Nash Bargaining and Proportional Fairness for Wireless Systems

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***Abstract*—Nash bargaining and proportional fairness are popular strategies for distributing resources among competing users. Under the conventional assumption of a convex compact utility set, both techniques yield the same unique solution. In this paper, we show that uniqueness is preserved for a broader class of logarithmically convex sets. Then, we study a scenario where the performance of each user is measured by its signal-to-interference ratio (SIR). The SIR is modeled by an axiomatic framework of log-convex interference functions. No power constraints are assumed. It is shown how existence and uniqueness of a proportionally fair optimizer depends on the interference coupling among the users. Finally, we analyze the feasible SIR set. Conditions are derived under which the Nash bargaining strategy has a single-valued solution.**

***Index Terms*—Author, please supply your own keywords or send a blank e-mail to keywords@ieee.org to receive a list of suggested keywords.**

I. Introduction

Wireless communication systems use cooperative resource allocation in order to efficiently exploit the available power and bandwidth. Cooperation is often facilitated by centralized architectures like cellular systems. However, cooperation can also be useful between decentralized system components. By letting users cooperate, they can efficiently distribute their resources while trying to avoid interference.

Consider a wireless system, with $K \geq 2$ users from an index set $\mathcal{K} = \{1, 2, \ldots, K\}$. If the users are coupled by interference, then there is a general tradeoff between the users' utilities $u_1, \ldots, u_K$. By $\mathcal{U}$, we denote the set of all feasible utility vectors $\boldsymbol{u} = [u_1, \ldots, u_K]^T$. A fundamental problem is to find a suitable operating point on the boundary of $\mathcal{U}$. Toward this end, there are various different resource allocation strategies, aiming at different "fairness" or "efficiency" goals.

In this paper, we focus on the particular strategy of *proportional fairness*[1], which is closely linked to the game-theoretic concept of *Nash bargaining*[2]–[4]. While the conventional approach is based on the assumption of a convex utility set, we will extend these theoretical frameworks to certain nonconvex sets, which are convex after a logarithmic transformation.

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Our approach is motivated by the particular needs of a wireless communication system, where the utility set can crucially depend on the way users are coupled by interference. In most of this paper, we use the framework of interference functions [5] to explicitly model the interuser interference as a function of the transmission powers. The utility of each user is given by its signal-to-interference ratio (SIR). The utility set $\mathcal{U}$ is determined by the underlying interference functions. No power constraints are assumed.

Such a "physical-layer-aware" approach complicates the task of resource allocation. For example, the SIR often depends on dynamic interference mitigation techniques. Therefore, the interference can depend on the transmission powers in a complicated, nonlinear way. Standard properties of utility sets, like convexity or compactness, cannot be taken for granted. Time-sharing or randomization arguments are typically invoked to justify convexity (e.g., [6] and [7]). However, this argumentation does not hold for performance measures like the SIR.

Our analysis will be carried out under the assumption of no power constraints. This leads to an unbounded SIR set, which is only limited by the effects of mutual interference. We will show how existence and uniqueness of a proportionally fair optimizer depend on the structure of the interference coupling. Studying the problem in the absence of power constraints provides valuable insight into how the behavior of the system depends on the structure of the interference coupling. This knowledge is also useful for certain power-constrained systems, as shown in [8].

Before giving a detailed problem formulation, we briefly summarize some known results and concepts from the literature.

A. Nash Bargaining and Proportional Fairness

We begin with some definitions. Matrices and vectors are denoted by bold capital letters and bold lowercase letters, respectively. Let $\boldsymbol{y}$ be a vector, then $y_l = [\boldsymbol{y}]_l$ is the lth component. The notation $\boldsymbol{y} \geq 0$ means that $y_l \geq 0$ for all components l. Also, $\boldsymbol{y} \geq \boldsymbol{x}$ means component-wise inequality, and $\boldsymbol{y} \neq \boldsymbol{x}$ means that inequality holds for at least one component. Component-wise vector multiplication is denoted by $\boldsymbol{x} \circ \boldsymbol{y}$. The set of nonnegative reals is denoted by $\mathbb{R}_+$, and the set of positive reals is denoted by $\mathbb{R}_{++}$.

Definition 1: A set $\mathcal{U} \subset \mathbb{R}^K_{++}$ is said to be *(downward) comprehensive* if for any $\boldsymbol{u} \in \mathcal{U}$ and $\boldsymbol{u}' \in \mathbb{R}^K_{++}$

$$\boldsymbol{u}' \leq \boldsymbol{u} \quad \text{implies} \quad \boldsymbol{u}' \in \mathcal{U}.$$

A bargaining solution is the unanimous agreement on utilities $\boldsymbol{u} = [u_1, \ldots, u_K]^T$ from a utility set $\mathcal{U}$. The Nash bargaining solution (NBS) corresponds to a Pareto-optimal point $\varphi(\mathcal{U})$ characterized by a set of axioms (Nash axioms). A more detailed description is given in Section II-A.

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If the region $\mathcal{U} \subset \mathbb{R}^K_{++}$ is compact[1] convex comprehensive, then the unique NBS fulfilling the Nash axioms is obtained by maximizing the product of utilities, i.e.,

$$\max_{\boldsymbol{u}\in\mathcal{U}} \prod_{k\in\mathcal{K}} u_k. \tag{1}$$

Since $\log \max \prod_k u_k = \max \log \prod_k u_k = \max \sum_k \log u_k$, the optimizer of (1) can be found equivalently by solving

$$\max_{\boldsymbol{u}\in\mathcal{U}} \sum_{k\in\mathcal{K}} \log u_k. \tag{2}$$

If $\mathcal{U}$ is *not compact convex*, then it is *a priori* unclear whether the maximum (2) or (1), respectively, exists. If it exists, then it is unclear whether this optimum really is the Nash bargaining solution characterized by the axiomatic framework.

In the following, we will show that it is indeed possible to extend the concepts of Nash bargaining and proportional fairness to certain nonconvex noncompact sets while preserving their main properties like existence and uniqueness of an optimizer. One such set is the SIR region, which will be introduced in the next section.

In the following, we will refer to strategy (2) as *proportional fairness* (PF). In its original definition [1], a vector $\boldsymbol{u}^*$ is said to be proportionally fair if, for any other feasible vector $\boldsymbol{u}$, the aggregate of proportional changes $(u_k - u_k^*)/u_k^*$ is nonpositive (see also [9]). For convex sets, this unique point is obtained as the optimizer of (2). In this case, Nash bargaining and proportional fairness are equivalent [1], [10] (see also [11]–[13]).

In this paper, we are interested in certain nonconvex sets that are strictly convex after a logarithmic transformation. It will be shown that this property is sufficient to ensure a unique optimizer of (2). This optimizer is also the proportionally fair operating point.

B. Wireless Utility Model—The Feasible SIR Set

In a wireless system, the users' utilities can strongly depend on the underlying physical layer, so the SIR is an important measure for the link performances. In most of the paper (except for Section II), the utility under consideration is

$$\mathrm{SIR}_k(\boldsymbol{p}) = \frac{p_k}{\mathcal{I}_k(\boldsymbol{p})}, \quad k \in \mathcal{K}. \tag{3}$$

The function $\mathcal{I}_k(\boldsymbol{p})$ yields the interference power experienced by the kth user. It depends on the *transmission powers* $\boldsymbol{p} = [p_1, \ldots, p_K]^T$. Many common performance measures, like capacity or bit error rate, can be modeled as a monotonic function of the SIR. Note that the effective path gain of user k can be incorporated in $\mathcal{I}_k(\boldsymbol{p})$ as an additional scaling factor. Then, (3) is the ratio of the *received power* to interference power.

In order to model the interference, we use the axiomatic framework proposed in [5].

[1]In this paper, *compact* and *closed* are defined relatively in $\mathbb{R}^K_{++}$. A set $\mathcal{U} \subset \mathbb{R}^K_{++}$ is said to be *relatively closed* in $\mathbb{R}^K_{++}$ if there exists a closed set $\mathcal{A} \subset \mathbb{R}^K$ such that $\mathcal{U} = \mathcal{A} \bigcap \mathbb{R}^K_{++}$. Focusing on positive sets will simplify the following analysis, which is based on logarithmic transformations of utilities. Note that this does not restrict the generality of the results since, for the problem under consideration, there cannot be a solution including zeros.

Definition 2: A function $\mathcal{I} : \mathbb{R}^K_+ \mapsto \mathbb{R}_+$ is said to be an *interference function* if it fulfills the axioms

A1	(non-negativeness)	$\mathcal{I}(\boldsymbol{p}) \geq 0$
A2	(scale invariance)	$\mathcal{I}(\alpha\boldsymbol{p}) = \alpha\mathcal{I}(\boldsymbol{p}) \ \forall \alpha \in \mathbb{R}_+$
A3	(monotonicity)	$\mathcal{I}(\boldsymbol{p}) \geq \mathcal{I}(\boldsymbol{p}')$ if $\boldsymbol{p} \geq \boldsymbol{p}'$.

In order to rule out the trivial case $\mathcal{I}(\boldsymbol{p}) = 0$ we make an additional assumption

$$\text{There exists a } \boldsymbol{p}' > 0 \text{ such that } \mathcal{I}(\boldsymbol{p}') > 0. \tag{4}$$

This means that each interference function depends on at least one transmitter. The notion of *dependency* will be introduced later in Section III-B. Also, we assume that each transmitter has impact on at least one interference function.

The axioms A1, A2, A3 are similar to Yates' framework of *standard interference functions* [14]. However, there are some important differences. In [14], *scalability* $\alpha J(\boldsymbol{p}) > J(\alpha\boldsymbol{p})$, for $\alpha > 1$, was required in order for J to be a standard interference function. This property was motivated by the presence of a constant noise power $\sigma_n^2 > 0$. A simple example is the linear function $J(\boldsymbol{p}) = \boldsymbol{p}^T\boldsymbol{v} + \sigma_n^2$, where $\boldsymbol{v} \geq 0$ is a vector of interference coupling coefficients.

In order to model noise with the framework A1, A2, A3, a more explicit approach is required. Defining an extended power vector $\underline{\boldsymbol{p}} = \begin{bmatrix} \boldsymbol{p} \\ \sigma^2 \end{bmatrix}$, the interference-plus-noise power can be modeled by an interference function $\mathcal{I}(\underline{\boldsymbol{p}})$. If $\mathcal{I}$ is strictly monotonic with respect to $\underline{p}_{K+1}$ and σ^2 is constant, then $J(\boldsymbol{p}) = \mathcal{I}(\underline{\boldsymbol{p}})$ is a standard interference function. Any standard interference function can be modeled this way.

In this paper, we assume that transmission powers are unconstrained, so noise has no impact and can be ignored. This not only simplifies the problem, it also has the advantage of bringing out clearly the effects of interference coupling. The results will help to better understand the analytical structure of the problem, thereby providing a basis for future research that includes noise and power constraints.

The axiomatic framework A1, A2, A3 was motivated by specific power control problems, like [15]. However, it is also useful for characterizing other types of coupling effects. An example is the min-max optimum

$$C(\boldsymbol{\gamma}) = \inf_{\boldsymbol{p}>0} \left(\max_{k\in\mathcal{K}} \frac{\gamma_k \cdot \mathcal{I}_k(\boldsymbol{p})}{p_k} \right). \tag{5}$$

The function $C(\boldsymbol{\gamma})$ is an indicator for the feasibility of SIR values $\boldsymbol{\gamma} = [\gamma_1, \ldots, \gamma_K]^T$. We have $C(\boldsymbol{\gamma}) \leq 1$ if and only if for any $\epsilon > 0$, there exists a $\boldsymbol{p}_\epsilon > 0$ such that $\mathrm{SIR}_k(\boldsymbol{p}_\epsilon) \geq \gamma_k - \epsilon$, for all $k \in \mathcal{K}$ (see, e.g., [5] for more details). The feasible SIR region is the sublevel set

$$\mathcal{S} = \{\boldsymbol{\gamma} \in \mathbb{R}^K_{++} : C(\boldsymbol{\gamma}) \leq 1\}. \tag{6}$$

Boundary points of $\mathcal{S}$ are characterized by $C(\boldsymbol{\gamma}) = 1$. If the infimum (5) is not attained, then the boundary point $\boldsymbol{\gamma}$ is only achievable in an asymptotic sense, as discussed in Section VII.

Observe that the function $C(\boldsymbol{\gamma})$ fulfills the axioms A1, A2, A3, so it can formally be regarded as an "interference function."

Note that this notion of interference abstracts away from its original physical meaning. Because of the properties A1, A2, A3, the set $\mathcal{S}$ is downward comprehensive (cf. Definition 1). It was shown in [16] that in fact *every* compact comprehensive utility set from $\mathbb{R}^K_{++}$ can be expressed as a sublevel set of an interference function.

C. Problem Formulation and Contributions

Consider the problem of proportionally fair resource allocation (2), where the utility set is the SIR region $\mathcal{S}$. Exploiting $-\sum_k \log \gamma_k = \sum_k \log \gamma_k^{-1}$, the problem can be formulated as

$$PF(\mathcal{I}) = \inf_{\boldsymbol{\gamma} \in \mathcal{S}} \sum_{k \in \mathcal{K}} \log \gamma_k^{-1}. \quad (7)$$

Using the parametrization (3), this can be rewritten as

$$PF(\mathcal{I}) = \inf_{\boldsymbol{p} \in \mathcal{P}} \sum_{k \in \mathcal{K}} \log \frac{\mathcal{I}_k(\boldsymbol{p})}{p_k} \quad (8)$$

where $\mathcal{P}$ is the set of power vectors. Since the SIR (3) is invariant with respect to a scaling of $\boldsymbol{p}$, we can define $\mathcal{P}$ as

$$\mathcal{P} = \left\{ \boldsymbol{p} \in \mathbb{R}^K_{++} : \|\boldsymbol{p}\|_1 = 1 \right\}. \quad (9)$$

Note that the optimization (7) is over the SIR region directly, whereas (8) is over the set of power vectors. This approach allows to model the impact of the physical layer on the interference. For example, $\mathcal{I}(\boldsymbol{p})$ can depend on $\boldsymbol{p}$ in a nonlinear way. Some examples will be given later in Section III-A.

Remark 1: For certain systems operating in a high-SIR regime, it is customary to approximate the data rate as $\log(1+\mathrm{SIR}) \approx \log(\mathrm{SIR})$ (see, e.g., [17]. Then, our problem (8) can be interpreted as the maximization of the sum rate $\sum_k \log(1+\mathrm{SIR}_k)$.

The SIR region $\mathcal{S}$ is generally nonconvex and noncompact (because no power constraints are assumed), so it is not clear whether the frameworks of Nash bargaining and proportional fairness can be applied or not. It is even not clear whether the infimum (8) is actually attained.

Nash bargaining for nonconvex regions was studied, e.g., in [6] and [18]–[24]. However, these papers either deal with different types of regions (typically, only comprehensiveness is required, in which case uniqueness may be lost) or additional axioms are introduced in order to guarantee uniqueness. Also, most of this work was done in a context other than wireless communications.

In Section II, we will extend the conventional Nash bargaining framework to a certain class of logarithmically convex (log-convex) utility sets. This is motivated by the special needs of a wireless communication system, where such regions can occur (examples will follow). In this respect, our approach is not directly linked to the previous game-theoretic literature [6], [18]–[24].

Note that the results of Section II apply to arbitrary log-convex utility sets. In Section III and in what follows, we will focus on a particular log-convex utility. Namely, we will study the SIR region resulting from *log-convex interference functions*. This section builds on recent results [25], where the structure of log-convex interference functions was investigated. We exploit that the interference coupling in the system can be characterized by a $K \times K$ *dependency matrix* $\boldsymbol{D}_{\mathcal{I}}$ (see Section III-B)

Assuming log-convex interference functions, we will study the existence and uniqueness of a proportionally fair optimizer (8). The following fundamental questions will be addressed:

1) *Boundedness:* When is $PF(\mathcal{I}) > -\infty$ fulfilled?
2) *Existence:* When does an optimizer $\hat{\boldsymbol{p}} > 0$ exist such that $PF(\mathcal{I}) = \sum_k \log \mathcal{I}_k(\hat{\boldsymbol{p}})/\hat{p}_k$?
3) *Uniqueness:* When is $\hat{\boldsymbol{p}} > 0$ the unique optimizer?

Property $PF(\mathcal{I}) > -\infty$ is necessary for the existence of $\hat{\boldsymbol{p}}$, but not sufficient. This justifies a separate treatment of problem 1) in Section IV. It is shown that $PF(\mathcal{I}) > -\infty$ implies the existence of a row or column permutation such that the dependency matrix $\boldsymbol{D}_{\mathcal{I}}$ has a strictly positive main diagonal. An additional condition is provided under which the converse holds as well.

In Section V, the existence of an optimizer $\hat{\boldsymbol{p}} > 0$ is studied. Under certain monotonicity conditions, an optimizer exists if and only if there exist row and column permutations such that the resulting matrix is block-irreducible [26] and its main diagonal is positive. Otherwise, no Pareto-optimal operating point can be found.

In Section VI we show that the uniqueness of an existing optimizer depends on the structure of the matrix $\boldsymbol{D}_{\mathcal{I}}\boldsymbol{D}_{\mathcal{I}}^T$. This extends recent results [27], which were carried out in the context of linear interference functions.

Finally, in Section VII, we study under which condition the SIR feasible set is strictly log-convex. If this is fulfilled, and if an optimizer exists, then it follows from the results of Section II that the proportionally fair operating point is obtained as the single-valued Nash bargaining solution.

II. Nash Bargaining for Log-Convex Utility Sets

We start by briefly reviewing some fundamentals of Nash bargaining for compact comprehensive convex utility sets from $\mathbb{R}^K_{++}$. Then, we will extend this framework to certain noncompact *log-convex* utility sets.

A. Conventional Nash Bargaining Solution (NBS)

One of the most popular bargaining strategies is the (symmetric) *Nash bargaining solution* (NBS), which was proposed by Nash [2] (see also [3], [4], and [28]). The applicability of the NBS for resource sharing in communication networks was studied in [7], [10]–[12], [29], and [30].

Let $\mathcal{D}^K$ denote the family of all compact comprehensive convex utility sets from $\mathbb{R}^K_{++}$. For any $\mathcal{U} \in \mathcal{D}^K$, the NBS is the unique (single-valued) solution outcome $\varphi(\mathcal{U})$ that fulfills the following axioms.

- *Weak Pareto Optimality (WPO).* The users should not be able to collectively improve upon the solution outcome, i.e.,

$$\varphi(\mathcal{U}) \in \{\boldsymbol{u} \in \mathcal{U} : \text{there is no } \boldsymbol{u}' \in \mathcal{U} \text{ with } \boldsymbol{u}' > \boldsymbol{u}\}.$$

- *Symmetry (SYM).* If $\mathcal{U}$ is symmetric, then the outcome only depends on the bargaining strategy and not on the identities of the users, i.e., $\varphi_1(\mathcal{U}) = \ldots = \varphi_K(\mathcal{U})$. This does not mean that the game is necessarily symmetric, but rather that all users have the same priorities.
- *Independence of Irrelevant Alternatives (IIA).* If the feasible set shrinks, but the solution outcome remains feasible,

then the solution outcome of the smaller set should be the same, i.e.,

$$\varphi(\mathcal{U}) \in \mathcal{U}', \text{ with } \mathcal{U}' \subseteq \mathcal{U} \Longrightarrow \varphi(\mathcal{U}') = \varphi(\mathcal{U}).$$

- *Scale Transformation Covariance (STC)*. The optimization strategy is invariant with respect to a component-wise scaling of the region. That is, for any $\mathcal{U} \in \mathcal{D}^K$ and $\boldsymbol{a} \in \mathbb{R}^K$ with $\boldsymbol{a} > 0$ and $\boldsymbol{a} \circ \mathcal{U} \in \mathcal{D}^K$, we have

$$\boldsymbol{\varphi}(\boldsymbol{a} \circ \mathcal{U}) = \boldsymbol{a} \circ \boldsymbol{\varphi}(\mathcal{U}).$$

For any convex set $\mathcal{U} \in \mathcal{D}^K$, these four axioms are fulfilled by a unique solution, obtained by solving (1) or (2).

Nash introduced the bargaining problem in [2] for convex compact sets and two players. Later, in [31], he extended this work by introducing the concept of a *disagreement point* (also known as *threat point*), which is the solution outcome in case the players are unable to reach a unanimous agreement. Some "nonstandard" variations of the Nash bargaining problem exist, including nonconvex regions (see, e.g., [6], [18], [21], and [22]) and problem formulations without a disagreement point (see, e.g., [4] and the references therein).

In this paper, we formulate the Nash bargaining problem without disagreement point. Therefore, the axiom STC differs slightly from its common definition used in game-theoretical literature (e.g., [3]), where an additional invariance with respect to a translation of the region is required. Omitting the disagreement point is justified by the special structure of the problem under consideration. We are interested in utility sets for which the existence of a solution is always guaranteed. From a mathematical point of view, zero utilities must be excluded because of the possibility of singularities (SIR tending to infinity). However, from a technical perspective, this corresponds to a bargaining game with disagreement point zero. The results are also relevant for certain games with nonzero disagreement point: If the zero of the utility scales does not matter, then we can reformulate the game within a transformed coordinate system.

B. *Extension of Nash Bargaining to Log-Convex Sets*

In the remainder of this paper, we will drop the customary assumption that $\mathcal{U}$ is compact convex. We will extend the above framework to a broader class of *log-convex* sets. Consider the bijective continuous mapping $\log(\boldsymbol{u}) = [\log u_1, \ldots, \log u_K]^T$, where $\boldsymbol{u} \in \mathcal{U} \subset \mathbb{R}^K_{++}$.

Definition 3: We say that a set $\mathcal{U} \subset \mathbb{R}^K_{++}$ is *log-convex* if the image set

$$\mathcal{L}og(\mathcal{U}) = \{\boldsymbol{q} = \log(\boldsymbol{u}) : \boldsymbol{u} \in \mathcal{U}\} \tag{10}$$

is convex.

Definition 4: By $\mathcal{ST}$, we denote the family of all closed comprehensive utility sets $\mathcal{U} \subset \mathbb{R}^K_{++}$ such that $\mathcal{L}og(\mathcal{U})$ is a strictly convex set in $\mathbb{R}^K$. By $\mathcal{ST}_c$, we denote the family of all $\mathcal{U} \in \mathcal{ST}$ that are additionally bounded, thus compact.

For bounded sets from $\mathcal{ST}_c$, it was shown in [8] that the unique solution fulfilling the Nash axioms is always the optimizer of (1) and (2), respectively. Here, we consider a possibly unbounded set $\mathcal{U} \in \mathcal{ST}$, for which the results [8] cannot be applied directly. We first need to study under which condition an optimizer exists. To this end, we introduce an auxiliary set

$$\underline{\mathcal{U}}(\lambda) = \mathcal{U} \cap \mathcal{G}(\lambda) \tag{11}$$

where

$$\mathcal{G}(\lambda) = \left\{\boldsymbol{\gamma} \in \mathbb{R}^K_{++} : \sum\nolimits_{k \in \mathcal{K}} \gamma_k \leq \lambda\right\}, \quad \lambda > 0. \tag{12}$$

Unlike $\mathcal{U}$, the set $\underline{\mathcal{U}}(\lambda)$ is always contained in $\mathcal{ST}_c$. Thus, there is a unique Nash bargaining solution $\varphi\big(\underline{\mathcal{U}}(\lambda)\big)$, given as the optimizer of the Nash product [8]. The associated utilities are denoted by $\boldsymbol{u}(\lambda)$.

The following theorem provides a necessary and sufficient condition for the existence of a unique solution. The result will be needed later in Section VII.

Theorem 1: Let $\mathcal{U} \in \mathcal{ST}$. Problems (1) and (2), respectively, have a unique solution $\hat{\boldsymbol{u}}$ if and only if there exists a $\hat{\lambda}$ such that for all $\lambda \geq \hat{\lambda}$

$$\varphi\left(\underline{\mathcal{U}}(\lambda)\right) = \arg\max_{\boldsymbol{u} \in \underline{\mathcal{U}}(\lambda)} \prod_{k \in \mathcal{K}} u_k = \boldsymbol{u}(\hat{\lambda}). \tag{13}$$

Then, $\hat{\boldsymbol{u}} = \boldsymbol{u}(\hat{\lambda})$.

Proof: Assume that there is a $\hat{\lambda}$ such that (13) holds for any $\lambda \geq \hat{\lambda}$. Then, $\boldsymbol{u}(\hat{\lambda})$ is the solution of (1) for the set $\underline{\mathcal{U}}(\lambda)$. The solution is unique because $\underline{\mathcal{U}}(\lambda) \in \mathcal{ST}_c$. Thus, $\boldsymbol{u}(\hat{\lambda})$ is also the unique optimizer of the larger set $\mathcal{U}$.

With $\underline{\mathcal{U}}(\lambda) \subseteq \mathcal{U}$, we have

$$\max_{\boldsymbol{u} \in \underline{\mathcal{U}}(\lambda)} \prod_{k \in \mathcal{K}} u_k \leq \sup_{\boldsymbol{u} \in \mathcal{U}} \prod_{k \in \mathcal{K}} u_k =: C. \tag{14}$$

We show by contradiction that the supremum is finite. If $C = +\infty$, then for any $\mu > 0$, there is a $\boldsymbol{u}^{(\mu)} \in \mathcal{U}$ such that $\prod_k u_l^{(\mu)} > \mu$. There always exists a $\lambda \geq \hat{\lambda}$ such that $\boldsymbol{u}^{(\mu)} \in \underline{\mathcal{U}}(\lambda)$. Thus, the value $\max_{\boldsymbol{u} \in \underline{\mathcal{U}}(\lambda)} \prod_k u_k$ could become arbitrarily large, which contradicts the assumption that (13) holds for arbitrary $\lambda \geq \hat{\lambda}$. This implies $C < +\infty$. Inequality (14) is satisfied with equality for all $\lambda \geq \hat{\lambda}$. Since $\boldsymbol{u}(\hat{\lambda}) \in \mathcal{U}$, we have $\sup_{\boldsymbol{u} \in \mathcal{U}} \prod_k (u_k) = \prod_k (\hat{u}_k)$. That is, the maximum (1) is attained by $\boldsymbol{u}(\hat{\lambda})$.

Conversely, assume that $\hat{\boldsymbol{u}}$ is the solution of the product maximization (1). For any $\lambda > 0$, we have

$$\max_{\boldsymbol{u} \in \underline{\mathcal{U}}(\lambda)} \prod_{k \in \mathcal{K}} (u_k) \leq \max_{\boldsymbol{u} \in \mathcal{U}} \prod_{k \in \mathcal{K}} (u_k) = \prod_{k \in \mathcal{K}} (\hat{u}_k). \tag{15}$$

There exists a $\hat{\lambda}$ for which this inequality is fulfilled with equality, with the maximizer $\boldsymbol{u}(\hat{\lambda}) = \hat{\boldsymbol{u}}$. This solution is also contained in any larger set $\underline{\mathcal{U}}(\lambda)$ where $\lambda \geq \hat{\lambda}$. ■

Theorem 1 shows that the Nash bargaining framework outlined in Section II-A also holds for certain noncompact nonconvex sets, provided that an optimizer exists.

We can even further extend this result by allowing that $\mathcal{U}$ has boundary segments parallel to some coordinate axis. Such segments are irrelevant for the solution outcome because no point on a parallel segment can be the solution of the product optimization (1). Parallel boundary segments translate to parallel segments in the log-transformed image set, which means that the image set is *not* strictly convex. Yet, Theorem 1 still holds since this irrelevant part of the boundary can be safely excluded.

Hence, the proposed framework extends the classical Nash bargaining framework to a broader family of utility sets. Any conventional (i.e., convex compact comprehensive) set from $\mathbb{R}_{++}$ has a log-convex image set with the required properties. However, the converse is not true.

III. Log-Convex Interference Functions and Interference Coupling

In this section and in the remainder of this paper, we will focus on a particular set from $\mathcal{ST}$. Namely, we will study the SIR feasible region $\mathcal{S}$ (cf. Section I-B) resulting from log-convex interference functions. By exploiting properties of the interference coupling, we will characterize boundedness and existence of an optimizer, as discussed in Section I-C.

We will begin by introducing log-convex interference functions along with some examples.

A. Log-Convex Interference Functions

It was shown in [32] that the SIR region $\mathcal{S}$ defined in (6) is a compact comprehensive convex set if and only if $C(\boldsymbol{\gamma})$ is a convex interference function, i.e., A1, A2, A3 are fulfilled and $\mathcal{I}(\boldsymbol{p})$ is convex on $\mathbb{R}^K_+$.

However, $C(\boldsymbol{\gamma})$ is generally not convex, so $\mathcal{S}$ can be non-convex. It can also be unbounded. It is therefore unclear whether or not the conventional Nash bargaining theory can be applied to the feasible SIR set $\mathcal{S}$. However, we can exploit that $\mathcal{S}$ is *log-convex* since the underlying interference functions $\mathcal{I}_1, \dots, \mathcal{I}_K$ are log-convex by assumption. This allows us to exploit results from Section II.

For the next definition we introduce a change of variable $\boldsymbol{p} = \exp\{\boldsymbol{s}\}$ (component-wise exponential).

Definition 5: An interference function $\mathcal{I} : \mathbb{R}^K_+ \mapsto \mathbb{R}_+$ is said to be a *log-convex interference function* if $\log \mathcal{I}(\exp\{\boldsymbol{s}\})$ is convex on $\mathbb{R}^K$.

Let $f(\boldsymbol{s}) := \mathcal{I}(\exp\{\boldsymbol{s}\})$. Then, a necessary and sufficient condition for log-convexity is [33]

$$f(\boldsymbol{s}(\lambda)) \le f(\hat{\boldsymbol{s}})^{1-\lambda} f(\check{\boldsymbol{s}})^{\lambda},\ \forall \lambda \in (0,1); \hat{\boldsymbol{s}}, \hat{\boldsymbol{s}} \in \mathbb{R}^K \tag{16}$$

where

$$\boldsymbol{s}(\lambda) = (1-\lambda)\hat{\boldsymbol{s}} + \lambda \check{\boldsymbol{s}}, \quad \lambda \in (0,1). \tag{17}$$

The change of variable $\boldsymbol{p} = \exp\{\boldsymbol{s}\}$ was already used by Sung [34] in the context of linear interference functions (see the following example), and later in [17] and [35]–[38].

In the following, we will discuss some examples of log-convex interference functions.

Example 1: Linear interference function:

$$\mathcal{I}_k(\boldsymbol{p}) = \boldsymbol{p}^T \boldsymbol{v}_k, \quad k \in \mathcal{K} \tag{18}$$

where $\boldsymbol{v}_k \in \mathbb{R}^K_+$ is a vector of coupling coefficients. All coupling vectors can be collected in a $K \times K$ *coupling matrix*

$$\boldsymbol{V} = [\boldsymbol{v}_1, \dots, \boldsymbol{v}_K]^T. \tag{19}$$

The function (18) is a log-convex interference function in the sense of Definition 5.

Example 2: The coefficients $\boldsymbol{v}$ can adapt to the current interference situation. An example is the *worst-case interference*

$$\mathcal{I}_k(\boldsymbol{p}) = \max_{c_k \in \mathcal{C}_k} \boldsymbol{p}^T \boldsymbol{v}_k(c_k), \quad k \in \mathcal{K}. \tag{20}$$

The parameter c_k can stand for some uncertainty, chosen from a compact uncertainty set $\mathcal{C}_k$. The source of uncertainty can be system imperfections or channel estimation errors. Examples can be found in the literature on robust power allocation [39].

Example 3: It was shown in [32] that $\mathcal{I}(\boldsymbol{p})$ is a convex interference function if and only if there exists a compact comprehensive convex set $\mathcal{W}(\mathcal{I}) \subset \mathbb{R}^K_+$ such that

$$\mathcal{I}(\boldsymbol{p}) = \max_{\boldsymbol{v} \in \mathcal{W}(\mathcal{I})} \boldsymbol{v}^T \boldsymbol{p}. \tag{21}$$

This is a maximum over linear (thus log-convex) interference functions, so $\mathcal{I}(\boldsymbol{p})$ is a log-convex interference function in the sense of Definition 5. Hence, any convex interference function is a log-convex interference function. The converse, however, is not true. Therefore, the class of log-convex interference functions is broader than the class of convex interference functions.

At first glance, this might seem contradictory since any log-convex function is convex, but not the other way round [33]. This apparent contradiction is explained by the special definition of a *log-convex interference function* (Definition 5) involving the change of variable $\boldsymbol{p} = \exp\{\boldsymbol{s}\}$.

The previous example (20) is a convex (thus log-convex) interference function, so it is a special case of representation (21). Other interpretations are possible. For example, (21) can be regarded as the optimum of a weighted sum-utility maximization problem over a utility region $\mathcal{W}(\mathcal{I})$ with individual weights $p_1, \dots, p_K$. This supports the discussion in Section I-B, where it was claimed that the applicability of the axiomatic framework A1, A2, A3 is not restricted to interference in a physical sense.

Example 4: Consider the indicator function $C(\boldsymbol{\gamma})$ defined in (5). If the underlying interference functions $\mathcal{I}_1, \dots, \mathcal{I}_K$ are log-convex, then $C(\boldsymbol{\gamma})$ is a log-convex interference function in the sense of Definition 5. This means that $C\big(\exp(\boldsymbol{q})\big)$ is log-convex with respect to the substitute variable $\boldsymbol{q} = \log \boldsymbol{\gamma}$, which is the SIR on a logarithmic scale. Since every log-convex function is convex, it follows that the resulting log-SIR region (6) is a convex set. This result was shown in [5]. It generalizes previous results on linear interference functions [34] (see also [35]–[38]).

Example 5: Consider a matrix $\boldsymbol{W} \ge 0$. The matrix is assumed to be stochastic, i.e., $\boldsymbol{W}\boldsymbol{1} = \boldsymbol{1}$, where $\boldsymbol{1}$ is the all-ones vector. Defining $w_{kl} = [\boldsymbol{W}]_{kl}$ and some constants $f_k > 0$, we can construct log-convex interference functions

$$\mathcal{I}_k(\boldsymbol{p}) = f_k \prod_{l \in \mathcal{K}} (p_l)^{w_{kl}}, \quad k \in \mathcal{K}. \tag{22}$$

It was shown in [40] that any log-convex interference function can be expressed as a maximum over elementary functions of the form (22). Hence, (22) can be regarded as a basic building block of log-convex interference functions.

B. Characterization of Interference Coupling

Now, we return our attention to the existence and uniqueness of a proportionally fair operating point, as discussed in Section I-C. To this end, consider a K-user system characterized by log-convex interference functions $\mathcal{I}_1,\ldots,\mathcal{I}_K$. The resulting SIR region $\mathcal{S}$ and possible operating points depend on the *interference coupling* in the system.

Interference coupling is a well-known concept in the context of linear interference functions (18), where the mutual cross-talk of transmission powers is characterized by a nonnegative *coupling matrix* $\boldsymbol{V}$. Modeling interference coupling by such a nonnegative link gain matrix is common in power control theory. For this case, the problem of proportional fairness was already successfully analyzed, e.g., in [41], [42].

However, the axiomatic framework A1, A2, A3 is more general and allows for adaptive strategies, where interference is rejected depending on $\boldsymbol{p}$ (see the examples in Section III-A). Therefore, a new approach is required for the characterization of interference coupling.

Independent of the actual choice of the power allocation, the interference coupling can be characterized by an asymptotic approach.

Definition 6: The *asymptotic coupling matrix* is

$$[\boldsymbol{A}_\mathcal{I}]_{kl} = \begin{cases} 1, & \text{if there exists a } \boldsymbol{p} > 0 \text{ such that} \\ & \quad \lim_{\delta\longrightarrow\infty} \mathcal{I}_k(\boldsymbol{p}+\delta \boldsymbol{e}_l) = +\infty \\ 0, & \text{otherwise} \end{cases} \tag{23}$$

where $\boldsymbol{e}_l$ is the all-zero vector with the lth component set to one, i.e.,

$$[\boldsymbol{e}_l]_n = \begin{cases} 1, & n = l \\ 0, & n \neq l. \end{cases} \tag{24}$$

The 1-entries in the kth row of $\boldsymbol{A}_\mathcal{I}$ mark the positions of the power components on which $\mathcal{I}_k$ depends. Notice that because of property A2, we have the following property [25].

Lemma 1: If there exists a $\hat{\boldsymbol{p}} > 0$ such that $\lim_{\delta\longrightarrow\infty} \mathcal{I}_k(\hat{\boldsymbol{p}}+\delta \boldsymbol{e}_l) = +\infty$, then

$$\lim_{\delta\longrightarrow\infty} \mathcal{I}_k(\boldsymbol{p}+\delta \boldsymbol{e}_l) = +\infty \quad \text{for all } \boldsymbol{p} > 0. \tag{25}$$

Hence, the condition in (23) does not depend on the choice of $\boldsymbol{p}$. That is, $\boldsymbol{A}_\mathcal{I}$ provides a general characterization of interference coupling for interference functions fulfilling A1, A2, A3. The matrix $\boldsymbol{A}_\mathcal{I}$ can be regarded as a generalization of the link gain matrix (19) commonly used in power control theory. In particular, $[\boldsymbol{A}_\mathcal{I}]_{kl} = 1 \Leftrightarrow [\boldsymbol{V}]_{kl} > 0$ and $[\boldsymbol{A}_\mathcal{I}]_{kl} = 0 \Leftrightarrow [\boldsymbol{V}]_{kl} = 0$.

This can be further extended to arbitrary convex interference functions, as discussed in Example 3. Since every convex interference function can be expressed as (21), it follows that there exists a $\boldsymbol{V} = [\boldsymbol{v}_1,\ldots,\boldsymbol{v}_K]^T$, with $\boldsymbol{v}_k \in \mathcal{W}(\mathcal{I}_k)$, such that $\mathcal{I}_k(\boldsymbol{p}) = \boldsymbol{v}_k^T\boldsymbol{p}$ for all k. Among all possible matrices $\boldsymbol{V}$, if there exists only one matrix such that $[\boldsymbol{V}]_{kl} > 0$, then this implies $[\boldsymbol{A}_\mathcal{I}]_{kl} > 0$.

Another interesting interpretation of $\boldsymbol{A}_\mathcal{I}$ is obtained for the special log-convex interference function (22) in Example 5. The coefficient matrix $\boldsymbol{W}$ can be regarded as a coupling matrix. In particular, $[\boldsymbol{W}]_{kl} > 0 \Leftrightarrow [\boldsymbol{A}_\mathcal{I}]_{kl} > 0$.

For the special case of log-convex interference functions, the condition in (23) can be weakened [25].

Lemma 2: For log-convex interference functions, we have $\boldsymbol{A}_\mathcal{I} = \boldsymbol{D}_\mathcal{I}$, where

$$[\boldsymbol{D}_\mathcal{I}]_{kl} = \begin{cases} 1, & \text{if there exists a } \boldsymbol{p} > 0 \text{ such that} \\ & \quad \mathcal{I}_k(\boldsymbol{p}+\delta \boldsymbol{e}_l) \text{ is not constant} \\ & \quad \text{for some values } \delta > 0 \\ 0, & \text{otherwise.} \end{cases} \tag{26}$$

The *dependency matrix* $\boldsymbol{D}_\mathcal{I}$ will play a central role in the following analysis of the proportionally fair operating point (8).

IV. Boundedness of the Cost Function

Having characterized the interference coupling, we are now in a position to study the existence of the proportionally fair infimum $PF(\mathcal{I})$ defined in (8). That is, we want to show under which conditions $PF(\mathcal{I}) > -\infty$. The following simple example shows that $PF(\mathcal{I})$ can be unbounded.

1) Example 6: Consider linear interference functions $\mathcal{I}_k(\boldsymbol{p}) = [\boldsymbol{V}\boldsymbol{p}]_k$, $k = 1, 2, 3$, with a coupling matrix

$$\boldsymbol{V} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}. \tag{27}$$

Without loss of generality, we can scale $\boldsymbol{p}$ such that $\|\boldsymbol{p}\|_1 = p_1 + p_2 + p_3 = 1$. Then, the cost function becomes

$$\sum_{k=1}^{3} \log \frac{\mathcal{I}_k(\boldsymbol{p})}{p_k} = \log\left(\frac{p_3}{p_1 p_2}\right). \tag{28}$$

Choose $p_2 = p_1$ and $p_3 = 1/n$, with $n > 1$. Since $\|\boldsymbol{p}\|_1 = 1$, we have $p_1 = 1/2 - 1/2n$. Thus,

$$PF(\mathcal{I}) = \inf_{n>1} \log\left(\frac{1}{n-1}\right) = -\infty.$$

Before deriving the first result, we need to discuss an important property of our objective $\sum_k \log \mathcal{I}_k(\boldsymbol{p})/p_k$. Consider an arbitrary row permutation $\boldsymbol{\sigma} = [\sigma_1,\ldots,\sigma_K]$ applied to the matrix $\boldsymbol{D}_\mathcal{I}$. This corresponds to a reordering of the indices of $\mathcal{I}_1,\ldots,\mathcal{I}_K$, but without changing the indices of the transmission powers $p_1,\ldots,p_K$. Such a reordering does not affect the objective function in problem (8). For an arbitrary $\boldsymbol{p} > 0$, we have

$$\sum_{k\in\mathcal{K}} \log \frac{\mathcal{I}_k(\boldsymbol{p})}{p_k} = \sum_{k\in\mathcal{K}} \log \mathcal{I}_k(\boldsymbol{p}) - \sum_{k\in\mathcal{K}} \log p_k \tag{29}$$

$$= \sum_{k\in\mathcal{K}} \log \frac{\mathcal{I}_{\sigma_k}(\boldsymbol{p})}{p_k}. \tag{30}$$

This follows from the fact that the summands in (29) can be arranged and combined arbitrarily.

This means that the optimization problem (8) is invariant with respect to permutations of powers or interference functions. Defining arbitrary permutation matrices $\boldsymbol{P}^{(1)}$, $\boldsymbol{P}^{(2)}$, the permuted dependency matrix $\tilde{\boldsymbol{D}}_\mathcal{I} = \boldsymbol{P}^{(1)}\boldsymbol{D}_\mathcal{I}\boldsymbol{P}^{(2)}$ can equivalently be used in order to characterize the behavior of proportional fairness. This fundamental observation is the basis for the following results.

The next Lemma, which will be needed later for the proof of Theorem 2, shows a connection between boundedness and the structure of the dependency matrix $\boldsymbol{D}_\mathcal{I}$.

Definition 7: We say that $K' \leq K$ interference functions with indices $\sigma_1, \dots, \sigma_{K'}$ depend on a power component with index l if at least one of these functions depends on this power, i.e., there exists a $k \in \{1, \dots, K'\}$ such that $[\boldsymbol{D}_\mathcal{I}]_{\sigma_k, l} = 1$.

Lemma 3: If $PF(\mathcal{I}) > -\infty$, then for every $r \in \mathcal{K}$, arbitrary log-convex interference functions $\mathcal{I}_{\sigma_1}, \dots, \mathcal{I}_{\sigma_r}$ depend on at least r components of the power vector $\boldsymbol{p}$.

Proof: The proof is by contradiction. Assume that there is a number $\hat{r}$ and interference functions $\mathcal{I}_{k_1}, \dots, \mathcal{I}_{k_{\hat{r}}}$, which only depend on powers $p_{l_1}, \dots, p_{l_n}$, with $n < \hat{r}$. From (30), we know that interference functions and powers can be permuted such that $\mathcal{I}_1, \dots, \mathcal{I}_{\hat{r}}$ only depend on $p_1, \dots, p_n$, with $n < \hat{r}$. Consider the vector $\boldsymbol{p}(\delta)$, defined as

$$[\boldsymbol{p}(\delta)]_l = \begin{cases} \delta, & l = 1, \dots, n \\ 1, & l = n+1, \dots, K \end{cases}$$

where $0 < \delta \leq 1$, i.e., $\boldsymbol{p}(\delta) \leq \boldsymbol{1}$. Axiom A3 implies $\mathcal{I}_k\big(\boldsymbol{p}(\delta)\big) \leq \mathcal{I}_k(\boldsymbol{1})$, so we have

$$\begin{aligned}\sum_{k=1}^{K} \log \frac{\mathcal{I}_k\big(\boldsymbol{p}(\delta)\big)}{p_k(\delta)} &= \sum_{k=1}^{n} \log \frac{\delta \mathcal{I}_k(\boldsymbol{1})}{\delta} + \sum_{k=n+1}^{\hat{r}} \log \frac{\delta \mathcal{I}_k(\boldsymbol{1})}{1} \\ &\quad + \sum_{k=\hat{r}+1}^{K} \log \frac{\mathcal{I}_k\big(\boldsymbol{p}(\delta)\big)}{1} \\ &\leq \sum_{k=1}^{n} \log \mathcal{I}_k(\boldsymbol{1}) + \sum_{k=n+1}^{\hat{r}} \log \mathcal{I}_k(\boldsymbol{1}) \\ &\quad + (\hat{r} - n) \log \delta + \sum_{k=\hat{r}+1}^{K} \log \mathcal{I}_k(\boldsymbol{1}).\end{aligned}$$

Therefore

$$PF(\mathcal{I}) \leq \sum_{k=1}^{K} \log \frac{\mathcal{I}_k\big(\boldsymbol{p}(\delta)\big)}{p_k(\delta)} \leq \sum_{k=1}^{K} \log \mathcal{I}_k(\boldsymbol{1}) + (\hat{r} - n) \log \delta.$$

This holds for all δ, thus letting $\delta \longrightarrow 0$; we obtain the contradiction $PF(\mathcal{I}) = -\infty$, thus concluding the proof. ■

A. Necessary and Sufficient Condition for Boundedness

Using Lemma 3, the following result is shown.

Theorem 2: Let $\mathcal{I}_1, \dots, \mathcal{I}_K$ be arbitrary log-convex interference functions. If

$$\inf_{\boldsymbol{p} > 0} \sum_{k \in \mathcal{K}} \log \frac{\mathcal{I}_k(\boldsymbol{p})}{p_k} > -\infty \tag{31}$$

then there exists a row permutation $\boldsymbol{\sigma} = [\sigma_1, \dots, \sigma_K]$ such that $[\boldsymbol{D}_\mathcal{I}]_{\sigma_k, k} > 0$ for all $k \in \mathcal{K}$. That is, the permuted matrix has a positive main diagonal.

Proof: Assume that (31) is fulfilled. Consider the function $\mathcal{I}_K$, which depends on L_K powers, with indices $\boldsymbol{k}^{(K)} = [k_1^{(K)}, \dots, k_{L_K}^{(K)}]$. The trivial case $L_K = 0$ is ruled out by (4). Consider the lth component $k_l^{(K)}$. The set $\mathcal{L}^{(K)}(k_l^{(K)})$ contains the indices $m \neq k_l^{(K)}$ on which $\mathcal{I}_1, \dots, \mathcal{I}_{K-1}$ depend. More precisely, $\mathcal{L}^{(K)}(k_l^{(K)})$ is the set of indices $m \neq k_l^{(K)}$ such that there exists a $k \in \{1, 2, \dots, K-1\}$ with $[\boldsymbol{D}_\mathcal{I}]_{km} \neq 0$. Let $\#\mathcal{L}^{(K)}(k_l^{(K)})$ denote the cardinality of this set. It follows from Lemma 3 that there exists at least one $\hat{l}$, $1 \leq \hat{l} \leq L_K$, such that

$$\#\mathcal{L}^{(K)}\left(k_{\hat{l}}^{(K)}\right) = K - 1. \tag{32}$$

Otherwise, K interference functions could not depend on K powers. Note that (32) need not be fulfilled for *all* indices $\boldsymbol{k}^{(K)}$. If (32) is fulfilled for multiple indices, then we can choose one. Because of (30), the powers can be arbitrarily permuted. Thus, we can choose a permutation $\boldsymbol{\sigma}$ such that $\sigma_K = k_{\hat{l}}^{(K)}$. That is, the interference function $\mathcal{I}_K$ depends on p_{σ_K}, thus $[\boldsymbol{D}_\mathcal{I}]_{K, \sigma_K} \neq 0$. This component σ_K is now kept fixed. It remains to consider the remaining functions $\mathcal{I}_1, \dots, \mathcal{I}_{K-1}$ which depend on powers $p_{\sigma_1}, \dots, p_{\sigma_{K-1}}$. These powers can still be permuted arbitrarily.

We continue with the interference function $\mathcal{I}_{K-1}$, which depends on $L_{K-1} > 0$ powers, with indices $\boldsymbol{k}^{(K-1)} = [k_1^{(K-1)}, \dots, k_{L_{K-1}}^{(K-1)}]$. We denote by $\mathcal{L}^{(K-1)}(k_l^{(K-1)})$ the set of indices m (excluding σ_K and $k_l^{(K-1)}$) such that there exists a $k \in \{1, 2, \dots, K-2\}$ with $[\boldsymbol{D}_\mathcal{I}]_{km} \neq 0$. There exists at least one $\hat{l}$, $1 \leq \hat{l} \leq L_{K-1}$ (no matter which one) such that

$$\#\mathcal{L}^{(K-1)}\left(k_{\hat{l}}^{(K-1)}\right) = K - 2. \tag{33}$$

The remaining $K-1$ powers (except for σ_K) can still be permuted arbitrarily, so we can choose $\sigma_{K-1} = k_{\hat{l}}^{(K-1)}$. Thus, $[\boldsymbol{D}_\mathcal{I}]_{K-1, \sigma_{K-1}} \neq 0$. This component is also kept fixed, and we focus on the remaining functions $\mathcal{I}_1, \dots, \mathcal{I}_{K-2}$ which depend on $p_{\sigma_1}, \dots, p_{\sigma_{K-2}}$.

By repeating this procedure for all remaining interference functions, the result follows. ■

Next, we are interested in the converse of Theorem 2. Under which condition does the existence of a permuted matrix with positive main diagonal imply the boundedness of $PF(\mathcal{I})$? In order to answer this question we introduce an additional property

$$[\boldsymbol{D}_\mathcal{I}]_{k,l} > 0 \text{ implies } \mathcal{I}_k(\boldsymbol{e}_l) > 0 \text{ for any } k, l \in \mathcal{K} \tag{34}$$

where $\boldsymbol{e}_k$ is defined in (24).

Theorem 3: Under the additional property (34), the condition in Theorem 2 is necessary and sufficient.

Proof: Assume that there exists a $\boldsymbol{\sigma}$ such that $[\boldsymbol{D}_\mathcal{I}]_{\sigma_k, k} > 0$ for all $k \in \mathcal{K}$. With (34) and properties A2, A3, we have

$$\mathcal{I}_{\sigma_k}(\boldsymbol{p}) \geq \mathcal{I}_{\sigma_k}(\boldsymbol{p} \circ \boldsymbol{e}_k) = p_k \cdot \mathcal{I}_{\sigma_k}(\boldsymbol{e}_k) = p_k \cdot C_k > 0 \tag{35}$$

for all $k \in \mathcal{K}$, where C_k are some positive values. The cost function is invariant with respect to a permutation of the indices of the interference functions, as can be seen from (29), so we have

$$\sum_{k \in \mathcal{K}} \log \frac{\mathcal{I}_k(\boldsymbol{p})}{p_k} \geq \sum_{k \in \mathcal{K}} \log C_k > -\infty$$

which completes the proof. ■

Note that property (34) is always fulfilled, e.g., for linear interference functions (18) or worst-case interference functions (20). However, there exist log-convex interference functions that do not fulfill (34). An example is the elementary log-convex interference function (22), for which $\mathcal{I}_k(\boldsymbol{e}_l)=0$.

In the following, it will be shown that the additional requirement (34) is justified. It is not possible to derive a sufficient condition for boundedness from $\boldsymbol{D}_\mathcal{I}$ alone, without further assumptions.

B. Elementary Log-Convex Interference Functions

It was shown in [25] that the elementary functions (22) play an important role in the analysis of log-convex interference functions. Therefore, in the remainder of this section, we will study boundedness for this special case. For some given coefficient matrix $\boldsymbol{W}$, our cost function can be rewritten as

$$\sum_{k\in\mathcal{K}}\log\frac{\mathcal{I}_k(\boldsymbol{p})}{p_k}=\log\left(\frac{\prod_l(p_l)^{(\sum_k w_{kl})}\cdot\prod_k f_k}{\prod_k p_k}\right). \quad (36)$$

The matrix $\boldsymbol{W}$ is row stochastic, i.e., $\boldsymbol{W}\mathbf{1}=\mathbf{1}$. This is an immediate consequence of axiom A2, as shown in [25]. The following theorem shows that in order for (36) to be bounded, $\boldsymbol{W}$ also needs to be *column stochastic*.

Theorem 4: For interference functions (22), the infimum (8) is bounded if and only if $\boldsymbol{W}$ is doubly stochastic, i.e.,

$$PF(\mathcal{I})=\inf_{\boldsymbol{p}>0}\sum_{k\in\mathcal{K}}\log\frac{\mathcal{I}_k(\boldsymbol{p})}{p_k}>-\infty \quad \Leftrightarrow \quad \boldsymbol{W}^T\mathbf{1}=\mathbf{1}. \quad (37)$$

Proof: Assume $\boldsymbol{W}^T\mathbf{1}=\mathbf{1}$, i.e., $\sum_k w_{kl}=1$ for all l. Then, it can be observed from (36) that, independent of the choice of $\boldsymbol{p}$, we have

$$\sum_{k\in\mathcal{K}}\log\frac{\mathcal{I}_k(\boldsymbol{p})}{p_k}=\log\left(\prod_{k\in\mathcal{K}}f_k\right)>-\infty.$$

Conversely, assume that $PF(\mathcal{I})>-\infty$. The proof is by contradiction: Assume that $\boldsymbol{W}^T\mathbf{1}\neq\mathbf{1}$. Since $\boldsymbol{W}\mathbf{1}=\mathbf{1}$, we have $K=\sum_k(\sum_l w_{kl})=\sum_l(\sum_k w_{kl})$. Therefore, $\boldsymbol{W}^T\mathbf{1}\neq\mathbf{1}$ implies the existence of a column index $\hat{l}$ such that $\sum_k w_{k\hat{l}}=\hat{w}_{\hat{l}}>1$. Consider a sequence $\boldsymbol{p}(n)=[p_1(n),\ldots,p_K(n)]^T$, defined as

$$p_l(n)=\begin{cases}1/n, & l=\hat{l}\\ \frac{1}{K-1}(1-\frac{1}{n}), & \text{otherwise.}\end{cases} \quad (38)$$

Using (36), (38), and $\sum_{l\neq\hat{l}}\sum_k w_{kl}=K-\hat{w}_{\hat{l}}$, we have

$$\sum_{k\in\mathcal{K}}\log\frac{\mathcal{I}_k\big(\boldsymbol{p}(n)\big)}{p_k(n)} =\log\left(\left(\frac{1}{n}\right)^{\hat{w}_{\hat{l}}-1}\cdot\left(\frac{1}{K-1}\left(1-\frac{1}{n}\right)\right)^{1-\hat{w}_{\hat{l}}}\cdot\prod_{k\in\mathcal{K}}f_k\right). \quad (39)$$

Letting $n\longrightarrow\infty$, it can be observed that the argument of the log-function tends to zero, so (39) tends to $-\infty$. This contradicts the assumption, thus concluding the proof. ■

Theorem 4 provides a necessary and sufficient condition for boundedness for a special log-convex interference function for which (34) is not fulfilled. It becomes apparent that in this case the boundedness does not depend on the structure of $\boldsymbol{D}_\mathcal{I}$. If $\boldsymbol{W}$ is chosen such that $\boldsymbol{W}^T\mathbf{1}\neq\mathbf{1}$, then the cost function is unbounded, even if $[\boldsymbol{D}_\mathcal{I}]_{kl}=1$ for $k\neq l$. Hence, it is not possible to show the converse of Theorem 2 without additional assumptions. This is illustrated by a simple example.

Example 7: Consider log-convex interference functions (22) with a coefficient matrix

$$\boldsymbol{W}=\begin{bmatrix}0&1&0\\ \frac{1}{2}&0&\frac{1}{2}\\ \frac{1}{2}&\frac{1}{2}&0\end{bmatrix}. \quad (40)$$

We have $\boldsymbol{W}^T\mathbf{1}=[1\quad 3/2\quad 1/2]^T\neq\mathbf{1}$, so the condition in Theorem 4 is not fulfilled. With $\mathcal{I}_1(\boldsymbol{p})=p_2$, $\mathcal{I}_2(\boldsymbol{p})=(p_1)^{1/2}\cdot(p_3)^{1/2}$, and $\mathcal{I}_3(\boldsymbol{p})=(p_1)^{1/2}\cdot(p_2)^{1/2}$, we have

$$\inf_{\boldsymbol{p}>0}\sum_{k=1}^{3}\log\frac{\mathcal{I}_k(\boldsymbol{p})}{p_k}=\inf_{\boldsymbol{p}>0}\log\frac{(p_2p_3)^{1/2}}{p_3}=-\infty. \quad (41)$$

The infimum is not bounded, even though there exists a column permutation $\boldsymbol{P}^{(1)}$ such that the main diagonal of $\boldsymbol{D}_\mathcal{I}\boldsymbol{P}^{(1)}$ is nonzero.

V. Existence of a Proportionally Fair Optimizer

In the previous section, it was shown that boundedness $PF(\mathcal{I})>-\infty$ is connected with the positivity of the main diagonal of a permuted dependency matrix. Now, we investigate under which condition the infimum $PF(\mathcal{I})>-\infty$ is actually attained by a power allocation $\boldsymbol{p}>0$. The next example shows that this is not always fulfilled, not even for the simple linear interference functions (18).

1) Example 8: Consider linear interference functions $\mathcal{I}_k(\boldsymbol{p})=[\boldsymbol{V}\boldsymbol{p}]_k$, $k=1,2,3$, with a coupling matrix

$$\boldsymbol{V}=\begin{bmatrix}0&1&0\\1&0&1\\1&1&0\end{bmatrix}. \quad (42)$$

We have

$$\begin{aligned}PF(\mathcal{I})&=\inf_{\boldsymbol{p}>0}\sum_{k=1}^{3}\log\frac{\mathcal{I}_k(\boldsymbol{p})}{p_k}\\ &=-\log\left[\frac{p_1\cdot p_3}{(p_1+p_3)(p_1+p_2)}\right]\\ &\geq-\log\left[\frac{p_1\cdot p_3}{p_3\cdot p_1}\right]=0. \quad (43)\end{aligned}$$

Next, we will show that this inequality is fulfilled with equality. Choosing $p_1=\lambda$, $p_2=\lambda^2$, and $p_3=1-\lambda-\lambda^2$, we have

$$\sum_{k=1}^{3}\log\frac{\mathcal{I}_k(\boldsymbol{p})}{p_k}=-\log\frac{(1-\lambda-\lambda^2)}{(1-\lambda^2)(1+\lambda)}.$$

This tends to zero as $\lambda\rightarrow 0$. Thus

$$PF(\mathcal{I})=\inf_{\boldsymbol{p}>0}\sum_{k=1}^{3}\log\frac{\mathcal{I}_k(\boldsymbol{p})}{p_k}\leq 0. \quad (44)$$

Combining (43) and (44), it follows that $PF(\mathcal{I})=0>-\infty$.

Now, we study whether this infimum is attained. Assume that there exists an optimizer $\boldsymbol{p}^* > 0$, then

$$\begin{aligned} 0 &= \log \frac{\mathcal{I}_1(\boldsymbol{p}^*)}{p_1^*} + \log \frac{\mathcal{I}_2(\boldsymbol{p}^*)}{p_2^*} + \log \frac{\mathcal{I}_3(\boldsymbol{p}^*)}{p_3^*} \\ &= -\log \left[\frac{p_1^* \cdot p_3^*}{(p_1^*+p_3^*)(p_1^*+p_2^*)} \right] \\ &> -\log \left[\frac{p_1^* \cdot p_3^*}{p_3^* p_1^*} \right] = 0. \end{aligned} \tag{45}$$

This is a contradiction, so the infimum $PF(\mathcal{I}) = 0$ is not attained.

Now, consider arbitrary log-convex interference functions $\mathcal{I}_1, \ldots, \mathcal{I}_K$. The mutual coupling is characterized by the dependency matrix $\boldsymbol{D}_\mathcal{I}$ defined in (26). We may assume, without loss of generality, that $\boldsymbol{D}_\mathcal{I}$ is in canonical form [26, p. 75]

$$\boldsymbol{D}_\mathcal{I} = \left[\begin{array}{ccc|ccc} \boldsymbol{D}^{(1,1)} & & 0 & 0 & \ldots & 0 \\ & \ddots & & \vdots & \ldots & \vdots \\ 0 & & \boldsymbol{D}^{(i,i)} & 0 & \ldots & 0 \\ \hline \boldsymbol{D}^{(i+1,1)} & \ldots & \boldsymbol{D}^{(i+1,i)} & \boldsymbol{D}^{(i+1,i+1)} & & 0 \\ \vdots & \ldots & \vdots & \vdots & \ddots & \\ \boldsymbol{D}^{(N,1)} & \ldots & \boldsymbol{D}^{(N,i)} & \boldsymbol{D}^{(N,2)} & \ldots & \boldsymbol{D}^{(N,N)} \end{array} \right]. \tag{46}$$

For any given dependency matrix $\boldsymbol{D}'_\mathcal{I}$, there always exists a permutation matrix $\boldsymbol{P}$ such that $\boldsymbol{D}_\mathcal{I} = \boldsymbol{P}\boldsymbol{D}'_\mathcal{I}\boldsymbol{P}^T$ has canonical form. This symmetric permutation preserves the relevant properties that will be exploited, so in the following, we can simplify the discussion by assuming that $\boldsymbol{D}_\mathcal{I}$ has the form (46). The matrix $\boldsymbol{D}_\mathcal{I}$ has N *irreducible* blocks $\boldsymbol{D}^{(n)} := \boldsymbol{D}^{(n,n)}$ along its main diagonal (shaded in gray). Recall that $\boldsymbol{D}^{(n)}$ is irreducible if and only if its associated directed graph is *strongly connected* [26]. If $\boldsymbol{D}_\mathcal{I}$ is irreducible, then it consists of one single block. We say that $\boldsymbol{D}_\mathcal{I}$ is *block-irreducible* if

$$\boldsymbol{D}_\mathcal{I} = \begin{bmatrix} \boldsymbol{D}^{(1)} & & 0 \\ & \ddots & \\ 0 & & \boldsymbol{D}^{(N)} \end{bmatrix}$$

where all subblocks $\boldsymbol{D}^{(n)}$ are irreducible.

Before stating the main result of this section (Theorem 5), we need some more definitions.

Definition 8 (Dependency Set): The *dependency set* $\mathsf{L}(k)$ is the index set of transmitters on which user k depends, i.e.,

$$\mathsf{L}(k) = \{l \in \mathcal{K} : [\boldsymbol{D}_\mathcal{I}]_{kl} = 1\}. \tag{47}$$

Definition 9 (Strict Monotonicity): $\mathcal{I}_k(\boldsymbol{p})$ is said to be strictly monotonic (on its dependency set) if $\boldsymbol{p}^{(1)} \geq \boldsymbol{p}^{(2)}$, with $p_l^{(1)} > p_l^{(2)}$ for some $l \in \mathsf{L}(k)$, implies $\mathcal{I}_k(\boldsymbol{p}^{(1)}) > \mathcal{I}_k(\boldsymbol{p}^{(2)})$.

In other words, $\mathcal{I}_k(\boldsymbol{p})$ is strictly increasing in at least one power component. Given this property, we can derive a necessary and sufficient condition for the existence of a proportionally fair optimizer.

Theorem 5: Let $\mathcal{I}_1, \ldots, \mathcal{I}_K$ be strictly monotonic log-convex interference functions. We assume that (34) is fulfilled. There exists a proportionally fair optimizer $\hat{\boldsymbol{p}} > 0$ if and only if there exist permutation matrices $\boldsymbol{P}^{(1)}$, $\boldsymbol{P}^{(2)}$ such that $\hat{\boldsymbol{D}}_\mathcal{I} := \boldsymbol{P}^{(1)}\boldsymbol{D}_\mathcal{I}\boldsymbol{P}^{(2)}$ is block-irreducible and its main diagonal is strictly positive.

Proof: The proof is given in the Appendix. ■

In the next section, we will study whether the optimizer characterized by Theorem 5 is unique.

VI. Uniqueness of the Solution

In the remainder of the paper, we assume that the interference functions $\mathcal{I}_1, \ldots, \mathcal{I}_K$ are log-convex in the sense of Definition 5. By the log-transformation, the line segment $\boldsymbol{s}(\lambda)$ defined in (17) is transformed to

$$\boldsymbol{p}(\lambda) = \log \boldsymbol{s}(\lambda) = \hat{\boldsymbol{p}}^{1-\lambda} \cdot \check{\boldsymbol{p}}^{\lambda} \tag{48}$$

With (16), it is clear that $\mathcal{I}_k(\mathrm{e}^{\boldsymbol{s}})$ is log-convex if and only if

$$\mathcal{I}_k\big(\boldsymbol{p}(\lambda)\big) \leq \big(\mathcal{I}_k(\hat{\boldsymbol{p}})\big)^{1-\lambda} \cdot \big(\mathcal{I}_k(\check{\boldsymbol{p}})\big)^{\lambda}, \quad \lambda \in (0,1). \tag{49}$$

Assume that there exists an optimizer for the problem of proportional fairness (8). Is this optimizer unique or not? In order to answer this question, we analyze the cost function

$$G(\boldsymbol{s}) = \sum_{k \in \mathcal{K}} \log \frac{\mathcal{I}_k(\mathrm{e}^{\boldsymbol{s}})}{\mathrm{e}^{s_k}}, \text{ on } \mathbb{R}^K \tag{50}$$

where we have used the substitution $\boldsymbol{p} = \mathrm{e}^{\boldsymbol{s}}$.

It is sufficient to show that the cost function $G(\boldsymbol{s})$ is strictly convex. Since $\boldsymbol{p} = \mathrm{e}^{\boldsymbol{s}}$ is a one-to-one mapping, uniqueness of an optimizer $\boldsymbol{s}$ implies uniqueness of the original problem (8). Note, that it is not necessary to show strict convexity of the SIR region, this will be done later in Section VII.

We start with the following lemma, which will be needed later for Theorem 6.

Lemma 4: The function $G(\boldsymbol{s})$ defined in (50) is strictly convex if and only if for arbitrary vectors $\hat{\boldsymbol{p}}, \check{\boldsymbol{p}} \in \mathbb{R}^K_{++}$, with $\hat{\boldsymbol{p}} \neq \mu\check{\boldsymbol{p}}$, $\mu \in \mathbb{R}_{++}$, there exists a $\lambda_0 \in (0,1)$ and at least one index k_0 such that

$$\mathcal{I}_{k_0}\left(\boldsymbol{p}(\lambda_0)\right) < \left(\mathcal{I}_{k_0}(\hat{\boldsymbol{p}})\right)^{1-\lambda_0} \cdot \left(\mathcal{I}_{k_0}(\check{\boldsymbol{p}})\right)^{\lambda_0}. \tag{51}$$

Proof: Assume that (51) holds for k_0. With $\hat{\boldsymbol{p}} = \mathrm{e}^{\hat{\boldsymbol{s}}}$ and $\check{\boldsymbol{p}} = \mathrm{e}^{\check{\boldsymbol{s}}}$, we have

$$\begin{aligned} G\big(\boldsymbol{s}(\lambda_0)\big) &= \sum_{k \in \mathcal{K}\setminus k_0} \log \frac{\mathcal{I}_k(\mathrm{e}^{\boldsymbol{s}(\lambda_0)})}{\mathrm{e}^{s_k(\lambda_0)}} + \log \frac{\mathcal{I}_{k_0}(\mathrm{e}^{\boldsymbol{s}(\lambda_0)})}{\mathrm{e}^{s_{k_0}(\lambda_0)}} \\ &\leq (1-\lambda_0) \sum_{k \in \mathcal{K}\setminus k_0} \log \frac{\mathcal{I}_k(\mathrm{e}^{\hat{\boldsymbol{s}}})}{\mathrm{e}^{\hat{s}_k}} \\ &\quad + \lambda_0 \sum_{k \in \mathcal{K}\setminus k_0} \log \frac{\mathcal{I}_k(\mathrm{e}^{\check{\boldsymbol{s}}})}{\mathrm{e}^{\check{s}}_k} + \log \frac{\mathcal{I}_{k_0}(\mathrm{e}^{\boldsymbol{s}(\lambda_0)})}{\mathrm{e}^{s_{k_0}(\lambda_0)}} \\ &< (1-\lambda_0) \sum_{k \in \mathcal{K}} \log \frac{\mathcal{I}_k(\mathrm{e}^{\hat{\boldsymbol{s}}})}{\mathrm{e}^{\hat{s}_k}} + \lambda_0 \sum_{k \in \mathcal{K}} \log \frac{\mathcal{I}_k(\mathrm{e}^{\check{\boldsymbol{s}}})}{\mathrm{e}^{\check{s}}_k} \\ &= (1-\lambda_0)G(\hat{\boldsymbol{s}}) + \lambda_0 G(\check{\boldsymbol{s}}) \end{aligned} \tag{52}$$

where the first inequality follows from the convexity of $G\big(\boldsymbol{s}(\lambda_0)\big)$ [25], and the second strict inequality is due to (51).

Conversely, assume that G is strictly convex. The proof is by contradiction: Suppose that there is $\hat{\boldsymbol{s}}, \check{\boldsymbol{s}} \in \mathbb{R}^K$ and $\lambda_0 \in (0,1)$, such that for all $k \in \mathcal{K}$,

$$\mathcal{I}_k\big(\mathrm{e}^{\boldsymbol{s}(\lambda_0)}\big) = \big(\mathcal{I}_k(\mathrm{e}^{\hat{\boldsymbol{s}}})\big)^{1-\lambda_0} \cdot \big(\big(\mathcal{I}_k(\mathrm{e}^{\check{\boldsymbol{s}}})\big)^{\lambda_0}. \tag{53}$$

With (53), we have

$$\begin{aligned} G\big(\boldsymbol{s}(\lambda_0)\big) &= \sum_{k\in\mathcal{K}} \log \frac{\big(\mathcal{I}_k(\mathrm{e}^{\hat{\boldsymbol{s}}})\big)^{1-\lambda_0} \cdot \big(\mathcal{I}_k(\mathrm{e}^{\check{\boldsymbol{s}}})\big)^{\lambda_0}}{\mathrm{e}^{(1-\lambda_0)\hat{s}_k} \cdot \mathrm{e}^{(\lambda_0)\check{s}_k}} \\ &= (1-\lambda_0)G(\hat{\boldsymbol{s}}) + \lambda_0 G(\check{\boldsymbol{s}}), \end{aligned} \tag{54}$$

which contradicts the assumption of strict convexity, thus concluding the proof. ■

Note that if (51) holds for a $\lambda_0 \in (0,1)$, then it holds for *all* $\lambda \in (0,1)$. This is a direct consequence of log-convexity (49).

In order to show the next Theorem 6, we need the Lemmas 5–7. Using the dependency set (47), we introduce the following definition.

Definition 10 (Strict Log-Convexity): A log-convex interference function $\mathcal{I}_k$ is said to be *strictly log-convex* if for all $\hat{\boldsymbol{p}}, \check{\boldsymbol{p}} \in \mathbb{R}^K_{++}$, with $\hat{p}_l \neq \check{p}_l$ for at least one $l \in \mathsf{L}_k$, we have

$$\mathcal{I}_k\big(\boldsymbol{p}(\lambda)\big) < \big(\mathcal{I}_k(\hat{\boldsymbol{p}})\big)^{1-\lambda} \cdot \big(\mathcal{I}_k(\check{\boldsymbol{p}})\big)^{\lambda} \tag{55}$$

where $\boldsymbol{p}(\lambda)$ is defined in (48).

We have the following result.

Lemma 5: Let $\mathcal{I}_k$ be a strictly log-convex interference function in the sense of Definition 10. For all $\lambda \in (0,1)$, we have

$$\mathcal{I}_k\big(\boldsymbol{p}(\lambda)\big) = \big(\mathcal{I}_k(\hat{\boldsymbol{p}})\big)^{1-\lambda} \cdot \big(\mathcal{I}_k(\check{\boldsymbol{p}})\big)^{\lambda} \tag{56}$$

if and only if for all $l \in \mathsf{L}(k)$

$$\hat{p}_l = \mu \check{p}_l, \quad \mu > 0. \tag{57}$$

Proof: Assume that (57) holds. We have

$$p_l(\lambda) = \hat{p}_l^{1-\lambda} \cdot \check{p}_l^{\lambda} = \mu^{1-\lambda} \cdot \check{p}_l, \quad \forall l \in \mathsf{L}(k) \tag{58}$$

and thus

$$\mathcal{I}_k\big(\boldsymbol{p}(\lambda)\big) = \mu^{1-\lambda} \cdot \mathcal{I}_k(\check{\boldsymbol{p}}). \tag{59}$$

With $\mathcal{I}_k(\hat{\boldsymbol{p}}) = \mu \mathcal{I}_k(\check{\boldsymbol{p}})$, we have

$$\begin{aligned} \mathcal{I}_k\big(\boldsymbol{p}(\lambda)\big) &= \big(\mathcal{I}_k(\hat{\boldsymbol{p}})\big)^{1-\lambda} \cdot \frac{\mathcal{I}_k(\check{\boldsymbol{p}})}{\big(\mathcal{I}_k(\check{\boldsymbol{p}})\big)^{1-\lambda}} \\ &= \big(\mathcal{I}_k(\hat{\boldsymbol{p}})\big)^{1-\lambda} \cdot \big(\mathcal{I}_k(\check{\boldsymbol{p}})\big)^{\lambda}. \end{aligned} \tag{60}$$

Conversely, assume that (56) is fulfilled. Then, strict log-convexity implies $\hat{p}_l = \mu \check{p}_l$ for all $l \in \mathsf{L}(k)$. ■

Based on Lemma 5, we can show the following result.

Lemma 6: Let $\mathcal{I}_1, \ldots, \mathcal{I}_K$ be strictly log-convex interference functions. Assume that $\boldsymbol{D}_{\mathcal{I}} \boldsymbol{D}_{\mathcal{I}}^T$ is irreducible. For arbitrary $\hat{\boldsymbol{p}}, \check{\boldsymbol{p}} \in \mathbb{R}^K_{++}$ and $\lambda_0 \in (0,1)$, the equality

$$\mathcal{I}_k\big(\boldsymbol{p}(\lambda_0)\big) = \big(\mathcal{I}_k(\hat{\boldsymbol{p}})\big)^{1-\lambda_0} \cdot \big(\mathcal{I}_k(\check{\boldsymbol{p}})\big)^{\lambda_0} \tag{61}$$

holds for all $k \in \mathcal{K}$, if and only if there exists a $\mu \in \mathbb{R}_{++}$ such that

$$\hat{\boldsymbol{p}} = \mu \check{\boldsymbol{p}}. \tag{62}$$

Proof: If (62) is fulfilled, then (61) is fulfilled for all $k \in \mathcal{K}$.

Conversely, assume that (61) is fulfilled, then it follows from Lemma 5 that

$$\hat{p}_l = \mu^{(k)} \cdot \check{p}_l, \quad \forall l \in \mathsf{L}(k) \tag{63}$$

where $\mu^{(k)} \in \mathbb{R}$ is associated with the $k\text{th}$ user. If $l \in \mathsf{L}(k_1) \cap \mathsf{L}(k_2)$, then (63) is fulfilled for both k_1 and k_2, i.e.,

$$\mu^{(k_1)} = \mu^{(k_2)}\ .$$

Since $\boldsymbol{D}_{\mathcal{I}} \boldsymbol{D}_{\mathcal{I}}^T$ is irreducible, for each k there is a sequence of indices $k_0 = 1, k_1, \ldots, k_r = k$, such that

$$\mathsf{L}(k_s) \cap \mathsf{L}(k_{s+1}) \neq \emptyset \tag{64}$$

for $s = 0, \ldots, r-1$. It can be concluded that

$$\mu^{(1)} = \mu^{(k_1)} = \ldots = \mu^{(k)} \tag{65}$$

which shows (62). ■

With Lemma 6 we can show the following result.

Lemma 7: Let $\mathcal{I}_1, \ldots, \mathcal{I}_K$ be strictly log-convex interference functions. There is at least one $k_0 \in \mathcal{K}$ such that the strict inequality (51) is fulfilled for $\hat{\boldsymbol{p}} \neq \mu \check{\boldsymbol{p}}$, if and only if $\boldsymbol{D}_{\mathcal{I}} \boldsymbol{D}_{\mathcal{I}}^T$ is irreducible.

Proof: From Lemma 6, we know that if $\boldsymbol{D}_{\mathcal{I}} \boldsymbol{D}_{\mathcal{I}}^T$ is irreducible, and $\hat{\boldsymbol{p}} \neq \mu \check{\boldsymbol{p}}$, for arbitrary $\hat{\boldsymbol{p}}, \check{\boldsymbol{p}} \in \mathbb{R}^K_{++}$, then there exists a $k_0 \in \mathcal{K}$ and a λ_0 such that (51) holds.

Conversely, assume that (51) is fulfilled. The proof is by contradiction. Suppose that $\boldsymbol{D}_{\mathcal{I}} \boldsymbol{D}_{\mathcal{I}}^T$ is not irreducible. Then, there are at least two indices $k_1, k_2 \in \mathcal{K}$, which are not connected (see [27, Definition 4 and Theorem 3]). Let $\mathcal{K}^{(1)}$ and $\mathcal{K}^{(2)}$ denote the sets of indices connected with k_1 and k_2, respectively. We have $\mathcal{K}^{(1)} \cap \mathcal{K}^{(2)} = \emptyset$. All other indices are collected in the (possibly) nonempty set $\mathcal{K}^{(3)} = \mathcal{K} \backslash (\mathcal{K}^{(1)} \cup \mathcal{K}^{(2)})$.

Consider a vector $\boldsymbol{p}^{(1)}$, and positive scalars $c^{(1)}$, $c^{(2)}$, where $c^{(1)} \neq c^{(2)}$. We define a vector $\boldsymbol{p}^{(2)}$ such that

$$p_k^{(2)} = \begin{cases} p_k^{(1)}, & \text{if } k \in \mathcal{K}^{(3)} \\ c^{(1)} p_k^{(1)}, & \text{if } k \in \mathcal{K}^{(1)} \\ c^{(2)} p_k^{(1)}, & \text{if } k \in \mathcal{K}^{(2)}. \end{cases} \tag{66}$$

Since $c^{(1)} \neq c^{(2)}$, we have $\boldsymbol{p}^{(1)} \neq \boldsymbol{p}^{(2)}$. Now, consider

$$p_k\left(\frac{1}{2}\right) := \left(p_k^{(1)}\right)^{1/2} \cdot \left(p_k^{(2)}\right)^{1/2}, \quad \forall k \in \mathcal{K}. \tag{67}$$

For $k \in \mathcal{K}^{(3)}$, we have $\mathsf{L}(k) \cap \mathcal{K}^{(1)} = \emptyset$ and $\mathsf{L}(k) \cap \mathcal{K}^{(2)} = \emptyset$. Therefore, $\mathcal{I}_k(\boldsymbol{p}^{(1)}) = \mathcal{I}_k(\boldsymbol{p}^{(2)})$, and thus

$$\mathcal{I}_k\big(\boldsymbol{p}(\tfrac{1}{2})\big) = \big(\mathcal{I}_k(\boldsymbol{p}^{(1)})\big)^{1/2} \cdot \big(\mathcal{I}_k(\boldsymbol{p}^{(2)})\big)^{1/2}. \tag{68}$$

For $k \in \mathcal{K}^{(1)}$, we have $p_l^{(2)} = c^{(1)} p_l^{(1)}$ for all $l \in \mathsf{L}(k)$, thus

$$\mathcal{I}_k\big(\boldsymbol{p}(\tfrac{1}{2})\big) = \big(\mathcal{I}_k(\boldsymbol{p}^{(1)})\big)^{1/2} \cdot \big(\mathcal{I}_k(\boldsymbol{p}^{(2)})\big)^{1/2}. \tag{69}$$

The corresponding result can be shown for $k \in \mathcal{K}^{(2)}$. Thus, (69) holds for *all* $k \in \mathcal{K}$. However, this contradicts the assumed strict convexity of the interference function. Hence, $\boldsymbol{D}_\mathcal{I}\boldsymbol{D}_\mathcal{I}^T$ must be irreducible. ■

This leads to the following result.

Theorem 6: Let $\mathcal{I}_1, \ldots, \mathcal{I}_K$ be strictly log-convex interference functions. The cost function $G(\boldsymbol{s})$ defined in (50) is strictly convex if and only if $\boldsymbol{D}_\mathcal{I}\boldsymbol{D}_\mathcal{I}^T$ is irreducible.

Proof: This follows from Lemma 4 and Lemma 7.

Hence, if a proportionally fair optimizer exists, and if $\boldsymbol{D}_\mathcal{I}\boldsymbol{D}_\mathcal{I}^T$ is irreducible, then we know from Theorem 6 that the solution is unique. However, $\boldsymbol{D}_\mathcal{I}\boldsymbol{D}_\mathcal{I}^T$ alone is not sufficient for the existence of an optimizer. This is shown by the next example.

Example 9: Consider the coupling matrix $\boldsymbol{V}$ defined in (42). The matrix $\boldsymbol{V}$ is irreducible. The product

$$\boldsymbol{V}\boldsymbol{V}^T = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}$$

is irreducible as well. The function $\sum_{k=1}^3 \log \frac{[\boldsymbol{V}\hat{\boldsymbol{p}}]_k}{\hat{p}_k}$ is strictly convex if we substitute $\boldsymbol{p} = \mathrm{e}^{\boldsymbol{s}}$. The resulting SIR region is strictly log-convex according to Theorem 6. However, the previous Example 8 shows that no optimizer exists. This is because the requirements in Theorem 5 are not satisfied.

Lemma 8: Consider an arbitrary dependency matrix $\hat{\boldsymbol{D}}_\mathcal{I}$ with a positive main diagonal. If $\hat{\boldsymbol{D}}_\mathcal{I}$ is irreducible, then $\hat{\boldsymbol{D}}_\mathcal{I}\hat{\boldsymbol{D}}_\mathcal{I}^T$ is irreducible as well.

Proof: Defining $\hat{\boldsymbol{D}}'_\mathcal{I} := \hat{\boldsymbol{D}}_\mathcal{I}\hat{\boldsymbol{D}}_\mathcal{I}^T$, we have

$$\left[\hat{\boldsymbol{D}}'_\mathcal{I}\right]_{kl} = \sum_{n=1}^{K} \left[\hat{\boldsymbol{D}}_\mathcal{I}\right]_{kn} \left[\hat{\boldsymbol{D}}_\mathcal{I}^T\right]_{nl} = \sum_{n=1}^{K} \left[\hat{\boldsymbol{D}}_\mathcal{I}\right]_{kn} \left[\hat{\boldsymbol{D}}_\mathcal{I}\right]_{ln}. \tag{70}$$

Consider the summand $n = l$. We have $[\hat{\boldsymbol{D}}'_\mathcal{I}]_{kl} \geq [\hat{\boldsymbol{D}}_\mathcal{I}]_{kl}[\hat{\boldsymbol{D}}_\mathcal{I}]_{ll} \geq 0$. By assumption of a positive main diagonal, we have $[\hat{\boldsymbol{D}}_\mathcal{I}]_{ll} > 0$. Thus, $[\hat{\boldsymbol{D}}_\mathcal{I}]_{kl} > 0$ implies that $[\hat{\boldsymbol{D}}'_\mathcal{I}]_{kl} > 0$ for an arbitrary choice of indices k, l. Hence, irreducibility of $\hat{\boldsymbol{D}}_\mathcal{I}$ implies irreducibility of $\hat{\boldsymbol{D}}_\mathcal{I}\hat{\boldsymbol{D}}_\mathcal{I}^T$. ■

Lemma 8 leads to the following Theorem 7, which complements Theorem 5. It provides a necessary and sufficient condition for the existence of a *unique* optimizer.

Theorem 7: Let $\mathcal{I}_1, \ldots, \mathcal{I}_K$ be strictly monotonic log-convex interference functions. We assume that (34) is fulfilled. Then, problem (8) has a unique optimizer $\hat{\boldsymbol{p}} > 0$, $\|\hat{\boldsymbol{p}}\|_1 = 1$, if and only if there exist permutation matrices $\boldsymbol{P}^{(1)}$, $\boldsymbol{P}^{(2)}$ such that $\hat{\boldsymbol{D}}_\mathcal{I} = \boldsymbol{P}^{(1)}\boldsymbol{D}_\mathcal{I}\boldsymbol{P}^{(2)}$ is irreducible and its main diagonal is strictly positive.

Proof: Assume that a unique optimizer $\hat{\boldsymbol{p}} > 0$ exists. Theorem 5 implies the existence of permutations such that $\hat{\boldsymbol{D}}_\mathcal{I}$ is block-irreducible with strictly positive main diagonal. That is, $\hat{\boldsymbol{D}}_\mathcal{I}$ is block-diagonal with $r \geq 1$ irreducible blocks $\hat{\boldsymbol{D}}_\mathcal{I}^{(1)}, \ldots, \hat{\boldsymbol{D}}_\mathcal{I}^{(r)}$. The optimization $\inf_{\boldsymbol{p}>0} \sum_k \log(\mathcal{I}_k(\boldsymbol{p})/p_k)$ is reduced to r independent subproblems with the respective dependency matrices. This leads to proportionally fair power allocations $\hat{\boldsymbol{p}}^{(1)}, \ldots, \hat{\boldsymbol{p}}^{(r)}$. Uniqueness of $\hat{\boldsymbol{p}}$ implies $r = 1$, i.e., $\hat{\boldsymbol{D}}_\mathcal{I}$ consists of a single irreducible block. To show this, suppose that $r > 1$. Since each power vector can be arbitrarily scaled, every vector

$$\hat{\boldsymbol{p}} = \begin{bmatrix} \mu_1 \cdot \hat{\boldsymbol{p}}^{(1)} \\ \vdots \\ \mu_r \cdot \hat{\boldsymbol{p}}^{(r)} \end{bmatrix}, \quad \text{with } \mu_1, \ldots, \mu_r > 0$$

is proportionally fair. Thus, $\hat{\boldsymbol{p}}$ is not unique. This contradicts the hypothesis and implies irreducibility.

Conversely, assume that there is an irreducible matrix $\hat{\boldsymbol{D}}_\mathcal{I}$ with a positive main diagonal. Since the requirements of Theorem 5 are fulfilled, we know that problem (8) has an optimizer $\hat{\boldsymbol{p}} > 0$. It remains to show that $\hat{\boldsymbol{p}} > 0$, with $\|\hat{\boldsymbol{p}}\|_1 = 1$, is unique. From Lemma 8, we know that $\hat{\boldsymbol{D}}_\mathcal{I}\hat{\boldsymbol{D}}_\mathcal{I}^T$ is irreducible. We have

$$\begin{aligned} \hat{\boldsymbol{D}}_\mathcal{I}\hat{\boldsymbol{D}}_\mathcal{I}^T &= \boldsymbol{P}^{(1)}\boldsymbol{D}_\mathcal{I}\boldsymbol{P}^{(2)}(\boldsymbol{P}^{(2)})^T\boldsymbol{D}_\mathcal{I}^T(\boldsymbol{P}^{(1)})^T \\ &= \boldsymbol{P}^{(1)}\boldsymbol{D}_\mathcal{I}\boldsymbol{D}_\mathcal{I}^T(\boldsymbol{P}^{(1)})^T. \end{aligned}$$

Thus, $\boldsymbol{D}_\mathcal{I}\boldsymbol{D}_\mathcal{I}^T$ is irreducible as well. It follows from Theorem 6 that the cost function $G(\boldsymbol{s})$ defined in (50) is strictly convex. Since the function $\exp\{\cdot\}$ is strictly monotonic, it can be concluded that the optimizer $\hat{\boldsymbol{p}}$ is unique. ■

VII. Equivalence of Nash Bargaining and Proportional Fairness

In the previous section, we have studied the existence and uniqueness of a proportionally fair optimizer directly, without analyzing the underlying SIR region.

In this section, we use the results of Section II-B, where the Nash bargaining theory was extended to the class of noncompact sets $\mathcal{ST}$. Next, we investigate conditions under which the SIR region is contained in $\mathcal{ST}$. If this is fulfilled, and if an optimizer exists, then we know that it is the unique NBS.

For the problem at hand, boundary points $\hat{\boldsymbol{\gamma}}$ with $C(\hat{\boldsymbol{\gamma}}) = 1$ need not be achievable. In order to guarantee the existence of a $\hat{\boldsymbol{p}} > 0$ such that

$$1 = C(\hat{\boldsymbol{\gamma}}) = \frac{\hat{\gamma}_k \mathcal{I}_k(\hat{\boldsymbol{p}})}{\hat{p}_k} \tag{71}$$

we need the additional requirement that $\boldsymbol{D}_\mathcal{I}$ is irreducible. This ensures the existence of a power allocation $\boldsymbol{p} > 0$ such that (71) is fulfilled [25]. Note that this solution is not required to be unique. An SIR boundary point may be associated with different power vectors. However, different SIR boundary points will always be associated with different power vectors.

Theorem 8: Let $\mathcal{I}_1, \ldots, \mathcal{I}_K$ be strictly log-convex and strictly monotonic interference functions. If $\boldsymbol{D}_\mathcal{I}$ and $\boldsymbol{D}_\mathcal{I}\boldsymbol{D}_\mathcal{I}^T$ are irreducible, then the SIR region $\mathcal{S}$ defined in (6) is contained in $\mathcal{ST}$.

Proof: Consider arbitrary boundary points $\hat{\boldsymbol{\gamma}}, \check{\boldsymbol{\gamma}}$ with $\hat{\boldsymbol{\gamma}} \neq \check{\boldsymbol{\gamma}}$ (at least one component). Since $\boldsymbol{D}_\mathcal{I}$ is irreducible, the points $\hat{\boldsymbol{\gamma}}$, $\check{\boldsymbol{\gamma}}$ are attained by power vectors $\hat{\boldsymbol{p}}, \check{\boldsymbol{p}}$, with $\hat{\boldsymbol{p}} \neq c\check{\boldsymbol{p}}$ for all $c > 0$, such that (71) is fulfilled. Next, consider $\boldsymbol{p}(\lambda)$ defined by (48). Defining $\boldsymbol{\gamma}(\lambda) = \hat{\boldsymbol{\gamma}}^{1-\lambda} \cdot \check{\boldsymbol{\gamma}}^{\lambda}$, we have [25]

$$\gamma_k(\lambda) \leq \frac{p_k(\lambda)}{\mathcal{I}_k\big(\boldsymbol{p}(\lambda)\big)}, \quad \forall k \in \mathcal{K}. \tag{72}$$

It can be observed that $\boldsymbol{\gamma}(\lambda)$ is feasible, i.e., $C\big(\boldsymbol{\gamma}(\lambda)\big) \leq 1$. Next, consider the image set $\mathcal{L}og(\mathcal{S})$, with boundary points $\log\check{\boldsymbol{\gamma}}$ and

$\log\hat{\boldsymbol{\gamma}}$. Since $\boldsymbol{\gamma}(\lambda)$ is contained in $\mathcal{S}$, all convex combinations $\log\boldsymbol{\gamma}(\lambda) = (1-\lambda)\log\hat{\boldsymbol{\gamma}} + \lambda\log\tilde{\boldsymbol{\gamma}}$ are contained in $\mathcal{L}og(\mathcal{S})$. Thus, $\mathcal{S}$ is log-convex. It remains to show strictness.

From Lemma 7, we know that there is at least one k_0 for which inequality (72) is strict. Following the same reasoning as in [27], we can successively reduce the powers of users for which strict inequality holds. Since $\boldsymbol{D}_\mathcal{I}\boldsymbol{D}_\mathcal{I}^T$ is irreducible, this reduces interference of other users, which in turn can reduce their power. The irreducibility of $\boldsymbol{D}_\mathcal{I}\boldsymbol{D}_\mathcal{I}^T$ ensures that all users benefit from this approach, so after a finite number of steps, we find a power vector $\tilde{\boldsymbol{p}} > 0$ such that

$$\gamma_k(\lambda) < \frac{\tilde{p}_k}{\mathcal{I}_k(\tilde{\boldsymbol{p}})}, \quad \forall k \in \mathcal{K}. \tag{73}$$

Thus, $C\big(\boldsymbol{\gamma}(\lambda)\big) < 1$, which proves strict log-convexity. ∎

Note that strict convexity of the SIR set does not imply that the PF problem (8) has an optimizer $\boldsymbol{p}^* > 0$. Example 9 in the previous section shows that $\boldsymbol{D}_\mathcal{I}$ and $\boldsymbol{D}_\mathcal{I}\boldsymbol{D}_\mathcal{I}^T$ can both be irreducible; however, no optimizer exists if the conditions in Theorem 5 are not fulfilled.

The following theorem links the previous results on the existence and uniqueness of a proportional fair optimizer with the Nash bargaining framework derived in Section II-B.

Corollary 1: Let $\mathcal{I}_1,\ldots,\mathcal{I}_K$ be strictly log-convex and strictly monotonic interference functions, and let $\boldsymbol{D}_\mathcal{I}$ and $\boldsymbol{D}_\mathcal{I}\boldsymbol{D}_\mathcal{I}^T$ be irreducible. There is a unique optimizer $\hat{\boldsymbol{p}} > 0$ to the problem of proportional fairness (8), with an associated SIR vector $\hat{\boldsymbol{\gamma}}$, if and only if there is a single-valued solution outcome φ satisfying the Nash axioms WPO, SYM, IIA, STC, and $\varphi = \hat{\boldsymbol{\gamma}}$.

Proof: This follows from Theorems 1 and 8. ∎

VIII. Conclusion

The classical requirement for Nash bargaining and proportional fairness is a compact comprehensive convex utility set. In this paper, we show that this can be generalized to certain strictly log-convex and noncompact sets. This result broadens the class of utility sets to which the framework of Nash bargaining and proportionally fairness can be applied.

A focus of the paper is on the SIR region resulting from log-convex interference functions. This region is log-convex, but not always strictly log-convex. Moreover, existence and uniqueness of a proportionally fair optimizer is generally not guaranteed.

Different aspects of this problem are studied. It turns out that existence and uniqueness is completely determined by the structure of the dependency matrix $\boldsymbol{D}_\mathcal{I}$, which characterizes the interference coupling for given axiomatic interference functions. The results show that only the "combinatorial structure" of $\boldsymbol{D}_\mathcal{I}$ matters, not the actual strength of the coupling coefficients.

An open problem for future work is to investigate the impact of additional power constraints. First results [8] already show that also in this case the dependency matrix $\boldsymbol{D}_\mathcal{I}$ plays an important role.

Appendix
Proof of Theorem 5

Assume that there exist permutation matrices $\boldsymbol{P}^{(1)}, \boldsymbol{P}^{(2)}$ such that $\hat{\boldsymbol{D}}_\mathcal{I} = \boldsymbol{P}^{(1)}\boldsymbol{D}_\mathcal{I}\boldsymbol{P}^{(2)}$ is block-irreducible with a nonzero main diagonal. We show that this implies the existence of an optimizer for problem (8). To this end, we first discuss the simpler case where $\hat{\boldsymbol{D}}_\mathcal{I}$ is irreducible. Then, this is extended to block-irreducibility.

Since (34) is fulfilled by assumption, Theorem 3 implies $PF(\mathcal{I}) > -\infty$, so for every $\epsilon > 0$ there exists a vector $\boldsymbol{p}(\epsilon) > 0$ such that

$$\sum_{k\in\mathcal{K}} \log\frac{\mathcal{I}_k\big(\boldsymbol{p}(\epsilon)\big)}{p_k(\epsilon)} \le PF(\mathcal{I}) + \epsilon. \tag{74}$$

Since $PF(\mathcal{I})$ is invariant with respect to a scaling of $\boldsymbol{p}(\epsilon)$, it can be assumed that $\max_k p_k(\epsilon) = 1$. Therefore, there exists a null sequence $\{\epsilon_n\}_{n\in\mathbb{N}}$ and a $\boldsymbol{p}^* \ge 0$, with $\max_k p_k^* = 1$, such that

$$\lim_{n\to\infty} \boldsymbol{p}(\epsilon_n) = \boldsymbol{p}^*.$$

We now show by contradiction that $\boldsymbol{p}^* > 0$. Assume that this is not fulfilled, then $\boldsymbol{p}^*$ has r zero components. Without loss of generality, we can assume that the user indices are chosen such that

$$\lim_{n\to\infty} p_l(\epsilon_n) = \begin{cases} 0, & l = 1,\ldots,r \\ p_l^* > 0, & l = r+1,\ldots,K. \end{cases} \tag{75}$$

The assumption of such an ordering is justified because for any permutation matrix $\boldsymbol{P}$, the product $\boldsymbol{P}\boldsymbol{D}_\mathcal{I}\boldsymbol{P}^T$ still has the properties of interest (irreducibility, existence of a positive main diagonal after row or column permutation). The first r components of $\boldsymbol{p}(\epsilon_n)$ tend to zero, so for any $C > 0$ and $1 \le k \le r$, we have that $\log\big(C/p_k(\epsilon_n)\big)$ tends to infinity. Therefore

$$\sum_{k\in\mathcal{K}} \log\frac{\mathcal{I}_k\big(\boldsymbol{p}(\epsilon_n)\big)}{p_k(\epsilon_n)} \le PF(\mathcal{I}) + \epsilon_n, \quad \text{for all } n \in \mathbb{N}$$

can only be fulfilled if

$$\lim_{n\to\infty} \mathcal{I}_k\big(\boldsymbol{p}(\epsilon_n)\big) = 0, \quad k = 1,\ldots,r. \tag{76}$$

Consider $\boldsymbol{e}_m$, as defined in (24). For any $m,k \in \mathcal{K}$ we have

$$\mathcal{I}_k\big(\boldsymbol{p}(\epsilon_n)\big) \ge \mathcal{I}_k\big(\boldsymbol{p}(\epsilon_n)\circ\boldsymbol{e}_m\big) = \mathcal{I}_k(\boldsymbol{e}_m)\cdot p_m(\epsilon_n). \tag{77}$$

Combining (75)–(77) yields

$$0 = \lim_{n\to\infty} \mathcal{I}_k\big(\boldsymbol{p}(\epsilon_n)\big) \ge \mathcal{I}_k(\boldsymbol{e}_m)\cdot p_m^*, \quad k = 1,\ldots,r, \quad m = r+1,\ldots,K.$$

Since $p_m^* > 0$ for $m = r+1,\ldots,K$, and $\mathcal{I}_k(\boldsymbol{e}_m) \ge 0$, it follows that $\mathcal{I}_k(\boldsymbol{e}_m) = 0$ for $m = r+1,\ldots,K$ and $k = 1,\ldots,r$. Consequently, $\mathcal{I}_1,\ldots,\mathcal{I}_r$ do not depend on $p_{r+1},\ldots,p_K$. This means that $\hat{\boldsymbol{D}}_\mathcal{I}$ is reducible, which contradicts the assumption, thus proving $\boldsymbol{p}^* > 0$. Since interference functions are continuous on $\mathbb{R}^K_{++}$[5], we have

$$\begin{aligned} PF(\mathcal{I}) &\le \sum_{k\in\mathcal{K}} \log\frac{\mathcal{I}_k(\boldsymbol{p}^*)}{p_k^*} \\ &= \lim_{n\to\infty} \sum_{k\in\mathcal{K}} \log\frac{\mathcal{I}_k\big(\boldsymbol{p}(\epsilon_n)\big)}{p_k(\epsilon_n)} \\ &\le PF(\mathcal{I}). \end{aligned}$$

Hence, the infimum $PF(\mathcal{I})$ is attained by $\boldsymbol{p}^* > 0$.

Next, we extend the proof to the case where $\hat{\boldsymbol{D}}_{\mathcal{I}}$ is *block-irreducible*. The lth block on the main diagonal has the dimension $K_l \times K_l$, and $\sum_{l=1}^{N} K_l = K$. By $\mathcal{I}_k^{(l)}$, we denote the kth interference function of the lth block, where $k = 1, \dots, K_l$. We have

$$\inf_{\boldsymbol{p}>0} \sum_{k\in\mathcal{K}} \log \frac{\mathcal{I}_k(\boldsymbol{p})}{p_k} = \sum_{l=1}^{N} PF(\mathcal{I}^{(l)}) \tag{78}$$

where

$$PF(\mathcal{I}^{(l)}) = \inf_{\boldsymbol{p}\in\mathbb{R}_{++}^{K_l}} \sum_{k=1}^{K_l} \log \frac{\mathcal{I}_k^{(l)}(\boldsymbol{p})}{p_k}. \tag{79}$$

By assumption, there exists a row or column permutation such that $\boldsymbol{D}_{\mathcal{I}}$ has a positive main diagonal. The same holds for each block $\hat{\boldsymbol{D}}_{\mathcal{I}}^{(l)}$ on the main diagonal. Since $\hat{\boldsymbol{D}}_{\mathcal{I}}^{(l)}$ is also irreducible, we know from the first part of the proof that there exists a $\hat{\boldsymbol{p}}^{(l)} \in \mathbb{R}_{++}^{K_l}$ such that

$$PF(\mathcal{I}^{(l)}) = \sum_{k=1}^{K_l} \log \frac{\mathcal{I}_k^{(l)}(\hat{\boldsymbol{p}}^{(l)})}{\hat{p}_k^{(l)}}.$$

Defining $\hat{\boldsymbol{p}} = [\hat{\boldsymbol{p}}^{(1)} \dots \hat{\boldsymbol{p}}^{(N)}]^T$, we have

$$PF(\mathcal{I}) = \sum_{l=1}^{N} PF(\mathcal{I}^{(l)}) = \sum_{k=1}^{K} \log \frac{\mathcal{I}_k(\hat{\boldsymbol{p}})}{\hat{p}_k} \tag{80}$$

which completes the first part of the proof.

In order to show the converse, assume that there exists an optimizer $\hat{\boldsymbol{p}} > 0$ that attains the infimum $PF(\mathcal{I}) > -\infty$. The proof is by contradiction. Assume that there are no permutation matrices $\boldsymbol{P}^{(1)}, \boldsymbol{P}^{(2)}$, such that $\boldsymbol{P}^{(1)}\boldsymbol{D}_{\mathcal{I}}\boldsymbol{P}^{(2)}$ is block-irreducible with strictly positive main diagonal. From Theorem 2, we know that there is a permutation matrix $\check{\boldsymbol{P}}$ such that $\check{\boldsymbol{D}}_{\mathcal{I}} = \boldsymbol{D}_{\mathcal{I}}\check{\boldsymbol{P}}$ has a nonzero main diagonal. There exists a permutation matrix $\boldsymbol{P}_1$ such that $\boldsymbol{P}_1\check{\boldsymbol{D}}_{\mathcal{I}}\boldsymbol{P}_1^T$ takes the canonical form (46), i.e.,

$$\boldsymbol{P}_1\check{\boldsymbol{D}}_{\mathcal{I}}\boldsymbol{P}_1^T = \begin{bmatrix} \tilde{\boldsymbol{D}}_{\mathcal{I}}^{(1)} & & \mathbf{0} \\ \vdots & \ddots & \\ \tilde{\boldsymbol{D}}_{\mathcal{I}}^{(r,N)} & \dots & \tilde{\boldsymbol{D}}_{\mathcal{I}}^{(N)} \end{bmatrix} = \tilde{\boldsymbol{D}}_{\mathcal{I}}.$$

Since $\check{\boldsymbol{D}}_{\mathcal{I}}$ has a positive main diagonal, $\tilde{\boldsymbol{D}}_{\mathcal{I}}$ also has a positive diagonal. Let $\tilde{\boldsymbol{p}} = \boldsymbol{P}_1\hat{\boldsymbol{p}}$ and $[\tilde{\mathcal{I}}_1(\tilde{\boldsymbol{p}}), \dots, \tilde{\mathcal{I}}_K(\tilde{\boldsymbol{p}})]^T = \boldsymbol{P}_1[\mathcal{I}_1(\boldsymbol{p}), \dots, \mathcal{I}_K(\boldsymbol{p})]^T$, then

$$\inf_{\boldsymbol{p}>0} \sum_{k\in\mathcal{K}} \frac{\mathcal{I}_k(\boldsymbol{p})}{p_k} = \sum_{k\in\mathcal{K}} \log \frac{\tilde{\mathcal{I}}_k(\tilde{\boldsymbol{p}})}{\tilde{p}_k} = PF(\tilde{\mathcal{I}}) = PF(\mathcal{I}).$$

Consider the first block $\tilde{\boldsymbol{D}}_{\mathcal{I}}^{(1)} \in \mathbb{R}_{+}^{K_1\times K_1}$ with interference functions $\tilde{\mathcal{I}}_1^{(1)}, \dots, \tilde{\mathcal{I}}_{K_1}^{(1)}$, depending on a power vector $\tilde{\boldsymbol{p}}^{(1)}$, given as the first K_1 components of $\tilde{\boldsymbol{p}}$. This block does not receive interference, so

$$\sum_{k=1}^{K_1} \log \frac{\tilde{\mathcal{I}}_k^{(1)}(\tilde{\boldsymbol{p}}^{(1)})}{\tilde{p}_k^{(1)}} = PF(\tilde{\mathcal{I}}^{(1)}) = \inf_{\boldsymbol{p}\in\mathbb{R}_{++}^{K_1}} \sum_{k=1}^{K_1} \log \frac{\tilde{\mathcal{I}}_k^{(1)}(\boldsymbol{p})}{p_k}.$$

Next, consider the second block $\tilde{\boldsymbol{D}}_{\mathcal{I}}^{(2)} \in \mathbb{R}_{+}^{K_2\times K_2}$. If $\tilde{\boldsymbol{D}}_{\mathcal{I}}^{(1,2)} = \mathbf{0}$, then

$$\sum_{k=1}^{K_2} \log \frac{\tilde{\mathcal{I}}_k^{(2)}(\tilde{\boldsymbol{p}}^{(2)})}{\tilde{p}_k^{(2)}} = PF(\tilde{\mathcal{I}}^{(2)}) = \inf_{\boldsymbol{p}\in\mathbb{R}_{++}^{K_2}} \sum_{k=1}^{K_2} \log \frac{\tilde{\mathcal{I}}_k^{(2)}(\boldsymbol{p})}{p_k}. \tag{81}$$

If $\boldsymbol{D}_{\mathcal{I}}^{(1,2)} \neq \mathbf{0}$, then at least one of the interference functions $\tilde{\mathcal{I}}_k^{(2)}(\boldsymbol{p})$, $1 \leq k \leq K_2$, depends on at least one $\tilde{p}_l^{(1)}$, $l = 1, \dots, K_1$. By scaling $\lambda \cdot \tilde{\boldsymbol{p}}^{(1)}$, $0 < \lambda < 1$, the optimum $PF(\tilde{\mathcal{I}}^{(1)})$ remains unaffected. However, the interference to the second block would be reduced because of the assumed strict monotonicity. Therefore, it would be possible to construct a new vector $\check{\boldsymbol{p}}$, with $\check{\boldsymbol{p}} \leq \tilde{\boldsymbol{p}}$, which achieves a better value

$$\sum_{k\in\mathcal{K}} \log \frac{\mathcal{I}_k(\check{\boldsymbol{p}})}{\check{p}_k} < \sum_{k\in\mathcal{K}} \log \frac{\mathcal{I}_k(\tilde{\boldsymbol{p}})}{\tilde{p}_k} = PF(\mathcal{I}).$$

However, this contradicts the assumption that $\tilde{\boldsymbol{p}}$ is an optimizer. It can be concluded that $\hat{\boldsymbol{D}}_{\mathcal{I}}$ is block-irreducible, with a strictly positive main diagonal.

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