{C} : |z| = 1\}$ with norm $\|f\|_\infty = \max_{\zeta\in\mathbb{T}} |f(\zeta)|$.
- ▷ Fourier series and Fourier coefficients for $f \in L^2(\mathbb{T})$:

$$f(\mathrm{e}^{\mathrm{i}\theta}) = \sum_{n\in\mathbb{Z}} c_n(f)\,\mathrm{e}^{\mathrm{i}n\theta} \qquad \text{with} \qquad c_n(f) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\mathrm{e}^{\mathrm{i}\theta})\,\mathrm{e}^{-\mathrm{i}n\theta}\,\mathrm{d}\theta$$

- ▷ Dirichlet energy of $f \in \mathscr{C}(\mathbb{T})$

$$\|f\|_{\mathrm{E}}^2 = \sum_{n\in\mathbb{Z}} |n|\,|c_n(f)|^2$$

  - *Physical energy* – related to *Dirichlet principle* and *Dirichlet problem* in potential theory
  - $\|\cdot\|_{\mathrm{E}}$ is a seminorm.

- ▷ With each $f \in L^2(\mathbb{T})$ we associate its conjugate function

$$\widetilde{f}(\mathrm{e}^{\mathrm{i}\theta}) = -\mathrm{i}\sum_{n\in\mathbb{Z}} \mathrm{sgn}(n)\,c_n(f)\,\mathrm{e}^{\mathrm{i}n\theta}\,, \qquad \theta \in [-\pi,\pi)\,,.$$

The mapping $\mathrm{H} : f \mapsto \widetilde{f}$ is known as Hilbert transform.

# Signal Spaces

▷ Continuous functions of finite Dirichlet energy $\sim$ Sobolev space $H^{1/2}$

$$H^{1/2} = \{f \in \mathscr{C}(\mathbb{T}) \ : \ \|f\|_{\mathrm{E}} < +\infty\} \qquad \text{with norm} \qquad \|f\|_{H^{1/2}} = \max\big(\|f\|_\infty, \|f\|_{\mathrm{E}}\big)$$

▷ Set of positive spectral densities

$$\mathscr{D} = \Big\{\phi \in H^{1/2} \ : \ \min_{\omega \in [-\pi,\pi]} \phi(\mathrm{e}^{\mathrm{i}\omega}) = s > 0\Big\}$$

$\Rightarrow$ There is no fancy densities in $\mathscr{D}$: continuous, finite energy, strictly positive

▷ Disk algebra

$$\mathscr{A}(\mathbb{D}) = \{f \ : \ \text{analytic in } \mathbb{D} \text{ and continuous in } \overline{\mathbb{D}} = \mathbb{D} \cup \mathbb{T}\} \qquad \text{with norm} \qquad \|f\|_\infty = \sup_{z \in \mathbb{D}} |f(z)|$$

▷ Densities with well behaving spectral factor

$$\mathscr{D}_+ = \{\phi \in \mathscr{D} \ : \ \phi_+ \in \mathscr{A}(\mathbb{D})\}$$

– $\mathscr{D}_+$ is our primary space for investigating the spectral factorization
– $\mathscr{D}$ and $\mathscr{D}_+$ are not linear spaces $\Rightarrow$ we will consider often

$$\log(\mathscr{D}_+) = \{f = \log(\phi) \ : \ \phi \in \mathscr{D}_+\}$$

# Signal Spaces

- Continuous functions of finite Dirichlet energy with continuous conjugate

$$\mathscr{B}_0 = \left\{ f \in H^{1/2}(\mathbb{T}) \,:\, \widetilde{f} \in \mathscr{C}(\mathbb{T}) \right\} \qquad \text{with norm} \qquad \|f\|_{\mathscr{B}_0} = \max\left( \|f\|_\infty, \|f\|_{\mathrm{E}}, \|f\|_\infty \right)$$

  - $\mathscr{B}_0$ is a separable Banach space

## Lemma

$$\mathscr{B}_0 = \log(\mathscr{D}_+) = \left\{ f = \log(\phi) \,:\, \phi \in \mathscr{D}_+ \right\}$$
$$\mathscr{D}_+ = \exp(\mathscr{B}_0) = \left\{ \phi = \exp(f) \,:\, f \in \mathscr{B}_0 \right\}$$

- This lemma allows us to consider the spectral factorization as a mapping $\mathrm{T} = \exp \circ \mathrm{S}$ on $\mathscr{B}_0$

$$(\mathrm{T}f)(z) = (S[\exp(f)])(z) = \exp\left( \frac{1}{4\pi} \int_{-\pi}^{\pi} f(\mathrm{e}^{\mathrm{i}\omega}) \frac{\mathrm{e}^{\mathrm{i}\omega} + z}{\mathrm{e}^{\mathrm{i}\omega} - z} \mathrm{d}\omega \right), \qquad z \in \mathbb{D}$$

- If $\phi \in \mathscr{D}_+$ is a spectral density and $f = \log(\phi) \in \mathscr{B}_0$, then

$$\phi_+(\mathrm{e}^{\mathrm{i}\theta}) = (\mathrm{S}\phi)(\mathrm{e}^{\mathrm{i}\theta}) = (\mathrm{T}f)(\mathrm{e}^{\mathrm{i}\theta}) = \exp\left( \tfrac{1}{2} \left[ f(\mathrm{e}^{\mathrm{i}\theta}) + \mathrm{i}\widetilde{f}(\mathrm{e}^{\mathrm{i}\theta}) \right] \right), \qquad \theta \in [-\pi, \pi)$$

# Sampling-based Computations

# Computation of the Spectral Factorization

- Often the given spectral factor $\phi(\zeta)$ is not known for all $\zeta \in \mathbb{T}$ but only on a finite sampling set $\{\phi(\zeta_n)\}_{n=1}^{N}$.
- To calculate $\phi_+$ on a digital computer only finitely many samples $\{\phi(\zeta_n)\}_{n=1}^{N}$ of $\phi$ can be taken into account.

## Example (A Two Step Procedure)

- Assume $\phi \in \mathscr{D}_+$ be an arbitrary spectral density.
- For each $N$, let $\mathscr{T}_N = \{\zeta_n \, : \, n = 1, 2, \dots, N\} \subset \mathbb{T}$ be a sampling set.
- Let $\{\phi(\zeta_n) \, : \, n = 1, 2, \dots, N\}$ be values of $\phi$ on the sampling set $\mathscr{T}_N$

1. Determine an approximation $\phi_N$ of the density $\phi$ (e.g. by spline interpolation) such that
$$\lim_{N \to \infty} \left\| \phi - \phi_N \right\|_\infty = 0 \, .$$
2. Determine the spectral factor $(\phi_N)_+$ of the spline $\phi_N$ using standard algorithms for polynomial spectral factorization.

## Question

Do we have $\lim_{N\to\infty} \left\| \phi_+ - (\phi_N)_+ \right\|_\infty = 0$ for all $\phi \in \mathscr{D}$?

# Computations – General Algorithms

## General Structure of Numerical Algorithms

Let $\mathscr{D}_+$ be the set of spectral densities.

- To every $N \in \mathbb{N}$ there is a finite sampling set $\mathscr{T}_N = \{\zeta_n \,:\, n = 1, 2, \dots, N\} \subset \mathbb{T}$.
- To every $N \in \mathbb{N}$ there exists an approximation operator

$$\mathrm{S}_N \,:\, \{\phi(\zeta_n)\}_{n=1}^{N} \mapsto \phi_+^{(N)}$$

  which determines an approximation of the spectral factor $\phi_+$ based on the samples of the given spectral density $\phi$ on the sampling set $\mathscr{T}_N$.
- Requirement:

$$\lim_{N\to\infty} \left\| \phi_+ - \mathrm{S}_N(\phi) \right\|_\infty = \lim_{N\to\infty} \left\| \phi_+ - \phi_+^{(N)} \right\|_\infty = 0\,, \qquad \text{for all } \phi \in \mathscr{D}_+\,.$$

## Question

- For a given set $\mathscr{D}$, is it always possible to find such computational procedure $\{\mathrm{S}_N\}_{N\in\mathbb{N}}$?
- For which sets $\mathscr{D}$ no such computational procedure $\{\mathrm{S}_N\}_{N\in\mathbb{N}}$ exists?
- For which sets $\mathscr{D}$ it will be possible to find such approximation sequences.

# Sampling-based Algorithms – Axiomatic

Let $\boldsymbol{S} = \{\mathrm{S}_N\}_{N\in\mathbb{N}}$ be a sequence of operators $\mathrm{S}_N : \mathscr{D}_+ \to \mathscr{A}(\mathbb{D})$. We say that $\boldsymbol{S}$ is a *sampling-based approximation method* of the spectral factorization mapping $\mathrm{S}$ if it satisfies the following properties

(A) To every $N \in \mathbb{N}$ there exists a finite sampling set $\mathscr{T}_N \subset \mathbb{T}$ such that for arbitrary $\phi_1, \phi_2 \in \mathscr{D}_+$

$$\begin{aligned} \phi_1(\zeta_n) &= \phi_2(\zeta_n) \qquad && \text{for all } \zeta_n \in \mathscr{T}_N \\ \text{implies } \big(\mathrm{S}_N\phi_1\big)(z) &= \big(\mathrm{S}_N\phi_2\big)(z) && \text{for all } z \in \overline{\mathbb{D}}\,. \end{aligned}$$

Remark: Note that the operators $\mathrm{S}_N$ might be non-linear.

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(B) There exists a dense subset $\mathscr{M} \subset \mathscr{B}_0$ such that

$$\lim_{N\to\infty} \|\phi_+ - \mathrm{S}_N(\phi)\|_{\mathscr{A}(\mathbb{D})} = 0 \qquad \text{for all } \phi \in \mathscr{D}_+ \text{ with } \log(\phi) \in \mathscr{M}\,.$$

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(C) $\mathrm{S}_N(\phi)$ is an outer function for every $N \in \mathbb{N}$ and for each $\phi \in \mathscr{D}_+$, i.e. there is a $\varphi_N \in L^1(\mathbb{T})$ such that

$$\big(\mathrm{S}_N\phi\big)(z) = \exp\left(\tfrac{1}{2\pi}\int_{-\pi}^{\pi} \log\varphi_N(\mathrm{e}^{\mathrm{i}\tau})\tfrac{\mathrm{e}^{\mathrm{i}\tau}+z}{\mathrm{e}^{\mathrm{i}\tau}-z}\mathrm{d}\tau\right)\,.$$

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(D) Let $\mathrm{T}_N : \mathscr{B}_0 \to \mathscr{A}(\mathbb{D})$ be defined for every $f \in \mathscr{B}_0$ by $\mathrm{T}_N(f) = \mathrm{S}_N(\exp f)$. We require that for every $N \in \mathbb{N}$, $\mathrm{T}_N$ is a continuous mapping, i.e. if $\{f_n\}_{n\in\mathbb{N}} \subset \mathscr{B}_0$ is a convergent sequence with limit $f \in \mathscr{B}_0$ then

$$\lim_{n\to\infty} \big\|\mathrm{S}_N\big[\exp(f)\big] - \mathrm{S}_N\big[\exp(f_n)\big]\big\|_{\mathscr{A}(\mathbb{D})} = 0\,.$$

Remark: Note that the operators $\mathrm{S}_N$ might be non-linear.

# Main Negative Result

# No Sampling Based Algorithms

## Theorem

*Let $\boldsymbol{S} = \{\mathrm{S}_N\}_{N\in\mathbb{N}}$ be a sampling-based approximation method having the four properties (A) – (D). Then the set $\mathscr{R}_0 \subset \mathscr{B}_0$ of all $f \in \mathscr{B}_0$ such that for $\phi = \exp(f)$*

$$\limsup_{N\to\infty} \|\phi_+ - \mathrm{S}_N(\phi)\|_{\mathscr{A}(\mathbb{D})} > 0$$

*holds, is a residual set in $\mathscr{B}_0$.*

## Remarks

- For any sampling-based approximation method $\boldsymbol{S} = \{\mathrm{S}_N\}_{N\in\mathbb{N}}$ there exist spectral densities $\phi \in \mathscr{D}_+$ such that $\mathrm{S}_N(\phi)$ does not converge to $\phi_+$.
- The divergence set is large in the sense that all $f = \log(\phi)$ for which $\mathrm{S}_N(\phi)$ does not converge to $\phi_+$ is a residual set in $\mathscr{B}_0$.
- The approximation error $\|\phi_+ - \mathrm{S}_N(\phi)\|_{\mathscr{A}(\mathbb{D})}$ does not necessarily diverges for $N \to \infty$ but it does not converges to zero.

## Recall

- $\mathscr{D}_+$ contains spectral densities with very descent analytic properties
- The approximation operators $\mathrm{S}_N$ might be non-linear

# Proof Idea

Recall: Let $\phi \in \mathscr{D}_+$ be a spectral density and $f = \log(\phi) \in \mathscr{B}_0$, then

$$\phi_+(\mathrm{e}^{\mathrm{i}\theta}) = \exp\left(\tfrac{1}{2}\left[f(\mathrm{e}^{\mathrm{i}\theta}) + \mathrm{i}\widetilde{f}(\mathrm{e}^{\mathrm{i}\theta})\right]\right), \qquad \theta \in [-\pi, \pi).$$

## Theorem (Boche, Pohl, ACHA 2020)

*Let $\{\mathrm{H}_N\}_{N\in\mathbb{N}}$ be a sequence of continuous operators $\mathrm{H}_N : \mathscr{B}_0 \to \mathscr{C}(\mathbb{T})$ which satisfies the following properties*

(I) *For every $N \in \mathbb{N}$ there exists finite subset $\mathscr{T}_N \subset \mathbb{T}$ such that for all $f_1, f_2 \in \mathscr{B}_0$*

$$\begin{aligned} f_1(\zeta_n) &= f_2(\zeta_n) && \text{for all } \zeta_n \in \mathscr{T}_N \\ \text{implies } (\mathrm{H}_N f_1)(\zeta) &= (\mathrm{H}_N f_2)(\zeta) && \text{for all } \zeta \in \mathbb{T}\,. \end{aligned}$$

(II) *There exists a subset $\mathscr{M} \in \mathscr{B}_0$ such that for all $f \in \mathscr{M}$ always $\lim_{N\to\infty} \|\mathrm{H}f - \mathrm{H}_N(f)\|_\infty = 0$ holds.*

*Then the set*

$$\left\{f \in \mathscr{B}_0 \ :\ \limsup_{N\to\infty} \|\mathrm{H}f - \mathrm{H}_N(f)\|_\infty > 0\right\}$$

*is a residual (i.e. non-meager and dense) subset of $\mathscr{B}_0$.*

# Extensions, Outlook

# Positive Result

▷ Let $f \in L^2(\mathbb{T})$ with Fourier coefficients $\{c_n(f)\}_{n\in\mathbb{Z}}$. For $\alpha > 0$ let

$$\|f\|_\alpha = \left(\sum_{n\in\mathbb{Z}} |n|^{2\alpha}\,|c_n(f)|^2\right)^{1/2}.$$

▷ Define the Banach spaces

$$H^\alpha = \{f \in \mathscr{C}(\mathbb{T}) \,:\, \|f\|_\alpha < \infty\} \qquad \text{with norm} \qquad \|f\|_{H^\alpha} = \max\left(\|f\|_\infty, \|f\|_\alpha\right)$$

– $H^\beta \subset H^\alpha \subset H^{1/2}$ for all $\beta > \alpha > 1/2$.

**Theorem**

*For all $\alpha > 1/2$ there exists a sampling-based approximation method $\boldsymbol{S} = \{\mathrm{S}_N\}_{N\in\mathbb{N}}$ such that*

$$\lim_{N\to\infty} \|\phi_+ - \mathrm{S}_N(\phi)\|_{\mathscr{A}(\mathbb{D})} = 0 \qquad \textit{for all } f \in \mathscr{D}_+ \textit{ with } \log(\phi) \in H^\alpha.$$

- Sharp characterization in terms of $\log(\phi)$ in the scale of Sobolev spaces $H^\alpha$.

# Computability Analysis

▷ Results can be reformulated in the framework of computability analysis.

## Theorem (Negative Result)

*There exist (Turing) computable spectral densities $\phi \in \mathscr{D}_+$ such that*

1. $\phi$ *and* $\phi_+$ *are absolute continuous, and* $\phi$ *and* $\phi_+$ *are in the Wiener algebra, and* $\|\phi_+\|_{\mathrm{E}} < \infty$
2. $\phi_+(1)$ *is not a computable number.*

## Theorem (Positive Result)

*Let* $\alpha > 1/2$ *and let* $\phi \in \mathscr{D}_+$ *be a computable spectral density such that* $\log(\phi) \in H^{\alpha}$ *is a computable. Then the spectral factor* $\phi_+$ *is a computable continuous function.*

# Summary

- Spectral factorization and the Hilbert transform can not always be computed on digital computers.
- There is a fundamental difference between continuous-time and discrete-time systems.
- Important to characterize signal spaces which allow digital implementation.
- Classes of problems which can be solved by analog computers but not by digital computers.

# Summary

- ▷ Spectral factorization and the Hilbert transform can not always be computed on digital computers.
- ▷ There is a fundamental difference between continuous-time and discrete-time systems.
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Thank you