A Calculus for Log-Convex Interference Functions

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***Abstract*—The behavior of certain interference-coupled multiuser systems can be modeled by means of logarithmically convex (log-convex) interference functions. In this paper, we show fundamental properties of this framework. A key observation is that any log-convex interference function can be expressed as an optimum over elementary log-convex interference functions. The results also contribute to a better understanding of certain quality-of-service (QoS) tradeoff regions, which can be expressed as sublevel sets of log-convex interference functions. We analyze the structure of the QoS region and provide conditions for the achievability of boundary points. The proposed framework of log-convex interference functions generalizes the classical linear interference model, which is closely connected with the theory of irreducible nonnegative matrices (Perron–Frobenius theory). We discuss some possible applications in robust communication, cooperative game theory, and max-min fairness.**

***Index Terms*—Achievable region, interference function, log-convex, max-min fairness, multiuser wireless communication, quality-of-service (QoS).**

I. Introduction

The performance limits of wireless point-to-point links are quite well understood. However, these results cannot always be transferred to multiuser wireless networks, which are more difficult to analyze because of possible interference between the communication links. In general, the achievable performance of one link can depend on the transmission strategy of other users. This leads to the notion of the *achievable* region, which characterizes the performance tradeoffs between the links or users.

An example is the region of signal-to-interference ratios (SIR) which was studied in the context of power control (see, e.g., [1]–[4] for an overview). Another example is the capacity region of the Gaussian multiple-input multiple-output (MIMO) broadcast channel, which was characterized in an information-theoretical context [5]–[7]. There are many more examples of multiuser systems where interference plays an important role for the design of optimal transmission.

Even though some achievable regions are relatively well-understood, there is no general theory for analyzing interference-coupled systems. For example, most results on the MIMO broadcast channel were derived under the assumption of an ideal encoding strategy and time-sharing. If these assumption are not fulfilled, e.g., because of practical implementation constraints, then the resulting capacity region can be nonconvex, and it is generally unclear how to achieve operating points on the boundary [8]. There are many further examples of interference-coupled systems for which the resulting achievable region is unknown. In order to analyze and optimize such systems, it therefore makes sense to aim at a fundamental understanding of performance tradeoffs and the resulting achievable regions.

In this paper, we propose a general and abstract framework for analyzing certain types of interference-coupled systems based on *log-convex interference functions* (as explained later). This theory is closely connected with the analysis of certain achievable regions.

We will start by discussing some motivating examples from the literature in the following section. These examples are mainly focused on power control theory because this is the historical background which has led to the theory of interference functions [3], [9], [10]. Power control theory also provides some intuitive examples which help to better understand the behavior of interference functions.

However, the proposed framework is more general and not confined to power control. The term "interference" should be understood as an abstract concept for modeling certain behaviors of multiuser systems. Examples of nonconventional interference functions will be discussed in Section II-C (spectral radius) and Section IV-E (cooperative game theory). The connection between our framework and Yates' *standard interference functions* [3] will be discussed in Section II-B. For an outline of our results see Section II-D.

II. Axiomatic Framework of Log-Convex Interference Functions

Some notational conventions are as follows: Matrices and vectors are denoted by bold capital letters and bold lowercase letters, respectively. Let $\boldsymbol{y}$ be a vector, then $y_l = [\boldsymbol{y}]_l$ is the lth component. Likewise, $A_{\min} = [\boldsymbol{A}]_{mn}$ is a component of the matrix $\boldsymbol{A}$. The notation $\boldsymbol{y} \geq 0$ means that $y_l \geq 0$ for all components l. Also, $\exp\{\boldsymbol{y}\}$ and $\log\{\boldsymbol{y}\}$ denotes component-wise exponential and logarithm, respectively. The set of nonnegative reals is denoted as $\mathbb{R}_+$. The set of positive reals is denoted as $\mathbb{R}_{++}$.

A. Linear Interference Functions and Perron–Frobenius Theory

We begin by discussing the conventional linear interference model. This is a special case of the axiomatic interference model which will be introduced in Section II-B.

Consider a multiuser system, with K independent users simultaneously transmitting on the same resource. The set of

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user indices is $\mathcal{K} = \{1, 2, \dots, K\}$. If each of the uncorrelated data streams is received by a single-user receiver (e.g., matched filter), then the interference power at user k can be written as

$$\mathcal{I}_k(\boldsymbol{p}) = \boldsymbol{p}^T \boldsymbol{v}_k, \quad k \in \mathcal{K} \tag{1}$$

where

- $\boldsymbol{p} \in \mathbb{R}_+^K$ is a vector of transmission powers,
- $\boldsymbol{v}_k \in \mathbb{R}_+^K$ is a vector of interference coupling coefficients.

The linear interference model (1) has a long-standing tradition in power control theory (see, e.g., [1], [2], [4] and the references therein). For given powers $\boldsymbol{p}$, the interference in the system is determined by the $K \times K$ *coupling matrix*

$$\boldsymbol{V} = [\boldsymbol{v}_1, \dots, \boldsymbol{v}_K]^T. \tag{2}$$

Consider a vector of requested SIR

$$\boldsymbol{\gamma} = [\gamma_1, \dots, \gamma_K]^T. \tag{3}$$

The point $\boldsymbol{\gamma}$ is said to be *feasible* if and only if

$$\rho_{\boldsymbol{V}}(\boldsymbol{\gamma}) = \inf_{\boldsymbol{p}>0} \left(\max_{1\le k\le K} \frac{[\boldsymbol{\Gamma}\boldsymbol{V}\boldsymbol{p}]_k}{p_k} \right) \le 1 \tag{4}$$

where $\boldsymbol{\Gamma} := \mathrm{diag}\{\boldsymbol{\gamma}\}$, and $\rho_{\boldsymbol{V}}(\boldsymbol{\gamma})$ is the *spectral radius* of the matrix $\boldsymbol{\Gamma}\boldsymbol{V}$. For further details on the theory of nonnegative matrices and the Collatz–Wielandt type characterization (4), see, e.g., [11]–[13].

The set of all feasible SIR vectors is

$$\mathcal{S} = \{\boldsymbol{\gamma} : \rho_{\boldsymbol{V}}(\boldsymbol{\gamma}) \le 1\}. \tag{5}$$

By $\partial\mathcal{S}$ we denote the boundary of $\mathcal{S}$, for which $\rho_{\boldsymbol{V}}(\boldsymbol{\gamma}) = 1$. If the matrix $\boldsymbol{V}$ is *irreducible* (i.e., the associated connected graph is fully connected), then the *Perron–Frobenius theorem* can be used to show that the infimum (4) is attained for any $\boldsymbol{\gamma} \in \partial\mathcal{S}$. In other words, there exists a power vector $\boldsymbol{p} > 0$ such that $p_k = \gamma_k \mathcal{I}_k(\boldsymbol{p})$ for all $k \in \mathcal{K}$.

For linear interference functions, the SIR region $\mathcal{S}$ is generally not a convex set (see, e.g., [14]). However, it was observed in [15] that the SIR region (5) is convex on a logarithmic scale. Later, this was extended in [16]–[19], where it was shown that convexity holds for arbitrary quality-of-service (QoS) measures $\mathrm{QoS}_k = \phi(\mathrm{SIR})$ for which the inverse mapping is log-convex.

In summary, it can be said that the linear interference model (1) has some interesting properties. With the typical assumption of a nonnegative irreducible coupling matrix $\boldsymbol{V}$, standard results from the Perron–Frobenius theory can be used. In the past, this framework has proved useful for the analysis of interference-coupled networks and it has provided the basis for many results and algorithms. For an overview see e.g., [1], [2], [4] and the references therein.

In this paper, we will extend many of these results to the more general model of *log-convex interference functions*, which will be introduced in the remainder of this section.

B. Axiomatic Approach to Interference Modeling

An axiomatic approach to interference modeling was proposed by Yates [3]. Instead of using a coefficient matrix, as in (1), the interference coupling was characterized by a framework of axioms. A function $J : \mathbb{R}_+^K \mapsto \mathbb{R}_{++}$ is said to be a *standard interference function* if the following properties are satisfied:

- *Positivity*: $J(\boldsymbol{p}) > 0$ for all power vectors $\boldsymbol{p} \in \mathbb{R}_+^K$;
- *Monotonicity*: If $\boldsymbol{p} > \boldsymbol{p}'$, then $J(\boldsymbol{p}) \ge J(\boldsymbol{p}')$;
- *Scalability*: For all $\alpha > 1$, $\alpha J(\boldsymbol{p}) > J(\alpha\boldsymbol{p})$.

Note, that the scalability property is motivated by the presence of a constant noise power σ_n^2. For example, the function $J(\boldsymbol{p}) = \boldsymbol{p}^T\boldsymbol{v} + \sigma_n^2$ is a standard interference function, whereas the linear interference function (1) is not standard.

The assumption of noise or scalability is often necessary. For example, the problem of signal-to-interference-plus-noise ratio (SINR)-constrained power minimization [3] would be meaningless without noise. However, there are other problems for which we would like to have a more general model. For example, the connection between the min-max balancing problem (4) and the Perron–Frobenius theory discussed in Section II-A would be more difficult to see with noise. Therefore, early classical results in power control theory (e.g., [20]–[22]) were derived without noise. This has eventually led to a deeper understanding of the subject. This proved useful, not only from a theoretic perspective, but also for the development of algorithmic solutions, where noise was included.

This need for a more general interference model has motivated the following axiomatic framework [10].

Definition 1: We say that $\mathcal{I} : \mathbb{R}_+^K \mapsto \mathbb{R}_+$ is an *interference function* if it fulfills the axioms:

A1	(nonnegativeness)	$\mathcal{I}(\boldsymbol{p}) \ge 0$	
A2	(scale invariance)	$\mathcal{I}(\alpha\boldsymbol{p}) = \alpha\mathcal{I}(\boldsymbol{p})$,	$\forall \alpha \in \mathbb{R}_+$
A3	(monotonicity)	$\mathcal{I}(\boldsymbol{p}) \ge \mathcal{I}(\boldsymbol{p}')$,	if $\boldsymbol{p} \ge \boldsymbol{p}'$.

In order to rule out the trivial case $\mathcal{I}(\boldsymbol{p}) = 0$, we make an additional assumption:

$$\text{There exists a } \boldsymbol{p}' > 0 \text{ such that } \mathcal{I}(\boldsymbol{p}') > 0. \tag{6}$$

It was shown in [10] that (6) implies $\mathcal{I}(\boldsymbol{p}) > 0$ for *all* $\boldsymbol{p} \in \mathbb{R}_{++}^K$.

Comparing A1–A3 to Yates' model, it is observed that "scalability" is replaced by "scale invariance." This allows to model interference functions of the type (1). But it can also be used to model interference plus noise $J(\boldsymbol{p}) = \boldsymbol{p}^T\boldsymbol{v} + \sigma_n^2$. To this end, we introduce an extended power vector

$$\underline{\boldsymbol{p}} = \begin{bmatrix} \boldsymbol{p} \\ \sigma_n^2 \end{bmatrix}. \tag{7}$$

In order to appropriately model the impact of the noise on the resulting interference, we need to require strict monotonicity with respect to the last component $\underline{p}_{K+1} = \sigma_n^2$ in addition to A1–A3. That is,

$$\mathcal{I}(\underline{\boldsymbol{p}}) > \mathcal{I}(\underline{\boldsymbol{p}}'), \qquad \text{if} \quad \underline{\boldsymbol{p}} \ge \underline{\boldsymbol{p}}' \text{ and } \underline{p}_{K+1} > \underline{p}'_{K+1}. \tag{8}$$

Keeping $\underline{p}_{K+1}$ constant, we obtain a model which has similar properties as the standard interference functions used in [3]. In particular, if $J(\boldsymbol{p})$ is a standard interference function, then there exists a $\mathcal{I}$ such that $J(\boldsymbol{p}) = \mathcal{I}(\underline{\boldsymbol{p}})$. Note, that property (8) is not required for most results of this paper, except for Section III-B, where some QoS regions will be studied with noise and power constraints.

This discussion shows that the axiomatic framework A1–A3 can be used as a fundamental basis for different types of interference-coupled systems. The standard interference functions [3] can be regarded as a special case. The linear interference model (1) is another special case. Further examples can be found in the literature, e.g., in the context of beamforming [23]–[25], [10], [26], code-division multiple access (CDMA) [27], [28], base-station assignment [29], [30], and robust designs [31].

More recently, there has been renewed interest in the axiomatic approach itself. Convergence properties of standard interference functions were studied in [9]. Properties of the framework A1–A3 were analyzed in [10]. A one-to-one correspondence between interference functions and comprehensive utility sets was shown in [32]. In [33], the framework A1–A3 was extended by the additional requirement of convexity (resp., concavity). Examples of convex or concave interference functions are [23]–[30].

Convexity is a useful property which should be exploited if possible. For example, Yates' power minimization problem [3] can be solved with superlinear convergence if the interference functions are convex or concave [34], [35]. The fixed-point iteration [3], which only exploits the properties of standard interference functions, only achieves linear convergence [36], [34].

In this paper, we study the class of *log-convex interference functions*, which will be introduced in the following section. It will be seen that some of the aforementioned interference functions are included as special cases. For example, any linear or convex interference function is log-convex, but the converse is not true. So the class of log-convex interference functions is broader than the discussed examples.

C. Log-Convex Interference Functions

Having introduced general interference functions in the previous section, we will now focus on the particular subclass of *log-convex interference functions*. To this end, we introduce a change of variable $\boldsymbol{p} = \exp\{\boldsymbol{s}\}$ (component-wise exponential).

Definition 2: We say that $\mathcal{I} : \mathbb{R}^K_+ \mapsto \mathbb{R}_+$ is a *log-convex interference function* if A1–A3 are fulfilled and in addition $\mathcal{I}(\exp\{\boldsymbol{s}\})$ is log-convex on $\mathbb{R}^K$.

Let $f(\boldsymbol{s}) := \mathcal{I}(\exp\{\boldsymbol{s}\})$. The function $f : \mathbb{R}^K \mapsto \mathbb{R}_+$ is log-convex on $\mathbb{R}^K$ if and only if $\log f$ is convex, or equivalently [37]

$$f((1-\lambda)\hat{\boldsymbol{s}} + \lambda\check{\boldsymbol{s}}) \le f(\hat{\boldsymbol{s}})^{1-\lambda} f(\check{\boldsymbol{s}})^{\lambda}, \quad \forall\lambda \in [0,1],\ \hat{\boldsymbol{s}}, \check{\boldsymbol{s}} \in \mathbb{R}^K. \tag{9}$$

Note, that the change of variable $\boldsymbol{p} = \exp\{\boldsymbol{s}\}$ was already used by Sung [15] for linear interference functions (1), and later in [16], [18], [19], [4]. It was also used in [38] in a different context.

Some examples of log-convex interference functions are as follows.

Example 1: The linear function (1) is a log-convex interference function in the sense of Definition 2.

Example 2: The coefficients $\boldsymbol{v}$ can adapt to the current interference situation. An example is the "worst case interference"

$$\mathcal{I}_k(\boldsymbol{p}) = \max_{c_k \in \mathcal{C}_k} \boldsymbol{p}^T \boldsymbol{v}_k(c_k), \quad k \in \mathcal{K}. \tag{10}$$

The parameter c_k can stand for some uncertainty, chosen from a compact uncertainty set $\mathcal{C}_k$. Such worst case interference functions are used, e.g., in the context of robust power control [31], [35]. The function (10) is a log-convex interference function.

Example 3: It was shown in [10] that any convex interference function is log-convex in the sense of Definition 2. That is, if $\mathcal{I}(\boldsymbol{p})$ is convex, then $\mathcal{I}(\mathrm{e}^{\boldsymbol{s}})$ is log-convex. The converse is not true, however. Therefore, the class of log-convex interference functions is broader than the class of convex interference functions. Special cases of convex interference functions include the linear function (1) and the worst case function (10). Hence, the requirement of log-convexity is relatively weak, as compared to many other existing interference models.

Examples of convex (thus log-convex) interference functions are found in [31], [35]. However, concave interference functions are generally not log-convex, so the examples [23]–[30] do not fall within the framework of this paper.

Example 4: Consider the spectral radius $\rho_{\boldsymbol{V}}(\boldsymbol{\gamma})$, as defined by (4). Assume that the desired quality of service is the logarithmic SIR, i.e., $\boldsymbol{q} = \log\boldsymbol{\gamma}$, then the spectral radius as a function of $\boldsymbol{q}$ is $\rho_{\boldsymbol{V}}(\mathrm{e}^{\boldsymbol{q}})$. It was shown in [16] (see also the related work [15], [18], [19], [4]), that the spectral radius is log-convex with respect to the variable $\boldsymbol{q}$. Every log-convex function is convex, so the log-SIR feasible region

$$\mathcal{S}_{\log} = \{\boldsymbol{q} \in \mathbb{R}^K : \rho_{\boldsymbol{V}}(\mathrm{e}^{\boldsymbol{q}}) \le 1\} \tag{11}$$

is a convex set. Moreover, $\rho_{\boldsymbol{V}}(\boldsymbol{\gamma})$ fulfills A1–A3, so the spectral radius is a log-convex interference function.

Note that this is an example where Yates' framework [3] is not appropriate. The interference function $\rho_{\boldsymbol{V}}(\boldsymbol{\gamma})$ is *scale invariant* but not *scalable* as required in [3].

Example 5: The function

$$\mathcal{I}(\boldsymbol{p}) = C \cdot \prod_{l\in\mathcal{K}} (p_l)^{w_l}, \quad C \in \mathbb{R}_{++}, w_l \in \mathbb{R}_+, \sum_{l\in\mathcal{K}} w_l = 1 \tag{12}$$

is a log-convex interference function. It will be shown in Section IV that (12) is a basic building block, which can be used to construct any other log-convex interference function.

D. Outline of Contributions

In Section III, we will discuss the connection between log-convex interference functions and certain QoS regions. Any comprehensive QoS region from $\mathbb{R}^K_{++}$ can be expressed as a sublevel set of an interference function [32]. An example is the SIR region (5) for linear interference functions, which was discussed in Section II-A. In the following, we will generalize these ideas to the framework of log-convex interference functions. For example, it will be shown in Section III-A that the spectral radius $\rho_{\boldsymbol{V}}$ is a special case of a min-max type log-convex interference function (16).

In Section IV, it will be shown that any log-convex interference function has an elementary structure (34). An interesting special case is the linear model (1), which leads to a decomposition involving the Kullback–Leibler distance, as shown in

Section IV-D. In Section IV-E, it will be shown that the elementary structure (34) occurs naturally in the context of cooperative game theory. This shows that log-convex interference functions are not only useful in the context of power control. The proposed framework is abstract enough to be applicable to various types of resource allocation problems. The results will also help to better understand the structure of log-convex comprehensive QoS regions, which can be expressed as sublevel sets of log-convex interference functions (see Section III).

In Section V, the results will be used for analyzing the interactions between multiple interference-coupled users in a network. Conditions will be derived under which weighted interference functions have a positive fixed point. This is closely connected with the question of whether a point on the boundary of the QoS region can be attained or not. In the context of linear interference functions, this problem is well-understood because of its close relationship with the theory of nonnegative matrices (e.g., Perron–Frobenius theorem). This paper extends many results and concepts known from the linear model to the more general axiomatic framework of log-convex interference functions.

III. QoS Regions of Log-Convex Interference Functions

In Example 4 it was observed that the spectral radius (4) is a log-convex interference function, so the SIR region (5) is a level set of a log-convex interference function.

This can be extended to QoS regions based on the more general axiomatic framework A1–A3. Consider K communication links (users) with log-convex interference functions $\mathcal{I}_1, \dots, \mathcal{I}_K$ depending on the same power vector $\boldsymbol{p} \in \mathbb{R}^K_{++}$. The SIR of the K users are

$$\mathrm{SIR}_k(\boldsymbol{p}) = \frac{p_k}{\mathcal{I}_k(\boldsymbol{p})}, \quad k \in \mathcal{K}. \tag{13}$$

The *quality-of-service* (QoS) is defined as a strictly monotonic and continuous function $\phi_k : \mathbb{R}_{++} \mapsto \mathbb{Q}$ of the SIR, i.e.,

$$\mathrm{QoS}_k(\boldsymbol{p}) = \phi_k(\mathrm{SIR}_k(\boldsymbol{p})), \quad k \in \mathcal{K}. \tag{14}$$

Let $\gamma_k : \mathbb{Q} \mapsto \mathbb{R}_{++}$ be the inverse function of ϕ_k, then $\gamma_k(q_k)$ is the minimum SIR level needed by the kth user to satisfy the QoS target q_k. Let $\boldsymbol{q} \in \mathbb{Q}^K$ be a vector of QoS values, then the associated SIR vector is

$$\boldsymbol{\gamma}(\boldsymbol{q}) = [\gamma_1(q_1), \dots, \gamma_K(q_K)]^T. \tag{15}$$

We will also use the notations $\boldsymbol{\gamma} := \boldsymbol{\gamma}(\boldsymbol{q})$ and $\boldsymbol{\Gamma} := \mathrm{diag}\{\boldsymbol{\gamma}\}$ in the following.

A. Max-Min SIR Balancing

Consider the weighted min-max optimum

$$C(\boldsymbol{\gamma}) = \inf_{\boldsymbol{p}>0} \left(\max_{k\in\mathcal{K}} \frac{\gamma_k \cdot \mathcal{I}_k(\boldsymbol{p})}{p_k} \right). \tag{16}$$

Note, that this problem formulation involves the inverse SIRs. Equivalently, the problem could be formulated as the supremum over the minimum (worst case) SIR. Thus, problem (16) corresponds to the problem of *max-min SIR balancing*, sometimes referred to as *max-min fairness*.

Similar to the spectral radius (4), the function $C(\boldsymbol{\gamma})$ provides a single measure for the feasibility of SIR values $\boldsymbol{\gamma} \in \mathbb{R}^K_{++}$. That is, QoS values $\boldsymbol{q} \in \mathbb{Q}^K$ are feasible if and only if $C(\boldsymbol{\gamma}(\boldsymbol{q})) \le 1$. The *QoS feasible region* is defined as the sublevel set

$$\mathcal{Q} = \{\boldsymbol{q} \in \mathbb{Q}^K : C(\boldsymbol{\gamma}(\boldsymbol{q})) \le 1\}. \tag{17}$$

In the following, we will analyze QoS regions of the type (17), with underlying log-convex interference functions $\mathcal{I}_1, \dots, \mathcal{I}_K$. Thanks to the one-to-one mapping between SIR and QoS values, most of the discussion, like the analysis of the boundary in Section V, can be confined to the SIR region. The results can immediately be transferred to the respective QoS region. It will turn out that the underlying log-convexity leads to a beneficial structure, and there are many parallels to the linear interference model.

In the remainder of this section, we will focus on the interesting special case of QoS functions $\phi(\mathrm{SIR})$ for which the inverse function $\gamma(\mathrm{QoS})$ is log-convex. Examples are

- capacity in the high signal-to-noise ratio (SNR) regime: $\phi(\mathrm{SIR}) \approx \alpha \log(\mathrm{SIR})$, with $\alpha \in \mathbb{R}_{++}$;
- bit-error rate (BER) in the high SNR regime: $\phi(\mathrm{SIR}) \approx (G_c \cdot \mathrm{SIR})^{-G_d}$, with coding gain G_c and diversity order G_d.

In this case, the function $C(\boldsymbol{\gamma}(\boldsymbol{q}))$ is log-convex on $\mathbb{Q}^K$. The proof is given in Appendix A. Since every log-convex function is convex [37], the QoS region $\mathcal{Q}$, as defined by (17), is a sublevel set of a convex function. Hence, $\mathcal{Q}$ is a convex set [10].

Moreover, $C(\boldsymbol{\gamma})$ fulfills the properties A1–A3, so it is an interference function itself. Since $C(\exp(\boldsymbol{q}))$ is log-convex, it follows that $C(\boldsymbol{\gamma})$ is a log-convex interference function in the sense of Definition 2.

This example shows that certain operations are closed within the framework of log-convex interference functions. That is, the properties of log-convex interference functions are preserved when these functions are combined to a new function. Starting with log-convex interference functions $\mathcal{I}_1, \dots, \mathcal{I}_K$ we obtain a new log-convex interference function $C(\boldsymbol{\gamma})$. Later, in Section IV, it will be shown that *every* log-convex interference function can be decomposed into elementary log-convex interference functions (12).

B. Power-Constrained QoS Regions

As discussed in Section II-B, the proposed framework of log-convex interference functions can also be applied to the analysis of SIR regions in the presence of noise and power constraints. This provides a link to the framework of standard interference functions introduced by Yates [3].

In order for transmit power constraints to have any effect on the SIR, we need to incorporate noise in our model. To this end, we use the $(K+1)$-dimensional extended power vector $\underline{\boldsymbol{p}}$, as defined by (7). We also assume that the interference functions $\mathcal{I}_1, \dots, \mathcal{I}_K$ are log-convex in the sense of Definition 2, and strict monotonicity (8) holds. Under this assumption, it will now be shown that the QoS region resulting from log-convex mappings $\gamma_k(q_k)$ is a convex set.

We start by considering a sum-power constraint $\|\boldsymbol{p}\|_1 \le P_{\max}$, which leads to a restricted QoS region $\mathcal{Q}(P_{\max}) \subset \mathcal{Q}$.

Since $\underline{p}_{K+1}$ is constant, we can redefine $\mathcal{I}_k$ as a function of the first K power variables, i.e., $J_k(\boldsymbol{p}) := \mathcal{I}_k(\underline{\boldsymbol{p}})$. Let $\mathcal{P}(\boldsymbol{q})$ be the set of power vectors that achieve a given target $\boldsymbol{q} \in \mathbb{Q}^K$, i.e.,

$$\mathcal{P}(\boldsymbol{q}) = \{\boldsymbol{p} > 0 : \mathrm{SIR}_k(\boldsymbol{p}) \geq \gamma_k(q_k), \ \forall\, k \in \mathcal{K}\}. \quad (18)$$

Because of the noise component, the set $\mathcal{P}(\boldsymbol{q})$ is nonempty if and only if $C(\boldsymbol{\gamma}(\boldsymbol{q})) < 1$. That is, $\boldsymbol{q}$ lies in the interior of $\mathcal{Q}$ (denoted as $\mathrm{int}\mathcal{Q}$). If $\mathcal{P}(\boldsymbol{q})$ is nonempty, then there is a unique vector

$$\boldsymbol{p}^{\min}(\boldsymbol{q}) = \arg\min_{\boldsymbol{p} \in \mathcal{P}(\boldsymbol{q})} \|\boldsymbol{p}\|_1 \quad (19)$$

achieving $\boldsymbol{q}$ with minimum total power. This is a consequence of $J_k(\boldsymbol{p})$ being standard [10], so the results [3] can be applied. The QoS region under a total power constraint is

$$\mathcal{Q}(P_{\max}) = \left\{ \boldsymbol{q} \in \mathbb{Q}^K : \mathcal{P}(\boldsymbol{q}) \neq \emptyset, \sum_{k \in \mathcal{K}} p_k^{\min}(\boldsymbol{q}) \leq P_{\max} \right\}. \quad (20)$$

Next, consider individual power limits

$$\boldsymbol{p}^{\max} = [p_1^{\max}, \ldots, p_K^{\max}]^T.$$

The vector $\boldsymbol{p}^{\min}(\boldsymbol{q})$ achieves the targets $\boldsymbol{q}$ not only with minimum total power, but also with individually minimum powers, as shown in [3]. Thus, we can use the function $\boldsymbol{p}^{\min}(\boldsymbol{q})$ also to characterize the QoS region under individual power limits

$$\mathcal{Q}(\boldsymbol{p}_{\max}) = \left\{ \boldsymbol{q} \in \mathbb{Q}^K : \mathcal{P}(\boldsymbol{q}) \neq \emptyset, p_k^{\min}(\boldsymbol{q}) \leq p_k^{\max}, \ \forall\, k \in \mathcal{K} \right\}. \quad (21)$$

Both regions $\mathcal{Q}(P_{\max})$ and $\mathcal{Q}(\boldsymbol{p}_{\max})$ depend on the underlying interference functions $\mathcal{I}_1, \ldots, \mathcal{I}_K$ which are assumed to be log-convex in the sense of Definition 2. If $\gamma_k(q_k)$ are log-convex in addition, then the function $\boldsymbol{p}^{\min}(\boldsymbol{q})$ is component-wise log-convex on $\mathrm{int}\mathcal{Q}$, as shown in Appendix B. Since the sum of log-convex functions is log-convex, also the sum power $\sum_{k \in \mathcal{K}} p_k^{\min}(\boldsymbol{q})$ is log-convex on $\mathrm{int}\mathcal{Q}$. Thus, both regions (20) and (21) are sublevel sets of log-convex indicator functions. Since every log-convex function is convex, it follows that the sum-power constrained region $\mathcal{Q}(P_{\max})$ and the individually constrained region $\mathcal{Q}(\boldsymbol{p}_{\max})$ are convex sets [10].

This discussion shows that the log-convex interference functions introduced in Section II-C can also be used to model interference with noise, thereby providing a link to standard interference functions [3]. However, it should be emphasized that *log-convexity* is the key property on which we focus here. Although most results readily extend to the special case of noise and power constraints (like the convexity of certain QoS regions), a more detailed discussion of this aspect is beyond the scope of this paper.

C. Weighted Utility and Cost Optimization

In this subsection, we consider another application example for the framework of log-convex interference functions. Assume that the SIR is related to the QoS by a function $\phi(x) = g(1/x)$, i.e.,

$$\mathrm{QoS} = g(1/\mathrm{SIR}).$$

The function g is assumed to be monotonically increasing and $g(\mathrm{e}^x)$ is convex with respect to x, like $g(x) = x$ or $g(x) = \log x$. We are interested in the optimization problem

$$\inf_{\boldsymbol{s} \in \mathbb{R}^K} \sum_{k \in \mathcal{K}} \alpha_k g(\mathcal{I}_k(\mathrm{e}^{\boldsymbol{s}})/\mathrm{e}^{s_k}) \quad \text{s.t. } \|\mathrm{e}^{\boldsymbol{s}}\|_1 \leq P_{\max} \quad (22)$$

where $\mathcal{I}_k(\mathrm{e}^{\boldsymbol{s}})$ is a log-convex interference function. The weights $\boldsymbol{\alpha} = [\alpha_1, \ldots, \alpha_K] > 0$ can model individual user requirements and possibly depend on system parameters like priorities, queue lengths, etc. By appropriately choosing $\boldsymbol{\alpha}$ it is possible to trade off overall efficiency against fairness.

We have the following result.

Theorem 1: Suppose that $\mathcal{I}_k(\mathrm{e}^{\boldsymbol{s}})$ is log-convex for all $k \in \mathcal{K}$ and g is monotonic increasing. Then problem (22) is convex if and only if $g(\mathrm{e}^x)$ is convex on $\mathbb{R}$.

Proof: This is shown in the Appendix C. □

If the optimization problem (22) is convex, then it can be solved by standard convex optimization techniques. Note, that the optimization is over the noncompact set $\mathbb{R}^K$, thus even if the problem is convex, it is not obvious that the optimum is achieved (e.g., $\boldsymbol{s} \rightarrow -\infty$ might occur). However, this case can be ruled out for a practical system with receiver noise $\sigma_n^2 > 0$, in which case $\mathrm{e}^{\boldsymbol{s}_k} \rightarrow 0$ can never happen, since otherwise the objective would tend to infinity, away from the minimum. Without noise, however, it can happen that one or more power components tend to zero, in which case the infimum is not achieved (see, e.g., the discussion in [10]).

A special case of problem (22) is (weighted) *proportional fairness* [39]

$$\sup_{\boldsymbol{p} > 0} \left(-\sum_{k \in \mathcal{K}} \alpha_k \log \frac{\mathcal{I}_k(\boldsymbol{p})}{p_k} \right) = \sup_{\boldsymbol{p} > 0} \left(\sum_{k \in \mathcal{K}} \alpha_k \log \frac{p_k}{\mathcal{I}_k(\boldsymbol{p})} \right). \quad (23)$$

Note that this problem (23) is also related to the problem of throughput maximization (see, e.g., [40], [38]). In the high SIR regime, we can approximate $\log(1 + \mathrm{SIR}) = \log(\mathrm{SIR})$, so (23) can be interpreted as the weighted sum throughput of the system.

Similar to the cost minimization problem (22), we formulate a utility maximization problem

$$\sup_{\boldsymbol{s} \in \mathbb{R}^K} \sum_{k \in \mathcal{K}} \alpha_k g(\mathcal{I}_k(\mathrm{e}^{\boldsymbol{s}})/\mathrm{e}^{s_k}) \quad \text{s.t. } \|\mathrm{e}^{\boldsymbol{s}}\|_1 \leq P_{\max}. \quad (24)$$

In this case, the function $g(\mathrm{e}^x)$ is required to be monotonic decreasing instead of increasing. As in Theorem 1, convexity of $g(\mathrm{e}^x)$ can be shown to be necessary and sufficient for (24) to be convex.

Notice that the supremum (24) can be written as a convex function $u(\boldsymbol{\alpha})$ of the weights $\boldsymbol{\alpha} = [\alpha_1, \ldots, \alpha_K]$. Moreover, $u(\boldsymbol{\alpha})$ fulfills the properties A1–A3, so it can be regarded as an "interference function." Using a substitution $\boldsymbol{\alpha} = \exp \boldsymbol{\beta}$, the function $u(\boldsymbol{\alpha})$ is a *log-convex interference function* in the sense of Definition 2. This is a further example, which shows that log-convex interference functions arise naturally in many different contexts. Even though our discussion is motivated by power control, the proposed theoretical framework provides a general tool, which is not limited to interference in a physical sense.

Also, (24) provides another example for a combination of log-convex interference functions resulting in a log-convex interference function. Again, it can be observed that certain operations are closed within the framework of log-convex interference functions.

IV. Structure and Representation of Log-Convex Interference Functions

In this section, we study elementary building blocks of log-convex interference functions. Every function $\mathcal{I}(\boldsymbol{p})$ can be expressed as a maximum of elementary interference functions (12). Conversely, log-convex interference functions can be synthesized from certain utility sets. The results allow for some interesting interpretations. For example, connections with the Kullback–Leibler distance and cooperative game theory will be shown. Some of the properties will be used later in Section V, where the boundary of the QoS region (17) will be analyzed.

A. Basic Properties

Consider the log-convex interference function $\xi(\boldsymbol{p}) = \prod_{l\in\mathcal{K}}(p_l)^{w_l}$, with fixed nonnegative coefficients $\boldsymbol{w} = [w_1,\dots,w_K]^T \in \mathbb{R}_+^K$ and $\|\boldsymbol{w}\|_1 = 1$. Using the substitution $\boldsymbol{p} = \mathrm{e}^{\boldsymbol{s}}$, it can be verified that $\xi(\mathrm{e}^{\boldsymbol{s}})$ is log-convex on $\mathbb{R}^K$. In addition, $\xi(\mathrm{e}^{\boldsymbol{s}})$ fulfills property A1 (nonnegativeness) because $\mathrm{e}^{\boldsymbol{s}} > 0$. Property A2 (scale-invariance) follows from the assumption $\|\boldsymbol{w}\|_1 = \sum_l w_l = 1$, which leads to

$$\begin{aligned}\xi(\alpha\boldsymbol{p}) &= \prod_{l\in\mathcal{K}}(\alpha p_l)^{w_l} = \alpha^{(\sum_l w_l)}\cdot\prod_{l\in\mathcal{K}}(p_l)^{w_l}\\ &= \alpha\prod_{l\in\mathcal{K}}(p_l)^{w_l} = \alpha\cdot\xi(\boldsymbol{p}).\end{aligned} \tag{25}$$

Finally, property A3 (monotonicity) follows from $\boldsymbol{w}\geq 0$. The property $\boldsymbol{w}\geq 0$ is even necessary since otherwise A3 would be violated. Also, $\|\boldsymbol{w}\|_1 = 1$ is necessary for A2 to hold, as can be seen from (25). So, $\xi(\boldsymbol{p})$ is a log-convex interference function if and only if $\|\boldsymbol{w}\|_1 = 1$ and $\boldsymbol{w}\geq 0$.

Now, it will be shown that $\xi(\boldsymbol{p})$ is a basic building block of any log-convex interference function. To this end, consider the function

$$f_{\mathcal{I}}(\boldsymbol{w}) = \inf_{\boldsymbol{p}>0}\frac{\mathcal{I}(\boldsymbol{p})}{\prod_{l\in\mathcal{K}}(p_l)^{w_l}},\quad \boldsymbol{w}\in\mathbb{R}_+^K. \tag{26}$$

The function $f_{\mathcal{I}}(\boldsymbol{w})$ has an interpretation in the context of convex analysis.

Lemma 1: The function $\log f_{\mathcal{I}}(\boldsymbol{w})$ is the conjugate of the convex function $\log\mathcal{I}(\mathrm{e}^{\boldsymbol{s}})$.

Proof: By monotonicity of the log function, we have

$$\log f_{\mathcal{I}}(\boldsymbol{w}) = \inf_{\boldsymbol{s}\in\mathbb{R}^K}\left(\log\mathcal{I}(\mathrm{e}^{\boldsymbol{s}}) - \sum_{l\in\mathcal{K}} w_l s_l\right) \tag{27}$$

which is the definition of the conjugate [37], [41]. □

In the following we will need $f_{\mathcal{I}}(\boldsymbol{w}) > 0$. This will become clear later from the first main result Theorem 2.

We begin by characterizing the set of coefficients $\boldsymbol{w}$ for which $f_{\mathcal{I}}(\boldsymbol{w}) > 0$ is fulfilled. The function $f_{\mathcal{I}}(\boldsymbol{w})$ was defined on $\mathbb{R}_+^K$. This is justified by the following lemma, which shows that only nonnegative coefficients are allowed.

Lemma 2: Let $\mathcal{I}$ be an interference function. If $\boldsymbol{w}$ has a negative component then $f_{\mathcal{I}}(\boldsymbol{w}) = 0$.

Proof: Consider an arbitrary $\boldsymbol{w}\in\mathbb{R}^K$, with a negative component $w_r < 0$ for some index r. Defining a power vector $\boldsymbol{p}(\lambda)$ with $p_l(\lambda) = 1$, $l\neq r$, and $p_r(\lambda) = \lambda$, with $\lambda > 0$, we have

$$f_{\mathcal{I}}(\boldsymbol{w}) \leq \frac{\mathcal{I}(\boldsymbol{p}(\lambda))}{\prod_{l\in\mathcal{K}}(p_l(\lambda))^{w_l}} = (\lambda)^{|w_r|}\cdot\mathcal{I}(\boldsymbol{p}(\lambda)).$$

Because $\mathcal{I}(\boldsymbol{p}(\lambda))\leq\mathcal{I}(\mathbf{1})$ for all $\lambda\in(0,1]$, we have

$$f_{\mathcal{I}}(\boldsymbol{w}) \leq \lim_{\lambda\to 0}(\lambda)^{|w_r|}\cdot\mathcal{I}(\mathbf{1}) = 0.$$

This can only be fulfilled with equality. □

With property A1, the function $f_{\mathcal{I}}(\boldsymbol{w})$ is always nonnegative. But we are only interested in the nontrivial case where $f_{\mathcal{I}}(\boldsymbol{w}) > 0$ is fulfilled.

Lemma 3: Let $\mathcal{I}$ be an interference function, and $\boldsymbol{w}\in\mathbb{R}_+^K$. If $f_{\mathcal{I}}(\boldsymbol{w}) > 0$ then $\|\boldsymbol{w}\|_1 = 1$.

Proof: The proof is by contradiction. Suppose that $f_{\mathcal{I}}(\boldsymbol{w}) > 0$ and $\|\boldsymbol{w}\|_1\neq 1$. From (26) we know that for an arbitrary constant $\hat{\boldsymbol{p}} > 0$ and a scalar $\lambda > 0$ we have

$$f_{\mathcal{I}}(\boldsymbol{w}) \leq \frac{\mathcal{I}(\lambda\hat{\boldsymbol{p}})}{\prod_{l\in\mathcal{K}}(\lambda\hat{p}_l)^{w_l}} = \frac{1}{\lambda^{(\|\boldsymbol{w}\|_1-1)}}\cdot C_1 \tag{28}$$

with a constant $C_1 = \mathcal{I}(\hat{\boldsymbol{p}})/\prod_l(\hat{p}_l)^{w_l}$. Inequality (28) holds for all $\lambda > 0$, thus

$$\begin{aligned}\|\boldsymbol{w}\|_1 > 1 &\Rightarrow 0 = \lim_{\lambda\to\infty}\frac{1}{\lambda^{(\|\boldsymbol{w}\|_1-1)}}\cdot C_1 \geq f_{\mathcal{I}}(\boldsymbol{w}) \geq 0\\ \|\boldsymbol{w}\|_1 < 1 &\Rightarrow 0 = \lim_{\lambda\to 0}\frac{1}{\lambda^{(\|\boldsymbol{w}\|_1-1)}}\cdot C_1 \geq f_{\mathcal{I}}(\boldsymbol{w}) \geq 0.\end{aligned}$$

This leads to the contradiction $f_{\mathcal{I}}(\boldsymbol{w}) = 0$, thus implying $\|\boldsymbol{w}\|_1 = 1$. □

From Lemmas 2 and 3 we know that the coefficients of interest are contained in the set

$$\mathcal{L}(\mathcal{I}) = \{\boldsymbol{w}\in\mathbb{R}_+^K : f_{\mathcal{I}}(\boldsymbol{w}) > 0\}. \tag{29}$$

The structure of $\mathcal{L}(\mathcal{I})$ can be further characterized.

Lemma 4: The function $f_{\mathcal{I}}(\boldsymbol{w})$, as defined by (26), is log-concave on $\mathbb{R}_+^K$.

Proof: The function $\prod_{l\in\mathcal{K}}(p_l)^{w_l}$ is log-convex and log-concave in $\boldsymbol{w}$, and so is its inverse. Point-wise minimization preserves log-concavity, so $f_{\mathcal{I}}(\boldsymbol{w})$ is log-concave. □

This can be used to prove the following result.

Lemma 5: The set $\mathcal{L}(\mathcal{I})$, as defined by (29), is convex.

Proof: Consider two points $\hat{\boldsymbol{w}},\check{\boldsymbol{w}}\in\mathcal{L}(\mathcal{I})$, and the line

$$\boldsymbol{w}(\lambda) = (1-\lambda)\hat{\boldsymbol{w}}_k + \lambda\check{\boldsymbol{w}}_k,\quad \lambda\in[0,1].$$

We have $\boldsymbol{w}(\lambda)\in\mathbb{R}_+^K$. The function $f_{\mathcal{I}}(\boldsymbol{w})$ is log-concave on $\mathbb{R}_+^K$, thus

$$f_{\mathcal{I}}(\boldsymbol{w}(\lambda)) \geq f_{\mathcal{I}}(\hat{\boldsymbol{w}})^{1-\lambda}\cdot f_{\mathcal{I}}(\check{\boldsymbol{w}})^{\lambda}. \tag{30}$$

Because $f_{\mathcal{I}_k}(\hat{\boldsymbol{w}}_k) > 0$ and $f_{\mathcal{I}_k}(\check{\boldsymbol{w}}_k) > 0$, we have $f_{\mathcal{I}}(\boldsymbol{w}(\lambda)) > 0$, thus $f_{\mathcal{I}}(\boldsymbol{w}(\lambda))\in\mathcal{L}(\mathcal{I})$. □

Another property will be needed later.

Lemma 6: The function $f_{\mathcal{I}}(\boldsymbol{w})$ is upper semi-continuous. That is, for every sequence $\boldsymbol{w}^{(n)} \geq 0$, with $\|\boldsymbol{w}^{(n)}\|_1 = 1$ and $\lim_{n\to\infty} \boldsymbol{w}^{(n)} = \boldsymbol{w}^*$, we have

$$f_{\mathcal{I}}(\boldsymbol{w}^*) \geq \limsup_{n\to\infty} f_{\mathcal{I}}\left(\boldsymbol{w}^{(n)}\right). \tag{31}$$

Proof: By definition (26), we have

$$\frac{\mathcal{I}(\boldsymbol{p})}{\prod_l (p_l)^{w_l^{(n)}}} \geq f_{\mathcal{I}}\left(\boldsymbol{w}^{(n)}\right), \qquad \forall\, \boldsymbol{p} > 0,\ \forall n \in \mathbb{N}. \tag{32}$$

The denominator in (32) is a continuous function of $\boldsymbol{w}$, thus

$$\frac{\mathcal{I}(\boldsymbol{p})}{\prod_l (p_l)^{w_l^*}} = \lim_{n\to\infty} \frac{\mathcal{I}(\boldsymbol{p})}{\prod_l (p_l)^{w_l^{(n)}}} \geq \limsup_{n\to\infty} f_{\mathcal{I}}\left(\boldsymbol{w}^{(n)}\right). \tag{33}$$

This holds for all $\boldsymbol{p} > 0$. The right side of this inequality is independent of $\boldsymbol{p}$, thus

$$\inf_{\boldsymbol{p}>0} \frac{\mathcal{I}(\boldsymbol{p})}{\prod_l (p_l)^{w_l^*}} = f_{\mathcal{I}}\left(\boldsymbol{w}^*\right) \geq \limsup_{n\to\infty} f_{\mathcal{I}}(\boldsymbol{w}^{(n)}). \qquad \square$$

To summarize, any strictly positive log-convex interference function $\mathcal{I}(\boldsymbol{p})$ is associated with a function $f_{\mathcal{I}}(\boldsymbol{w}) > 0$, with the following properties.

- $f_{\mathcal{I}}(\boldsymbol{w})$ is log-concave and upper semi-continuous. The resulting superlevel set $\mathcal{L}(\mathcal{I})$ is convex.
- $f_{\mathcal{I}}(\boldsymbol{w}) > 0$ implies $\|\boldsymbol{w}\|_1 = 1$, so all elements of $\mathcal{L}(\mathcal{I})$ have this property.

Additional properties and interpretations of the function $f_{\mathcal{I}}(\boldsymbol{w})$ will be discussed later in Sections IV-D and IV-E.

B. Analysis of Log-Convex Interference Functions

With the results of the previous section, we are now in a position to state the main representation theorem.

Theorem 2: Every log-convex interference function $\mathcal{I}(\boldsymbol{p})$, on $\mathbb{R}^K_{++}$, can be represented as

$$\mathcal{I}(\boldsymbol{p}) = \max_{\boldsymbol{w}\in\mathcal{L}(\mathcal{I})} \left(f_{\mathcal{I}}(\boldsymbol{w}) \cdot \prod_{l\in\mathcal{K}} (p_l)^{w_l} \right). \tag{34}$$

Proof: According to (26), we have for all $\boldsymbol{p} > 0$ and $\boldsymbol{w} \in \mathcal{L}(\mathcal{I})$

$$\mathcal{I}(\boldsymbol{p}) \geq f_{\mathcal{I}}(\boldsymbol{w}) \cdot \prod_{l\in\mathcal{K}} (p_l)^{w_l} \tag{35}$$

Thus

$$\sup_{\boldsymbol{w}\in\mathcal{L}(\mathcal{I})} \left(f_{\mathcal{I}}(\boldsymbol{w}) \cdot \prod_{l\in\mathcal{K}} (p_l)^{w_l} \right) \leq \mathcal{I}(\boldsymbol{p}). \tag{36}$$

It will turn out later that the supremum (36) is actually attained.

The function $\log \mathcal{I}(\mathrm{e}^{\boldsymbol{s}})$ is convex, so for any $\hat{\boldsymbol{s}} \in \mathbb{R}^K$, there is a finite $\hat{\boldsymbol{w}} \in \mathbb{R}^K$ such that (see, e.g., [41, Theorem 1.2.1, p. 77])

$$\log \mathcal{I}(\mathrm{e}^{\boldsymbol{s}}) - \log \mathcal{I}(\mathrm{e}^{\hat{\boldsymbol{s}}}) \geq \sum_{l\in\mathcal{K}} \hat{w}_l (s_l - \hat{s}_l), \quad \text{for all } \boldsymbol{s} \in \mathbb{R}^K.$$

Using $\boldsymbol{p} = \mathrm{e}^{\boldsymbol{s}}$, this can be rewritten as

$$\frac{\mathcal{I}(\boldsymbol{p})}{\prod_{l\in\mathcal{K}} (p_l)^{\hat{w}_l}} \geq \frac{\mathcal{I}(\hat{\boldsymbol{p}})}{\prod_{l\in\mathcal{K}} (\hat{p}_l)^{\hat{w}_l}} = \hat{C}_1, \quad \forall\, \boldsymbol{p} > 0 \tag{37}$$

with a constant $\hat{C}_1 \in \mathbb{R}_{++}$. With (26) we have $f_{\mathcal{I}}(\boldsymbol{w}) \geq \mathrm{e}^{\hat{C}_1} > 0$, thus, $\hat{\boldsymbol{w}} \in \mathcal{L}(\mathcal{I})$. We can rewrite (37) as

$$\mathcal{I}(\hat{\boldsymbol{p}}) \leq \frac{\mathcal{I}(\boldsymbol{p})}{\prod_{l\in\mathcal{K}} (p_l)^{\hat{w}_l}} \prod_{l\in\mathcal{K}} (\hat{p}_l)^{\hat{w}_l}, \quad \forall\, \boldsymbol{p} > 0. \tag{38}$$

Inequality (38) holds for all $\boldsymbol{p} > 0$, thus

$$\mathcal{I}(\hat{\boldsymbol{p}}) \leq f_{\mathcal{I}}(\hat{\boldsymbol{w}}) \cdot \prod_{l\in\mathcal{K}} (\hat{p}_l)^{\hat{w}_l}, \tag{39}$$

which shows that inequality (36) must be fulfilled with equality, thus

$$\mathcal{I}(\boldsymbol{p}) = \sup_{\boldsymbol{w}\in\mathcal{L}(\mathcal{I})} \left(f_{\mathcal{I}}(\boldsymbol{w}) \cdot \prod_{l\in\mathcal{K}} (p_l)^{w_l} \right). \tag{40}$$

It remains to show that this supremum is attained. Consider an arbitrary $\boldsymbol{p} > 0$. From (40) we know that there is a sequence $\boldsymbol{w}^{(n)} \in \mathcal{L}(\mathcal{I})$, $n \in \mathbb{N}$, such that

$$\mathcal{I}(\boldsymbol{p}) - \frac{1}{n} \leq f_{\mathcal{I}}\left(\boldsymbol{w}^{(n)}\right) \cdot \prod_{l\in\mathcal{K}} (p_l)^{w_l^{(n)}}, \quad \forall n \in \mathbb{N}. \tag{41}$$

There is a subsequence $\boldsymbol{w}^{(n_m)}$, $m \in \mathbb{N}$, which converges to a limit $\boldsymbol{w}^* = \lim_{m\to\infty} \boldsymbol{w}^{(n_m)}$. Now, we show that $\boldsymbol{w}^*$ is also contained in $\mathcal{L}(\mathcal{I})$. With $p_l \leq \|\boldsymbol{p}\|_\infty$ we can bound (41)

$$\mathcal{I}(\boldsymbol{p}) - \frac{1}{n} \leq f_{\mathcal{I}}\left(\boldsymbol{w}^{(n)}\right) \cdot (\|\boldsymbol{p}\|_\infty)^{\sum_l w_l^{(n)}}. \tag{42}$$

Exploiting $\|\boldsymbol{w}^{(n)}\|_1 = 1$, we have

$$f_{\mathcal{I}}\left(\boldsymbol{w}^{(n)}\right) \geq \frac{\mathcal{I}(\boldsymbol{p}) - \frac{1}{n}}{\|\boldsymbol{p}\|_\infty}, \quad \text{for all } n \in \mathbb{N}. \tag{43}$$

The function $\mathcal{I}$ is positive by assumption (6), thus

$$\liminf_{m\to\infty} f_{\mathcal{I}}\left(\boldsymbol{w}^{(n_m)}\right) \geq \frac{\mathcal{I}(\boldsymbol{p})}{\|\boldsymbol{p}\|_\infty} > 0. \tag{44}$$

By combining Lemma 6 and (44) we obtain $f_{\mathcal{I}}(\boldsymbol{w}^*) > 0$, thus $\boldsymbol{w}^* \in \mathcal{L}(\mathcal{I})$. With (40) we have

$$\begin{aligned}\mathcal{I}(\boldsymbol{p}) &\geq f_{\mathcal{I}}(\boldsymbol{w}^*) \cdot \prod_{l\in\mathcal{K}} (p_l)^{w_l^*} \\ &\geq \liminf_{m\to\infty} \left(f_{\mathcal{I}}\left(\boldsymbol{w}^{(n_m)}\right) \cdot \prod_{l\in\mathcal{K}} (p_l)^{w_l^{(n_m)}} \right) \\ &\geq \liminf_{m\to\infty} \left(\mathcal{I}(\boldsymbol{p}) - \frac{1}{n_m} \right) = \mathcal{I}(\boldsymbol{p}) \end{aligned} \tag{45}$$

where the last inequality follows from (41). Hence

$$\begin{aligned}\mathcal{I}(\boldsymbol{p}) &= f_{\mathcal{I}}(\boldsymbol{w}^*) \cdot \prod_{l\in\mathcal{K}} (p_l)^{w_l^*} \\ &= \max_{\boldsymbol{w}\in\mathcal{L}(\mathcal{I})} \left(f_{\mathcal{I}}(\boldsymbol{w}) \cdot \prod_{l\in\mathcal{K}} (p_l)^{w_l} \right). \qquad \square\end{aligned}$$

Theorem 2 shows that every log-convex interference function can be represented as (34). From Lemma 4 we know that $f_{\mathcal{I}}(\boldsymbol{w})$ is log-concave. The product of log-concave functions is log-concave, so $f_{\mathcal{I}}(\boldsymbol{w}) \cdot \prod_{l\in\mathcal{K}} (p_l)^{w_l}$ is log-concave in $\boldsymbol{w}$. Thus, problem (34) consists of maximizing a log-concave function over a convex set $\mathcal{L}(\mathcal{I})$.

C. Synthesis of Log-Convex Interference Functions

In the previous section we have analyzed log-convex interference functions. Any log-convex interference function can be broken down into elementary building blocks. Now we will study the reverse approach: the *synthesis* of a log-convex interference function.

To this end, consider the coefficient set

$$\mathcal{M} = \{\boldsymbol{w} \in \mathbb{R}_+^K : \|\boldsymbol{w}\|_1 = 1\} \tag{46}$$

and an arbitrary nonnegative bounded function $g(\boldsymbol{w}) : \mathcal{M} \mapsto \mathbb{R}_+$. We can synthesize a function

$$\mathcal{I}_g(\mathrm{e}^{\boldsymbol{s}}) = \sup_{\boldsymbol{w}\in\mathcal{M}:g(\boldsymbol{w})>0} \left(g(\boldsymbol{w}) \prod_{l\in\mathcal{K}} (\mathrm{e}^{s_l})^{w_l} \right). \tag{47}$$

Notice, that $g(\boldsymbol{w}) \prod_{l\in\mathcal{K}} (\mathrm{e}^{s_l})^{w_l}$ is log-convex in $\boldsymbol{s}$ for any choice of $\boldsymbol{w}$. Maximization preserves log-convexity, so $\mathcal{I}_g(\boldsymbol{p})$ is a log-convex interference function in the sense of Definition 2.

Lemma 7: The convex function $\log \mathcal{I}_g(\mathrm{e}^{\boldsymbol{s}})$ is the conjugate of the function $\log(1/g(\boldsymbol{w}))$.

Proof: Because of the monotonicity of the logarithm, we can exchange the order of sup and log, thus

$$\begin{aligned}\log \mathcal{I}_g(\mathrm{e}^{\boldsymbol{s}}) &= \sup_{\boldsymbol{w}\in\mathcal{M}:g(\boldsymbol{w})>0} \left(\log g(\boldsymbol{w}) + \sum_{l\in\mathcal{K}} w_l s_l \right) \\ &= \sup_{\boldsymbol{w}\in\mathcal{M}:g(\boldsymbol{w})>0} \left(\sum_{l\in\mathcal{K}} w_l s_l - \log \frac{1}{g(\boldsymbol{w})} \right)\end{aligned} \tag{48}$$

which is the definition of the conjugate function [41]. □

Now, consider the analysis of the function $\mathcal{I}_g(\mathrm{e}^{\boldsymbol{s}})$, for which there exists a function $f_{\mathcal{I}_g}(\boldsymbol{w})$, as defined by (26). An interesting question is: when does $g = f_{\mathcal{I}_g}$ hold? In other words, are analysis and synthesis reverse operations?

Theorem 3: $g = f_{\mathcal{I}_g}$ if and only if $g(\boldsymbol{w})$ is log-concave on $\mathcal{M}$ and upper semicontinuous.

Proof: The function $\mathcal{I}_g$ is a log-convex interference function, thus $f_{\mathcal{I}_g}$ is log-concave and upper semicontinuous. The result follows from Corollary 1.3.6 in [41, p. 219]. □

In the remainder of this section, we will show application examples and additional interpretations of $f_{\mathcal{I}}(\boldsymbol{w})$.

D. Connection With the Kullback–Leibler Distance

In Section II-A we have discussed the example of the linear interference function $\mathcal{I}(\boldsymbol{p}) = \boldsymbol{v}^T\boldsymbol{p}$. For this special log-convex interference function, we will now show that the function $f_{\mathcal{I}}(\boldsymbol{w})$ has an interesting interpretation. With the definition (26) we have

$$f_{\mathcal{I}}(\boldsymbol{w}) = \inf_{\boldsymbol{p}>0} \frac{\sum_{l\in\mathcal{K}} v_l p_l}{\prod_{l\in\mathcal{K}} (p_l)^{w_l}}. \tag{49}$$

If two or more components of $\boldsymbol{v}$ are nonzero, then the optimization (49) is strictly convex after the substitution $\boldsymbol{p} = \mathrm{e}^{\boldsymbol{s}}$, as shown in [42]. Thus, there exists a unique optimizer $\boldsymbol{p}^*$, which is found by computing the partial derivatives and setting the result to zero. A necessary and sufficient condition for optimality is

$$p_r^* = \frac{w_r}{v_r} \cdot \sum_{l\in\mathcal{K}} v_l p_l^*, \quad \forall\, r \in \mathcal{K}. \tag{50}$$

With (50), the minimum (49) can be written as

$$\begin{aligned}f_{\mathcal{I}}(\boldsymbol{w}) &= \frac{\sum_l v_l\, p_l^*}{\prod_r \left(\frac{w_r}{v_r} \cdot \sum_l v_l\, p_l^* \right)^{w_r}} \\ &= \frac{\sum_l v_l\, p_l^*}{\prod_r \left(\frac{w_r}{v_r} \right)^{w_r} \cdot (\sum_l v_l\, p_l^*)^{\sum_r w_r}}.\end{aligned} \tag{51}$$

Exploiting $\sum_r w_r = 1$, we have

$$\log f_{\mathcal{I}}(\boldsymbol{w}) = \log \prod_{l\in\mathcal{K}} \left(\frac{w_l}{v_l} \right)^{-w_l} = -\sum_{l\in\mathcal{K}} w_l \log \frac{w_l}{v_l}. \tag{52}$$

It can be observed that $-\log f_{\mathcal{I}}(\boldsymbol{w})$ is the Kullback–Leibler distance between the vectors $\boldsymbol{v}$ and $\boldsymbol{w}$. This connects the function $f_{\mathcal{I}}(\boldsymbol{w})$ with a known measure. For related results on the connection between the Kullback–Leibler distance and the Perron root of nonnegative matrices, see [43].

Next, consider K users with coupling coefficients $\boldsymbol{V} = [\boldsymbol{v}_1, \ldots, \boldsymbol{v}_K]^T$, and a spectral radius $\rho_{\boldsymbol{V}}(\boldsymbol{\gamma})$, as defined by (4). The SIR region $\mathcal{S}$ is defined in (5). Since $\rho_{\boldsymbol{V}}(\boldsymbol{\gamma})$ is a log-convex interference function (see the discussion in Example 4 in Section II-C), all properties derived so far can be applied. The following corollary follows directly from the structure result Theorem 2.

Corollary 1: Consider an arbitrary square irreducible matrix $\boldsymbol{V} \geq 0$ with interference functions $\mathcal{I}_k^{(\boldsymbol{V})}$, as defined by (1). Then there exists a log-concave function $f_{\boldsymbol{V}}(\boldsymbol{w})$, defined on $\mathbb{R}_+$, with $\|\boldsymbol{w}\|_1 = 1$, such that

$$\rho_{\boldsymbol{V}}(\boldsymbol{\gamma}) = \max_{\boldsymbol{w}\in\mathcal{L}(\mathcal{I}^{(\boldsymbol{V})})} \left(f_{\boldsymbol{V}}(\boldsymbol{w}) \prod_{l\in\mathcal{K}} (\gamma_l)^{w_l} \right). \tag{53}$$

As an example, consider the two-user case, with

$$\rho_{\boldsymbol{V}}(\boldsymbol{\gamma}) = \rho\left(\begin{bmatrix} 0 & \gamma_1 V_{12} \\ \gamma_1 V_{21} & 0 \end{bmatrix} \right) = \sqrt{\gamma_1\gamma_2 V_{12} V_{21}}. \tag{54}$$

The spectral radius of an irreducible nonnegative matrix is given by its maximal eigenvalue. For $K = 2$, we obtain the function (54), which is log-convex after a substitution $\gamma_k = \exp q_k$ [19]. Here, we assume that there is no self-interference, so the main diagonal is set to zero. Comparing (53) with (54) we have

$$f_{\boldsymbol{V}}(\boldsymbol{w}) = \begin{cases} \sqrt{V_{12}V_{21}}, & w_1 = w_2 = 1/2 \\ 0, & \text{otherwise.} \end{cases} \tag{55}$$

This shows how (54) can be understood as a special case of the more general representation (53).

E. Connection With Cooperative Game Theory

The function $f_{\mathcal{I}}(\boldsymbol{w})$, as defined by (26), has another interesting interpretation in the context of the asymmetric Nash bargaining problem [44], [45]

$$\mathcal{N}(\boldsymbol{w}) = \max_{\boldsymbol{u}\in\mathcal{U}} \prod_{k\in\mathcal{K}} (u_k)^{w_k} \tag{56}$$

where $\boldsymbol{u} = [u_1, \dots, u_K]^T$ is a utility vector from a convex utility set $\mathcal{U}$, and $\boldsymbol{w} = [w_1, \dots, w_K]^T \in \mathbb{R}_+^K$ are weighting factors, with $\|\boldsymbol{w}\|_1 = 1$.

It was shown in [32] that for any convex compact downward-comprehensive utility region $\mathcal{U} \subseteq \mathbb{R}_{++}^K$, there exists a convex interference function $\mathcal{I}_{\mathcal{U}}(\boldsymbol{u})$, such that

$$\mathcal{U} = \{\boldsymbol{u} > 0 : \mathcal{I}_{\mathcal{U}}(\boldsymbol{u}) \le 1\}. \tag{57}$$

The function $\mathcal{I}_{\mathcal{U}}(\boldsymbol{u})$ fulfills A1–A3 and it is convex on $\mathbb{R}_+^K$ (see [33] for details). Every convex interference function is log-convex in the sense of Definition 2. The bargaining optimum (56) is attained on the boundary of $\mathcal{U}$, which is characterized by $\mathcal{I}_{\mathcal{U}}(\boldsymbol{u}) = 1$. Thus, (56) can be rewritten as [46]

$$\begin{aligned}\mathcal{N}(\boldsymbol{w}) &= \max_{\{\boldsymbol{u}>0:\mathcal{I}_{\mathcal{U}}(\boldsymbol{u})=1\}} \prod_{l\in\mathcal{K}} (u_l)^{w_l} \\ &= \sup_{\boldsymbol{u}>0} \frac{\prod_{l\in\mathcal{K}} (u_l)^{w_l}}{\mathcal{I}_{\mathcal{U}}(\boldsymbol{u})} = \frac{1}{f_{\mathcal{I}_{\mathcal{U}}}(\boldsymbol{w})}.\end{aligned} \tag{58}$$

For given weights $\boldsymbol{w}$, the asymmetric Nash bargaining optimum $\mathcal{N}(\boldsymbol{w})$ is determined by the function $f_{\mathcal{I}_{\mathcal{U}}}(\boldsymbol{w})$, as defined by (26). This shows an interesting connection between cooperative game theory and the theory of log-convex interference functions.

V. Analysis of the Boundary of QoS Regions Based on Log-Convex Interference Functions

We now show how the results of the previous sections can be applied to the analysis of QoS regions with no power constraints, as introduced in Section III. While the discussion in Section III has focused on convexity properties, we will now focus on the *achievability* of the boundary.

A. Fixed-Point Characterization and Achievability

The QoS is assumed to be a bijective mapping of the SIR, as in (14), so we can confine the discussion to the feasible SIR region

$$\mathcal{S} = \{\boldsymbol{\gamma} \in \mathbb{R}_{++}^K : C(\boldsymbol{\gamma}) \le 1\} \tag{59}$$

where $\boldsymbol{\gamma}$ is a vector of SIR values, and $C(\boldsymbol{\gamma})$ is the min-max optimum as defined by (16). The function $C(\boldsymbol{\gamma})$ is an indicator for the feasibility of a point $\boldsymbol{\gamma}$. The boundary $\partial\mathcal{S}$ is

$$\partial\mathcal{S} = \{\boldsymbol{\gamma} \in \mathbb{R}_{++}^K : C(\boldsymbol{\gamma}) = 1\}. \tag{60}$$

By definition, $\boldsymbol{\gamma} \in \partial\mathcal{S}$ is feasible, at least in an asymptotic sense. That is, for any $\epsilon > 0$ there exists a $\boldsymbol{p}_\epsilon > 0$ such that $\mathrm{SIR}_k(\boldsymbol{p}_\epsilon) \ge \boldsymbol{\gamma}_k - \epsilon$ for all $k \in \mathcal{K}$. If this holds for $\epsilon = 0$, then we say that $\boldsymbol{\gamma}$ is achievable.

Achievability is important, e.g., to ensure numerical stability for resource allocation algorithms operating on the boundary of the region. Algorithms are usually derived under the premise that the boundary is achievable. However, wireless systems are often parametrized with respect to the transmission powers, which can result in a QoS region with a complicated structure. It is thus important to analyze the boundary and to show under which conditions achievability holds.

A general characterization of achievability is complicated, as shown in [47]. Thus, in this paper we will focus on the practically relevant special case when $\boldsymbol{\gamma} \in \partial\mathcal{S}$ is achieved with equality, i.e.,

$$\mathrm{SIR}_k(\boldsymbol{p}) = \gamma_k, \quad \text{for all } k \in \mathcal{K}. \tag{61}$$

If there exists a $\boldsymbol{p}^* > 0$ such that (61) is fulfilled, then $\boldsymbol{p}^*$ is the optimizer of the min-max balancing problem (16), with an optimum $C(\boldsymbol{\gamma}) = 1$.

In the remainder of this paper, we will use a slightly more general definition of achievability. An arbitrary $\boldsymbol{\gamma} > 0$ is said to be "achievable" (with equality) if there exists a $\boldsymbol{p}^* > 0$ such that

$$C(\boldsymbol{\gamma}) = \frac{\gamma_k \mathcal{I}_k(\boldsymbol{p}^*)}{p_k^*} = \frac{\gamma_k}{\mathrm{SIR}_k(\boldsymbol{p})}, \quad \text{for all } k \in \mathcal{K}. \tag{62}$$

Introducing the vector notation $\boldsymbol{\mathcal{I}}(\boldsymbol{p}) = [\mathcal{I}_1(\boldsymbol{p}), \dots, \mathcal{I}_K(\boldsymbol{p})]^T$ and $\boldsymbol{\Gamma} := \mathrm{diag}\{\boldsymbol{\gamma}\}$, the system of (62) can be rewritten as

$$\boldsymbol{p}^* = \frac{1}{C(\boldsymbol{\gamma})} \boldsymbol{\Gamma} \boldsymbol{\mathcal{I}}(\boldsymbol{p}^*). \tag{63}$$

Definition 3: A positive power vector $\boldsymbol{p}^* > 0$ is said to be a *fixed point* if it satisfies (63), i.e., if it is a fixed point of the function $\frac{1}{C(\boldsymbol{\gamma})} \boldsymbol{\Gamma} \boldsymbol{\mathcal{I}}(\boldsymbol{p})$.

For any boundary point $\boldsymbol{\gamma} \in \partial\mathcal{S}$ we have $C(\boldsymbol{\gamma}) = 1$, in which case (63) is equivalent to (61). For arbitrary $\boldsymbol{\gamma} > 0$, the existence of a fixed point $\boldsymbol{p}^* > 0$ implies that the infimum (16) is attained, and scaled SIR values $\gamma_k / C(\boldsymbol{\gamma})$ are achieved for all $k \in \mathcal{K}$. To simplify the discussion and to be consistent with previous work, we say that the (scaled) "targets" $\boldsymbol{\gamma} > 0$ are "achievable" if (63) is fulfilled.

For general interference functions characterized by A1–A3, which are not necessarily log-convex, the existence of a fixed point was studied in [10]. For this general case, only a few basic properties were shown.

Lemma 8: Let $\mathcal{I}_1, \dots, \mathcal{I}_K$ be interference functions characterized by A1–A3, then

1) there always exists a $\boldsymbol{p}^* \ge 0$, $\boldsymbol{p}^* \ne 0$, such that (63) is fulfilled;
2) if $\boldsymbol{\Gamma}\boldsymbol{\mathcal{I}}(\boldsymbol{p}^*) = \mu \boldsymbol{p}^*$ for some $\boldsymbol{p}^* > 0$ and $\mu > 0$, then $\mu = C(\boldsymbol{\gamma})$ and $\boldsymbol{p}^*$ is an optimizer of (16).

The existence of a positive fixed point is best understood for linear interference functions (1). In [47] conditions were derived based on the theory of nonnegative matrices [12]. Also in [47], this was extended to the more general class of interference functions with adaptive receiver designs. Both models have in common that the interference is characterized by means of a coupling matrix.

The axiomatic framework of log-convex interference functions is not based on a coupling matrix. Only axioms A1–A3 plus log-convexity is required, so it is *a priori* unclear under which conditions achievability is ensured, and whether previous results can be extended to this model.

Fortunately, log-convex interference functions have a rich analytical structure, and it will turn out that most of the properties known for the linear case can be extended to the axiomatic framework. Based on the structure result derived in Section IV, we will show in the remainder of this section that the interference coupling can be characterized by means of the coefficient vectors $\boldsymbol{w}$. Thereby, conditions for the existence of a fixed point $\boldsymbol{p}^* > 0$ fulfilling (63) can be derived.

B. Existence of a Fixed Point for Constant $\boldsymbol{W}$

It was shown in Section IV that every log-convex interference function can be represented as (34), based on coupling coefficients $\boldsymbol{w} \geq 0$, with $\|\boldsymbol{w}\|_1 = 1$. Now, we study the interactions between K log-convex interference functions. By $\boldsymbol{w}_k$ we denote a coefficient vector associated with user k. All coefficients are collected in a matrix

$$\boldsymbol{W} = [\boldsymbol{w}_1, \ldots, \boldsymbol{w}_K]^T \geq 0, \quad \text{with } \|\boldsymbol{w}_k\|_1 = 1, \ \forall\, k \in \mathcal{K}.$$

Only in this section, it will be assumed that $\boldsymbol{W}$ is *constant*. This approach simplifies the analysis and reveals some characteristic properties. Arbitrary log-convex interference functions will be studied later in Sections V-E and V-F.

Because of the property $\sum_l w_{kl} = 1$, the matrix $\boldsymbol{W}$ is (row) stochastic. Let $\mathbf{1}$ be the all-one vector, then

$$\boldsymbol{W}\mathbf{1} = \mathbf{1}. \tag{64}$$

For arbitrary constants $f_k > 0$, we obtain interference functions

$$\mathcal{I}_k(\boldsymbol{p}, \boldsymbol{W}) = f_k \cdot \prod_{l \in \mathcal{K}} (p_l)^{w_{kl}}, \quad k \in \mathcal{K}. \tag{65}$$

The resulting min-max optimum for a constant $\boldsymbol{W}$ is

$$C(\boldsymbol{\gamma}, \boldsymbol{W}) = \inf_{\boldsymbol{p} > 0} \left(\max_{k \in \mathcal{K}} \frac{\gamma_k \mathcal{I}_k(\boldsymbol{p}, \boldsymbol{W})}{p_k} \right). \tag{66}$$

We are now interested in the existence of a fixed point $\boldsymbol{p}^* > 0$ fulfilling

$$C(\boldsymbol{\gamma}, \boldsymbol{W})\, \boldsymbol{p}^* = \boldsymbol{\Gamma} \boldsymbol{\mathcal{I}}(\boldsymbol{p}^*, \boldsymbol{W}). \tag{67}$$

The next lemma provides a necessary and sufficient condition for strict positivity of the fixed point. This basic property will be used later, e.g., in the Proof of Theorem 4.

Lemma 9: Let $\boldsymbol{t} := (\gamma_1 f_1, \ldots, \gamma_K f_K)^T$. Equation (67) has a solution $\boldsymbol{p}^* > 0$ if and only if an additive translation of $\log \boldsymbol{t}$ (component-wise logarithm) lies in the range of the matrix $\boldsymbol{I} - \boldsymbol{W}$. That is, iff there exists a $\mathcal{C} = \mathcal{C}(\boldsymbol{\gamma}, \boldsymbol{W}) = \log C(\boldsymbol{\gamma}, \boldsymbol{W}) \in \mathbb{R}$ such that we can find an $\boldsymbol{s}^* \in \mathbb{R}^K$ with

$$(\boldsymbol{I} - \boldsymbol{W})\boldsymbol{s}^* = \log \boldsymbol{t} - \mathcal{C}\mathbf{1} \tag{68}$$

where $\boldsymbol{p}^* = \exp\{\boldsymbol{s}^*\}$ (component-wise).

Proof: Suppose there exists an $\boldsymbol{s}^* \in \mathbb{R}^K$ and a $\mathcal{C} \in \mathbb{R}$ such that (68) is fulfilled. Taking $\exp\{\cdot\}$ of both sides of (68), we have for all $k \in \mathcal{K}$

$$\exp\left\{ s_k^* - \sum_{l \in \mathcal{K}} w_{kl} s_l^* \right\} = \frac{p_k^*}{\prod_{l \in \mathcal{K}} (p_l^*)^{w_{kl}}} = \gamma_k f_k \frac{1}{C(\boldsymbol{\gamma}, \boldsymbol{W})}.$$

With (65) it follows that $\boldsymbol{p}^* = \exp\{\boldsymbol{s}^*\} > 0$ is a fixed point of (67), i.e., the infimum $C(\boldsymbol{\gamma}, \boldsymbol{W})$ is achieved.

Conversely, assume that there exists a solution $\boldsymbol{p}^* > 0$ such that (67) is fulfilled. By taking the logarithm of both sides we obtain (68). □

To conclude, if there exists a $\mathcal{C} \in \mathbb{R}$ such that $\log \boldsymbol{t} - \mathcal{C}\mathbf{1}$ lies in the range of $\boldsymbol{I} - \boldsymbol{W}$, then there is an $\boldsymbol{s}^* \in \mathbb{R}^K$ such that (68) holds. Thus, the existence of a fixed point $\boldsymbol{p}^* > 0$ depends on the subspace structure of $\boldsymbol{I} - \boldsymbol{W}$.

Corollary 2: If there exists a $\mathcal{C} \in \mathbb{R}$ such that (68) holds, then $\mathcal{C}$ is unique.

Proof: This follows from Lemmas 8 and 9. □

Next, we show how the existence of a strictly positive fixed point depends on the structure of the nonnegative square row stochastic matrix $\boldsymbol{W}$. We may assume, without loss of generality, that after simultaneous permutations of rows and columns, $\boldsymbol{W}$ is reduced to the canonical form shown in (69) at the bottom of the page (see e.g., [12, p. 75]), with irreducible blocks along the main diagonal.

The dimension of each square block $\boldsymbol{W}^{(n)} := \boldsymbol{W}^{(n,n)}$ along the main diagonal is equal or greater than two. This is a consequence of (6), which implies that each user is interfered by at least one other user. If $\boldsymbol{W}$ is irreducible, then it consists of one single block. Note that the off-diagonal blocks need not be square.

$$\boldsymbol{W} = \begin{bmatrix} \boldsymbol{W}^{(1,1)} & & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} \\ & \ddots & & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & & \boldsymbol{W}^{(i,i)} & \mathbf{0} & \cdots & \mathbf{0} \\ \boldsymbol{W}^{(i+1,1)} & \cdots & \boldsymbol{W}^{(i+1,i)} & \boldsymbol{W}^{(i+1,i+1)} & \mathbf{0} & \mathbf{0} \\ \vdots & \cdots & \vdots & \vdots & \ddots & \mathbf{0} \\ \boldsymbol{W}^{(N,1)} & \cdots & \boldsymbol{W}^{(N,i)} & \boldsymbol{W}^{(N,2)} & \cdots & \boldsymbol{W}^{(N,N)} \end{bmatrix}. \tag{69}$$

Definition 4: A diagonal block $\boldsymbol{W}^{(n)}$ is called *isolated* if $\boldsymbol{W}^{(n,m)} = \boldsymbol{0}$ for $m = 1, 2, \ldots, n-1$. We assume, without loss of generality, that the first i blocks are isolated.

Definition 5: A diagonal block is called *maximal* if its spectral radius equals the overall spectral radius $\rho(\boldsymbol{W})$.

From the results of Section IV we know that the matrix $\boldsymbol{W}$ is stochastic, i.e., (64) is fulfilled. Therefore we have the following.

- $\rho(\boldsymbol{W}) = 1$, which is a consequence of (64) and the Perron–Frobenius theorem. We have $\rho(\boldsymbol{W}) = \max_{1\le n\le N} \rho(\boldsymbol{W}^{(n)}) = 1$.
- A diagonal block is maximal if and only if it is isolated. This follows from (64) and the results [47]. For all nonisolated blocks, we have $\rho(\boldsymbol{W}^{(n)}) < 1$.
- $\boldsymbol{I} - \boldsymbol{W}$ is singular, which becomes evident when rewriting (64) as $(\boldsymbol{I} - \boldsymbol{W})\boldsymbol{1} = \boldsymbol{0}$.

We begin with the simple case where $\boldsymbol{W}$ consists of a single irreducible block.

Theorem 4: Let $\boldsymbol{W} \geq 0$ be row-stochastic and irreducible, then there exists a unique (up to a scaling) fixed point $\boldsymbol{p}^* > 0$ fulfilling (67).

Proof: The proof is given in the Appendix D. □

Next, we will address the more general case where $\boldsymbol{W}$ can be reducible. Without loss of generality, the canonical form (69) can be assumed. We exploit the special properties of stochastic matrices. In particular, each isolated block has a spectral radius one, and the nonisolated blocks have a spectral radius strictly less than one.

Let K_n denote the number of users belonging to the nth block $\boldsymbol{W}^{(n)}$, and $\mathcal{K}_n$ is the set of associated user indices. Also, $\boldsymbol{\gamma}^{(n)} \in \mathbb{R}^{K_n}_{++}$ is the vector of SIR targets associated with this block.

For each isolated block n, with $1 \le n \le i$, we define

$$C\left(\boldsymbol{\gamma}^{(n)}, \boldsymbol{W}^{(n)}\right) = \inf_{\boldsymbol{p}\in\mathbb{R}^K_{++}} \left(\max_{k\in\mathcal{K}_n} \frac{\gamma_k \mathcal{I}_k(\boldsymbol{p}, \boldsymbol{W})}{p_k} \right) \tag{70}$$

$$\le C(\boldsymbol{\gamma}, \boldsymbol{W}). \tag{71}$$

This inequality is a consequence of definition (66), where a larger set $\mathcal{K}$ is used instead of $\mathcal{K}_n$. Each isolated block n only depends on powers from the same block, so the users associated with this block form an independent subsystem.

The next lemma shows that $C(\boldsymbol{\gamma}, \boldsymbol{W})$ only depends on the isolated blocks. Inequality (71) is fulfilled with equality for at least one isolated block.

Lemma 10: Let $\boldsymbol{W}$ be a row-stochastic matrix in canonical form (69), and $\boldsymbol{W}^{(1)}, \ldots, \boldsymbol{W}^{(i)}$ be the isolated irreducible blocks on the main diagonal, then

$$C(\boldsymbol{\gamma}, \boldsymbol{W}) = \max_{1\le n\le i} C\left(\boldsymbol{\gamma}^{(n)}, \boldsymbol{W}^{(n)}\right). \tag{72}$$

Proof: The proof is given in the Appendix E. □

The Proof of Lemma 10 shows that there always exists a vector $\hat{\boldsymbol{p}} > 0$ such that

$$\max_{k\in\mathcal{K}} \frac{\gamma_k \mathcal{I}_k(\hat{\boldsymbol{p}}, \boldsymbol{W})}{\hat{p}_k} = C(\boldsymbol{\gamma}, \boldsymbol{W}). \tag{73}$$

That is, the infimum (66) is always achieved.

Theorem 5: There exists a fixed point $\boldsymbol{p}^* > 0$ satisfying (67) if and only if

$$C(\boldsymbol{\gamma}, \boldsymbol{W}) = C\left(\boldsymbol{\gamma}^{(n)}, \boldsymbol{W}^{(n)}\right), \quad 1 \le n \le i. \tag{74}$$

Proof: Suppose that there exists a $\boldsymbol{p}^* > 0$ such that (67) holds. Then, for all isolated blocks $\boldsymbol{W}^{(n)}$, with $1 \le n \le i$, we have

$$\gamma_k \mathcal{I}_k(\boldsymbol{p}, \boldsymbol{W}) = C(\boldsymbol{\gamma}, \boldsymbol{W}) \cdot p_k, \quad k \in \mathcal{K}_n. \tag{75}$$

Because of uniqueness (Lemma 8, part 2) we know that $C(\boldsymbol{\gamma}, \boldsymbol{W}) = C(\boldsymbol{\gamma}^{(n)}, \boldsymbol{W}^{(n)})$ holds for all n with $1 \le n \le i$.

Conversely, assume that (74) holds. Then the Proof of Lemma 10 shows that there is a $\boldsymbol{p}^* > 0$ such that (67) is fulfilled. For the isolated blocks, this follows from Theorem 4. For the nonisolated blocks, a vector can be constructed as in the Proof of Lemma 10. □

The results show that the existence of a fixed point $\boldsymbol{p}^*$ only depends on the isolated blocks. However, $\boldsymbol{p}^*$ is generally not unique since different scalings are possible for the isolated blocks. Arbitrary SIR can be achieved by users with nonisolated blocks, as shown in the Proof of Lemma 10.

C. Min-Max and Max-Min Balancing

In the previous section we have exploited that the min-max optimum $C(\boldsymbol{\gamma})$ characterizes the boundary of the SIR region (59). Now, an interesting question is whether an equivalent indicator function is obtained by max-min balancing, i.e.,

$$c(\boldsymbol{\gamma}) = \sup_{\boldsymbol{p}>0} \left(\min_{k\in\mathcal{K}} \frac{\gamma_k \mathcal{I}_k(\boldsymbol{p})}{p_k} \right). \tag{76}$$

In general, we have [10]

$$c(\boldsymbol{\gamma}) \le C(\boldsymbol{\gamma}). \tag{77}$$

Note that (77) is not a simple consequence of Fan's minimax inequality since we do not only interchange the optimization order, but also the domain. Inequality (77) was derived in [10] by exploiting the special properties of interference functions. Even for simple linear interference functions, equality does not need to hold [10].

Now, we extend these results by showing special properties for log-convex interference functions.

Theorem 6: Consider an arbitrary row-stochastic matrix $\boldsymbol{W} \in \mathbb{R}^{K\times K}_{+}$ with resulting log-convex interference functions $\mathcal{I}_k(\boldsymbol{p}, \boldsymbol{W})$, $k \in \mathcal{K}$. We have

$$c(\boldsymbol{\gamma}, \boldsymbol{W}) = C(\boldsymbol{\gamma}, \boldsymbol{W}) \tag{78}$$

if and only if for all isolated blocks $n = 1, 2, \ldots, i$

$$C\left(\boldsymbol{\gamma}^{(n)}, \boldsymbol{W}^{(n)}\right) = C(\boldsymbol{\gamma}, \boldsymbol{W}). \tag{79}$$

Proof: If (79) holds, then it follows from Theorem 5 that there is a fixed point $\boldsymbol{p}^* > 0$ fulfilling (67), thus implying (78). Conversely, assume that (78) holds. With (71) we have $C(\boldsymbol{\gamma}^{(n)}, \boldsymbol{W}^{(n)}) \le C(\boldsymbol{\gamma}, \boldsymbol{W})$ for all isolated blocks $1 \le n \le i$.

In a similar way, we can use definition (76) in order to show $c(\boldsymbol{\gamma}^{(n)}, \boldsymbol{W}^{(n)}) \geq c(\boldsymbol{\gamma}, \boldsymbol{W})$. With (77) we have

$$c(\boldsymbol{\gamma}, \boldsymbol{W}) \leq C\left(\boldsymbol{\gamma}^{(n)}, \boldsymbol{W}^{(n)}\right) \leq C(\boldsymbol{\gamma}, \boldsymbol{W}), \ \ \forall\, n \in \{1, 2, \ldots, i\}.$$

With (78) this is fulfilled with equality, so (79) holds. □

The following corollary is a direct consequence of Theorems 5 and 6.

Corollary 3: Consider an arbitrary row-stochastic matrix $\boldsymbol{W} \in \mathbb{R}_+^{K\times K}$. There exists a strictly positive fixed point $\boldsymbol{p}^* > 0$ satisfying (67) if and only if $c(\boldsymbol{\gamma}, \boldsymbol{W}) = C(\boldsymbol{\gamma}, \boldsymbol{W})$.

Note that Corollary 3 is derived under the assumption of particular interference functions (65), where $\boldsymbol{W}$ and f_k are constant. The result cannot be transferred to general log-convex interference functions with adaptive $\boldsymbol{W}$. Even for simple linear interference functions (1), the condition $c(\boldsymbol{\gamma}) = C(\boldsymbol{\gamma})$ does not always ensure the existence of a fixed point (63), as shown in [48], [49].

In the next section, we will study a more general class of log-convex interference functions where $\boldsymbol{W}$ is chosen adaptively. It will be shown (Theorem 7) that $c(\boldsymbol{\gamma}) = C(\boldsymbol{\gamma})$ holds if all possible $\boldsymbol{W}$ are irreducible.

D. Generalization to Adaptive $\boldsymbol{W}$

In the previous subsection we have considered a special class of log-convex interference functions (65), which depend on a fixed coefficient matrix $\boldsymbol{W}$. Now, the results will be extended by maximizing with respect to $\boldsymbol{W}$. The coefficients f_k are still assumed to be constant. General log-convex interference functions will be addressed later in Sections V-E and V-F.

Consider a coefficient set

$$\mathcal{W} = \{\boldsymbol{W} = [\boldsymbol{w}_1, \ldots, \boldsymbol{w}_K]^T : \boldsymbol{w}_k \in \mathcal{L}_k, \ \forall\, k \in \mathcal{K}\} \quad (80)$$

where $\mathcal{L}_k \subseteq \mathbb{R}_+^K$ is an arbitrary closed and bounded set such that any $\boldsymbol{w} \in \mathcal{L}_k$ fulfills $\|\boldsymbol{w}\|_1 = 1$. The set $\mathcal{W}$ is also closed and bounded.

Based on $\mathcal{W}$ and (65), we define log-convex interference functions

$$\mathcal{I}_k(\boldsymbol{p}) = \max_{\boldsymbol{W}\in\mathcal{W}} \mathcal{I}_k(\boldsymbol{p}, \boldsymbol{W}), \quad \forall\, k \in \mathcal{K}. \quad (81)$$

Note, that $\mathcal{I}_k(\boldsymbol{p}, \boldsymbol{W})$ only depends on $\boldsymbol{w}_k \in \mathcal{L}_k$, so we have K independent optimization problems. We will also use the vector notation

$$\boldsymbol{\mathcal{I}}(\boldsymbol{p}) = \begin{bmatrix} \max_{\boldsymbol{W}\in\mathcal{W}} \mathcal{I}_1(\boldsymbol{p}, \boldsymbol{W}) \\ \vdots \\ \max_{\boldsymbol{W}\in\mathcal{W}} \mathcal{I}_K(\boldsymbol{p}, \boldsymbol{W}) \end{bmatrix}. \quad (82)$$

Theorem 7: Consider a set $\mathcal{W}$, as defined by (80), with the additional requirement that all elements $\boldsymbol{W} \in \mathcal{W}$ are irreducible, with resulting interference functions (82). Then $c(\boldsymbol{\gamma}) = C(\boldsymbol{\gamma})$ and there exists a fixed point $\boldsymbol{p}^* > 0$ satisfying (63).

Proof: The proof is given in the Appendix G. □

The next theorem provides a necessary and sufficient condition for the existence of a strictly positive fixed point.

Theorem 8: Let $\boldsymbol{\mathcal{I}}(\boldsymbol{p})$ be defined as by (82). A vector $\boldsymbol{p}^* > 0$ is a fixed point satisfying (63) if and only if there exists a stochastic matrix $\boldsymbol{W}^* \in \mathcal{W}$ and a $\mu > 0$ such that

$$\mathcal{I}_k(\boldsymbol{p}^*) = \max_{\boldsymbol{W}\in\mathcal{W}} \mathcal{I}_k(\boldsymbol{p}^*, \boldsymbol{W}) = \mathcal{I}_k(\boldsymbol{p}^*, \boldsymbol{W}^*), \ \forall\, k \in \mathcal{K} \quad (83)$$

$$\boldsymbol{\Gamma}\boldsymbol{\mathcal{I}}(\boldsymbol{p}^*, \boldsymbol{W}^*) = \mu \cdot \boldsymbol{p}^*. \quad (84)$$

Then

$$\mu = C(\boldsymbol{\gamma}) = C(\boldsymbol{\gamma}, \boldsymbol{W}^*). \quad (85)$$

Proof: If $\boldsymbol{p}^* > 0$ is a fixed point satisfying (63) then (83) and (84) are fulfilled. From (84) we know that $\boldsymbol{p}^*$ is also a fixed point of $\boldsymbol{\Gamma}\boldsymbol{\mathcal{I}}(\cdot, \boldsymbol{W}^*)$. Because $\boldsymbol{p}^* > 0$, we known from Lemma 8 (part 2) that (85) is fulfilled.

Conversely, assume that (84) and (83) are fulfilled. Then

$$\begin{aligned}[\boldsymbol{\Gamma}\boldsymbol{\mathcal{I}}(\boldsymbol{p}^*)]_k &= \gamma_k \max_{\boldsymbol{W}\in\mathcal{W}} \mathcal{I}_k(\boldsymbol{p}^*, \boldsymbol{W}) = \gamma_k \mathcal{I}_k(\boldsymbol{p}^*, \boldsymbol{W}^*) \\ &= \mu[\boldsymbol{p}^*]_k. \end{aligned} \quad (86)$$

That is, $\boldsymbol{p}^* > 0$ is a fixed point of $\boldsymbol{\Gamma}\boldsymbol{\mathcal{I}}(\boldsymbol{p})$. Lemma 8 (part 2) yields (85). □

For the special case that all $\boldsymbol{W}$ are irreducible, we have the following result.

Theorem 9: Consider a the set $\mathcal{W}$, as defined by (80), such that all $\boldsymbol{W} \in \mathcal{W}$ are irreducible. Then

$$\max_{\boldsymbol{W}\in\mathcal{W}} C(\boldsymbol{\gamma}, \boldsymbol{W}) = C(\boldsymbol{\gamma}) \quad (87)$$

and there is a $\boldsymbol{p}^* > 0$ such that $\boldsymbol{\Gamma}\boldsymbol{\mathcal{I}}(\boldsymbol{p}^*) = C(\boldsymbol{\gamma})\boldsymbol{p}^*$, where $\boldsymbol{\mathcal{I}}$ is defined by (82).

Proof: The proof is given in the Appendix F. □

E. Characterization of Interference Coupling

Thus far, the interference coupling between the users has been characterized by a coefficient matrix $\boldsymbol{W} \in \mathcal{W}$. The structure of $\boldsymbol{W}$ determines whether there is a fixed point or not. This shows some similarities to the conventional power control model, where a *link gain matrix* is often used to characterize interference coupling.

In the remainder of this paper, we will return our attention to the general log-convex interference functions introduced in Section II. For this axiomatic model, there is no clear definition of the notion of interference coupling. We will therefore begin by introducing an asymptotic definition. The results will be used later in Section V-F, where conditions for the existence of a fixed point will be derived.

Let $\boldsymbol{e}_l$ be the all-zero vector with the lth component set to one, i.e.,

$$[\boldsymbol{e}_l]_n = \begin{cases} 1, & n = l \\ 0, & n \neq l. \end{cases}$$

We have the following result.

Lemma 11: Assume there exists a $\hat{\boldsymbol{p}} > 0$ such that $\lim_{\delta\to\infty} \mathcal{I}_k(\hat{\boldsymbol{p}} + \delta\boldsymbol{e}_l) = +\infty$, then

$$\lim_{\delta\to\infty} \mathcal{I}_k(\boldsymbol{p} + \delta\boldsymbol{e}_l) = +\infty, \quad \text{for all } \boldsymbol{p} > 0. \quad (88)$$

Proof: Let $\boldsymbol{p} > 0$ be arbitrary. There exists a $\lambda > 0$ such that $\lambda \boldsymbol{p} \geq \hat{\boldsymbol{p}}$. Thus, A3 implies

$$\lim_{\delta\to\infty} \mathcal{I}_k(\lambda\boldsymbol{p} + \delta\boldsymbol{e}_l) \geq \lim_{\delta\to\infty} \mathcal{I}_k(\hat{\boldsymbol{p}} + \delta\boldsymbol{e}_l) = +\infty. \quad (89)$$

With A2 we have $\mathcal{I}_k(\lambda\boldsymbol{p} + \delta\boldsymbol{e}_l) = \lambda\mathcal{I}_k(\boldsymbol{p} + \frac{\delta}{\lambda}\boldsymbol{e}_l)$. This implies $\lim_{\delta\to\infty} \mathcal{I}_k(\boldsymbol{p} + \frac{\delta}{\lambda}\boldsymbol{e}_l) = +\infty$, from which (88) follows. The interference function $\mathcal{I}_k$ is unbounded and monotonic increasing (axiom A3), hence the existence of the limits is guaranteed. □

For arbitrary interference functions satisfying A1–A3, condition (88) formalizes the notion of "user l causing interference to user k." With Lemma 11 we obtain a relatively simple definition of interference coupling by means of a matrix.

Definition 6: We refer to $\boldsymbol{A}_\mathcal{I}$ as the *asymptotic matrix* of $\mathcal{I}$

$$[\boldsymbol{A}_\mathcal{I}]_{kl} = \begin{cases} 1, & \text{if there exists a } \boldsymbol{p} > 0 \text{ such that} \\ & \lim_{\delta\to\infty} \mathcal{I}_k(\boldsymbol{p} + \delta\boldsymbol{e}_l) = +\infty \\ 0, & \text{otherwise.} \end{cases} \quad (90)$$

The matrix $\boldsymbol{A}_\mathcal{I}$ characterizes the way users are connected by interference. Notice that because of Lemma 11, the condition in (90) does not depend on the choice of $\boldsymbol{p}$.

In addition to the asymptotic matrix $\boldsymbol{A}_\mathcal{I}$, we introduce a further definition based on a weaker condition.

Definition 7: $\boldsymbol{D}_\mathcal{I}$ is called dependency matrix. We define $[\boldsymbol{D}_\mathcal{I}]_{kl}$ in

$$[\boldsymbol{D}_\mathcal{I}]_{kl} = \begin{cases} 1, & \text{if there exists a } \boldsymbol{p} > 0 \text{ such that } \mathcal{I}_k(\boldsymbol{p} + \delta\boldsymbol{e}_l) \\ & \text{ is not constant for some values } \delta > 0 \\ 0, & \text{otherwise.} \end{cases} \quad (91)$$

Evidently, $[\boldsymbol{A}_\mathcal{I}]_{kl} = 1$ implies $[\boldsymbol{D}_\mathcal{I}]_{kl} = 1$, but the converse is generally not true. However, both characterizations are indeed equivalent if the underlying interference functions are log-convex.

Theorem 10: Let $\mathcal{I}_1, \ldots, \mathcal{I}_K$ be log-convex interference functions, then both characterizations are equivalent, i.e., $\boldsymbol{A}_\mathcal{I} = \boldsymbol{D}_\mathcal{I}$.

Proof: The proof is given in the Appendix H □

Finally, we will derive a condition under which the asymptotic matrix is irreducible. To this end we introduce the set

$$\mathcal{W}_\mathcal{I} = \{\boldsymbol{W} = [\boldsymbol{w}_1, \ldots, \boldsymbol{w}_K]^T : \boldsymbol{w}_k \in \mathcal{L}(\mathcal{I}_k),\ \forall\, k \in \mathcal{K}\}. \quad (92)$$

Note that $\mathcal{W}_\mathcal{I}$ is based on the sets $\mathcal{L}(\mathcal{I}_k)$, as defined by (29). So it depends on the log-convex interference functions $\mathcal{I}_1, \ldots, \mathcal{I}_K$, which are arbitrary. In this respect it differs from the previously used set $\mathcal{W}$. Any $\boldsymbol{W} \in \mathcal{W}_\mathcal{I}$ is stochastic because of Lemma 3.

Theorem 11: The asymptotic matrix $\boldsymbol{A}_\mathcal{I}$ (equivalently, $\boldsymbol{D}_\mathcal{I}$) is irreducible if and only if there exists an irreducible stochastic matrix $\hat{\boldsymbol{W}} \in \mathcal{W}_\mathcal{I}$, and constants $C_1, \ldots, C_K > 0$, such that for all $\boldsymbol{p} > 0$

$$\mathcal{I}_k(\boldsymbol{p}) \geq C_k \prod_{l\in\mathcal{K}} (p_l)^{\hat{w}_{kl}}, \quad \forall\, k \in \mathcal{K}, \quad \forall\, \boldsymbol{p} > 0. \quad (93)$$

Proof: The proof is given in Appendix I □

Theorem 11 links irreducibility with the existence of nonzero lower bounds for the interference functions $\mathcal{I}_1, \ldots, \mathcal{I}_K$. This will be used in the next section.

F. Fixed-Point Analysis for General Log-Convex Interference Functions

In this subsection, we will study the existence of a fixed point $\boldsymbol{p}^* > 0$ satisfying (63) for general log-convex interference functions as introduced in Definition 2. Consider the coefficient set $\mathcal{W}_\mathcal{I}$ as defined by (92). The Theorem 12 shows that the existence of one irreducible coefficient matrix from $\mathcal{W}_\mathcal{I}$ is sufficient.

Theorem 12: Let $\boldsymbol{\mathcal{I}} = [\mathcal{I}_1, \ldots, \mathcal{I}_K]^T$ be a vector of log-convex interference functions, such that there exists a stochastic irreducible matrix $\hat{\boldsymbol{W}} \in \mathcal{W}_\mathcal{I}$. Then for all $\boldsymbol{\gamma} > 0$ there exists a fixed point $\boldsymbol{p}^* > 0$ such that

$$\boldsymbol{\Gamma}\boldsymbol{\mathcal{I}}(\boldsymbol{p}^*) = C(\boldsymbol{\gamma})\boldsymbol{p}^*. \quad (94)$$

Proof: The proof is given in the Appendix J. □

In Theorem 12 we have required $\hat{\boldsymbol{W}} \in \mathcal{W}_\mathcal{I}$, which means that $\hat{\boldsymbol{W}}$ is stochastic and $f_{\mathcal{I}_k}(\hat{\boldsymbol{w}}_k) > 0$ for all $k \in \mathcal{K}$. In this case, we know from (81) that

$$\mathcal{I}_k(\boldsymbol{p}) \geq f_{\mathcal{I}_k}(\hat{\boldsymbol{w}}_k) \prod_{l\in\mathcal{K}} (p_l)^{\hat{w}_{kl}}, \quad \forall k \in \mathcal{K},\ \forall \boldsymbol{p} > 0. \quad (95)$$

Conversely, consider a stochastic matrix $\hat{\boldsymbol{W}}$ such that (93) is fulfilled for some $C_1, \ldots, C_K > 0$. Then

$$\frac{\mathcal{I}_k(\boldsymbol{p})}{\prod_{l\in\mathcal{K}} (p_l)^{\hat{w}_{kl}}} \geq C_k > 0, \quad \forall k \in \mathcal{K},\ \forall \boldsymbol{p} > 0. \quad (96)$$

Thus, $f_{\mathcal{I}_k}(\hat{\boldsymbol{w}}_k) > 0, \forall k \in \mathcal{K}$, which implies $\hat{\boldsymbol{W}} \in \mathcal{W}_\mathcal{I}$. Both conditions are equivalent, so Theorem 12 leads to the following corollary.

Corollary 4: Assume there exist $C_1, \ldots, C_K > 0$ and a stochastic irreducible matrix $\hat{\boldsymbol{W}} \in \mathcal{W}_\mathcal{I}$ such that (93) holds, then for all $\boldsymbol{\gamma} > 0$ there exists a fixed point $\boldsymbol{p}^* > 0$ such that (94) holds.

With Theorem 11 we can reformulate this result as another corollary, which shows that irreducibility of the dependency matrix is always sufficient for the existence of a fixed point.

Corollary 5: If the dependency matrix $\boldsymbol{D}_\mathcal{I}$ (equivalently, $\boldsymbol{A}_\mathcal{I}$) is irreducible, then for all $\boldsymbol{\gamma} > 0$ there exists a fixed point $\boldsymbol{p}^* > 0$ such that (94) holds.

The next theorem addresses the case where the dependency matrix is *not irreducible*. Without loss of generality, we can choose the user indices such that $\boldsymbol{D}_\mathcal{I}$ has the canonical form (69). If an additional assumption is fulfilled, then there is at least one SIR vector which is not achievable.

Theorem 13: Assume that the dependency matrix $\boldsymbol{D}_\mathcal{I}$ (equivalently, $\boldsymbol{A}_\mathcal{I}$) is reducible, so it can be written in canonical form (69). Let $1, \ldots, l_1$ be the user indices associated with the isolated blocks. If

$$\inf_{\boldsymbol{p}>0} \max_{k>l_1} \frac{\gamma_k \mathcal{I}_k(\boldsymbol{p})}{p_k} = \underline{C}_1(\boldsymbol{\gamma}) > 0, \quad \forall\, \boldsymbol{\gamma} > 0 \quad (97)$$

then there exists a $\boldsymbol{\gamma} > 0$ such that there is no fixed point $\boldsymbol{p}^* > 0$ fulfilling (94).

Proof: The proof is given in the Appendix K. □

Note that condition (97) in Theorem 13 is not redundant. In the remainder of this subsection we will discuss examples of log-convex interference functions with reducible $\boldsymbol{D}_{\mathcal{I}}$ where all $\boldsymbol{\gamma} > 0$ have a corresponding fixed point (94). But in these cases we have a trivial lower bound $\underline{C}_1(\boldsymbol{\gamma}) = 0$. In this sense, Theorem 13 is best possible.

A result corresponding to Theorem 13 is known from the theory of nonnegative matrices [12], which is closely connected with the linear interference functions. For example, consider linear interference functions (1) based on a nonnegative coupling matrix $\boldsymbol{V}$. Without loss of generality, we can assume that $\boldsymbol{V}$ has canonical form (69). This is a special case of the log-convex interference model studied in this paper. We have $\boldsymbol{D}_{\mathcal{I}} = \boldsymbol{V}$. Let $\rho(\boldsymbol{\Gamma}^{(n)}\boldsymbol{V}^{(n)})$ be the spectral radius of the nth (weighted) block on the main diagonal, then it can be shown that

$$\underline{C}_1(\boldsymbol{\gamma}) = \max_{n>i} \rho(\boldsymbol{\Gamma}^{(n)}\boldsymbol{V}^{(n)}) \tag{98}$$

where i is the number of isolated blocks.

Consider the example

$$\boldsymbol{\Gamma V} = \mathrm{diag}[\gamma_1, \dots, \gamma_K] \cdot \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix}. \tag{99}$$

The isolated block is zero, so $\underline{C}_1(\boldsymbol{\gamma}) = 0$. The overall spectral radius is $\rho(\boldsymbol{\Gamma V}) = \sqrt{\gamma_1\gamma_2}$. It can easily be checked that for any $\boldsymbol{\Gamma}$ there is a $\boldsymbol{p}_\Gamma > 0$ such that $\boldsymbol{\Gamma V p}_\Gamma = \rho(\boldsymbol{\Gamma V})\boldsymbol{p}_\Gamma$. This also follows from [47], where it was shown that an arbitrary $\boldsymbol{\gamma} > 0$ is associated with a positive fixed point $\boldsymbol{p}_\Gamma > 0$ if and only if the set of maximal blocks equals the set of isolated blocks, i.e.,

$$\rho(\boldsymbol{\Gamma V}) = \rho\left(\boldsymbol{\Gamma}^{(n)}\boldsymbol{V}^{(n)}\right), \quad 1 \le n \le i \tag{100}$$

$$\text{and} \quad \rho(\boldsymbol{\Gamma V}) > \rho\left(\boldsymbol{\Gamma}^{(n)}\boldsymbol{V}^{(n)}\right), \quad n > i. \tag{101}$$

These conditions are fulfilled for the example (99), because $\rho(\boldsymbol{\Gamma}^{(1)}\boldsymbol{V}^{(1)}) = \sqrt{\gamma_1\gamma_2}$ and $\rho(\boldsymbol{\Gamma}^{(2)}\boldsymbol{V}^{(2)}) = 0$.

With (100) and (101) we can also derive simple sufficient conditions for the nonexistence of a fixed point. For example, we can choose a reducible matrix $\boldsymbol{\Gamma V}$ such that a nonisolated block $\boldsymbol{\Gamma}^{(n)}\boldsymbol{V}^{(n)}$, $n > i$, is maximal. Or we can choose $\boldsymbol{\gamma}$ such that an isolated block $\boldsymbol{\Gamma}^{(n)}\boldsymbol{V}^{(n)}$, $n \le i$, is not maximal. In both cases, there is no solution to the fixed-point equation $\boldsymbol{\Gamma V p} = \rho(\boldsymbol{\Gamma V})\boldsymbol{p}$. Note that both cases require that at least one nonisolated block has a nonzero spectral radius, so $\underline{C}_1(\boldsymbol{\gamma}) > 0$.

Discussing linear interference functions helps to better understand Theorem 13. However, the actual value of the theorem—as well as the other results—lies in its applicability to a broader class of interference functions. All results hold for arbitrary log-convex interference functions as introduced by Definition 2.

As a further illustration, consider the log-convex interference functions $\mathcal{I}_k(\boldsymbol{p}, \boldsymbol{W})$, as defined by (65), based on an arbitrary reducible stochastic matrix $\boldsymbol{W}$. We assume that there is at least one nonisolated block and a single isolated block. Every nonzero entry in $\boldsymbol{W}$ corresponds to a nonzero entry in $\boldsymbol{A}_{\mathcal{I}}$ and $\boldsymbol{D}_{\mathcal{I}}$ with the same position. From Lemma 10 and Theorem 5 we know that for any $\boldsymbol{\gamma} > 0$ we have $C(\boldsymbol{\gamma}, \boldsymbol{W}) = C(\boldsymbol{\gamma}^{(1)}, \boldsymbol{W}^{(1)})$ and there is a fixed point $\boldsymbol{p}^* > 0$. This is a consequence of $\boldsymbol{W}$ having a single isolated block. Arbitrary γ_k can be achieved by the nonisolated users (see Proof of Theorem 5), so $\underline{C}_1(\boldsymbol{\gamma}) = 0$ for all $\boldsymbol{\gamma} > 0$. That is, $\boldsymbol{D}_{\mathcal{I}}$ can be reducible and all $\boldsymbol{\gamma} > 0$ are associated with a fixed point, but in this case $\underline{C}_1(\boldsymbol{\gamma}) = 0$. This is another example showing that the requirement $\underline{C}_1(\boldsymbol{\gamma}) > 0$ is generally important and cannot be omitted.

The results of this section show that the special properties of log-convex interference functions are very useful for the analysis of the fixed point (94), which is closely connected with the achievability of boundary points of the QoS region. In particular, the irreducibility of the dependency matrix $\boldsymbol{D}_{\mathcal{I}}$ is sufficient for the achievability of the *entire boundary*. This shows an interesting analogy to the theory of linear interference functions (Perron–Frobenius theory), where an irreducible "link gain matrix" is typically assumed to ensure the existence of a min-max optimal power vector. Linear interference functions are a special case of the axiomatic framework of log-convex interference functions. Note, that log-convexity is the key property which is exploited here. A similar characterization of the boundary can be more complicated for other classes of interference functions (see, e.g., [10]). This is still an open problem for general interference functions being solely characterized by A1–A3.

VI. Conclusion

This paper provides an axiomatic framework for log-convex interference functions. Log-convexity is a useful property with interesting applications in multiuser communications. We have discussed the examples of robust designs, utility optimization, cooperative game theory, and max-min fairness. The results are also useful for the analysis of QoS regions: many QoS regions can be expressed as a sublevel set of a log-convex interference function. By analyzing the structure of interference functions, we are able to better understand the structure of the associated QoS region.

It has been shown that properties of log-convex interference functions are closed under certain operations. For example, if the underlying functions $\mathcal{I}_1, \dots, \mathcal{I}_K$ are log-convex interference functions then the min-max optimum $C(\boldsymbol{\gamma})$ is a log-convex interference function as well. The same holds for the sum of log-convex interference functions. The results in Section IV show that *every* log-convex interference function can be expressed as an optimum over elementary log-convex interference functions. This justifies the name "calculus" used in the title.

Finally, the results show that log-convex interference functions offer rich analytical possibilities, similar to linear interference functions. For example, the achievability of the entire boundary of the SIR region (existence of a fixed point) can be completely characterized by means of a single "dependency matrix." Similar results are known from the theory of linear interference functions, which is based on an irreducible link gain matrix. In this case, the Perron–Frobenius theorem states the existence of a positive eigenvector (fixed point). Hence, log-convex interference functions can be regarded as a natural generalization of linear interference functions.

APPENDIX

A. Log-Convexity of $C(\boldsymbol{\gamma}(\boldsymbol{q}))$, Used in (17)

Proof: Consider two arbitrary points $\hat{\boldsymbol{q}}, \check{\boldsymbol{q}} \in \mathbb{Q}^K$, being connected by a line

$$\boldsymbol{q}(\lambda) = (1-\lambda)\hat{\boldsymbol{q}} + \lambda\check{\boldsymbol{q}}, \quad \lambda \in [0,1]. \tag{102}$$

Consider the point $\hat{\boldsymbol{q}}$. The definition (16) implies the existence of an $\epsilon > 0$ and a vector $\hat{\boldsymbol{p}} := \hat{\boldsymbol{p}}(\epsilon) > 0$ such that

$$\max_{k\in\mathcal{K}} \log \frac{\gamma_k(\hat{q}_k)\cdot\mathcal{I}_k(\hat{\boldsymbol{p}})}{[\hat{\boldsymbol{p}}]_k} \le \log C(\boldsymbol{\gamma}(\hat{\boldsymbol{q}})) + \epsilon. \tag{103}$$

A similar inequality holds for the point $\check{\boldsymbol{q}}$, with $\check{\boldsymbol{p}} > 0$. Next, we introduce the substitutions $\hat{\boldsymbol{q}} = \mathrm{e}^{\hat{\boldsymbol{s}}}$ and $\check{\boldsymbol{q}} = \mathrm{e}^{\check{\boldsymbol{s}}}$, with

$$\boldsymbol{s}(\lambda) = (1-\lambda)\hat{\boldsymbol{s}} + \lambda\check{\boldsymbol{s}}, \quad \lambda \in [0,1]. \tag{104}$$

Now, we can exploit that the functions $\gamma_k(q_k)$ and $\mathcal{I}_k(\mathrm{e}^{\boldsymbol{s}})$ are log-convex by assumption. Since e^{s_k} is log-convex and log-concave, and the point-wise product of two log-convex functions is log-convex [37], the function $\mathcal{I}_k(\mathrm{e}^{\boldsymbol{s}})/\mathrm{e}^{s_k}$ is log-convex. Thus

$$\begin{aligned}
&\log\left(\gamma_k(q_k(\lambda))\cdot\frac{\mathcal{I}_k\left(\mathrm{e}^{\boldsymbol{s}(\lambda)}\right)}{\mathrm{e}^{s_k(\lambda)}}\right)\\
&= \log\gamma_k(q_k(\lambda)) + \log\frac{\mathcal{I}_k\left(\mathrm{e}^{\boldsymbol{s}(\lambda)}\right)}{\mathrm{e}^{s_k(\lambda)}}\\
&\quad \le (1-\lambda)\log\gamma_k(\hat{q}_k) + \lambda\log\gamma_k(\check{q}_k)\\
&\qquad + (1-\lambda)\log\frac{\mathcal{I}_k(\mathrm{e}^{\hat{\boldsymbol{s}}})}{\mathrm{e}^{\hat{s}_k}} + \lambda\log\frac{\mathcal{I}_k(\mathrm{e}^{\check{\boldsymbol{s}}})}{\mathrm{e}^{\check{s}_k}}\\
&\quad = (1-\lambda)\log\frac{\gamma_k(\hat{q}_k)\cdot\mathcal{I}_k(\mathrm{e}^{\hat{\boldsymbol{s}}})}{\mathrm{e}^{\hat{s}_k}} + \lambda\log\frac{\gamma_k(\check{q}_k)\cdot\mathcal{I}_k(\mathrm{e}^{\check{\boldsymbol{s}}})}{\mathrm{e}^{\check{s}_k}}\\
&\quad \le (1-\lambda)\log C(\boldsymbol{\gamma}(\hat{\boldsymbol{q}})) + \lambda\log C(\boldsymbol{\gamma}(\check{\boldsymbol{q}})) + 2\epsilon
\end{aligned}$$

where the last inequality follows from (103). Consequently

$$\begin{aligned}
&\log C(\boldsymbol{q}(\lambda))\\
&\quad = \inf_{\boldsymbol{s}\in\mathbb{R}^K_+}\left(\max_{k\in\mathcal{K}}\log\frac{\gamma_k(q_k(\lambda))\cdot\mathcal{I}_k(\mathrm{e}^{\boldsymbol{s}})}{\mathrm{e}^{s_k}}\right)\\
&\quad \le (1-\lambda)\log C(\boldsymbol{\gamma}(\hat{\boldsymbol{q}})) + \lambda\log C(\boldsymbol{\gamma}(\check{\boldsymbol{q}})) + 2\epsilon.
\end{aligned} \tag{105}$$

This holds for any $\epsilon > 0$. The left-hand side of (105) does not depend on ϵ, so letting $\epsilon \to 0$ it can be concluded that $C(\boldsymbol{\gamma}(\boldsymbol{q}))$ is log-convex on $\mathbb{Q}^K$. □

B. Log-Convexity of $\boldsymbol{p}^{\min}(\boldsymbol{q})$, as Defined by (19)

Proof: Consider two arbitrary feasible QoS points $\hat{\boldsymbol{q}}, \check{\boldsymbol{q}} \in \mathrm{int}\mathcal{Q}$, connected by a line $\boldsymbol{q}(\lambda)$, as defined by (102). Log-convexity implies

$$\gamma_k(q_k(\lambda)) \le \gamma_k(\hat{q}_k)^{1-\lambda}\cdot\gamma_k(\check{q}_k)^{\lambda}, \quad \forall\, k\in\mathcal{K}. \tag{106}$$

By $\hat{\boldsymbol{p}} := \boldsymbol{p}^{\min}(\hat{\boldsymbol{q}})$ and $\check{\boldsymbol{p}} := \boldsymbol{p}^{\min}(\check{\boldsymbol{q}})$ we denote the power vectors solving the power minimization problem (19) for given targets $\hat{\boldsymbol{q}}$ and $\check{\boldsymbol{q}}$, respectively. It was shown in [3] that these vectors are characterized by fixed-point equations

$$\gamma_k(\hat{q}_k)\cdot\mathcal{I}_k(\hat{\boldsymbol{p}}) = \hat{p}_k, \quad \forall\, k\in\mathcal{K} \tag{107}$$

$$\gamma_k(\check{q}_k)\cdot\mathcal{I}_k(\check{\boldsymbol{p}}) = \check{p}_k, \quad \forall\, k\in\mathcal{K}. \tag{108}$$

Now, we introduce substitutions $\hat{\boldsymbol{p}} = \exp\hat{\boldsymbol{s}}$ (component-wise) and $\check{\boldsymbol{p}} = \exp\check{\boldsymbol{s}}$. The points $\hat{\boldsymbol{s}}$ and $\check{\boldsymbol{s}}$ are connected by a line $\boldsymbol{s}(\lambda)$, as defined by (104). Because $\mathcal{I}_k(\mathrm{e}^{\boldsymbol{s}})$ is log-convex on $\mathbb{R}^K$ by assumption

$$\mathcal{I}_k(\exp\boldsymbol{s}(\lambda)) \le \mathcal{I}_k(\exp\hat{\boldsymbol{s}})^{1-\lambda}\cdot\mathcal{I}_k(\exp\check{\boldsymbol{s}})^{\lambda}, \quad \forall\, k\in\mathcal{K}. \tag{109}$$

Defining

$$\boldsymbol{p}(\lambda) := \exp\boldsymbol{s}(\lambda) = \hat{\boldsymbol{p}}^{1-\lambda}\cdot\check{\boldsymbol{p}}^{\lambda} \tag{110}$$

inequality (109) can be rewritten as

$$\mathcal{I}_k(\boldsymbol{p}(\lambda)) \le \mathcal{I}_k(\hat{\boldsymbol{p}})^{1-\lambda}\cdot\mathcal{I}_k(\check{\boldsymbol{p}})^{\lambda}. \tag{111}$$

With (106), (110), and (111), we have

$$\begin{aligned}
&\frac{\gamma_k(q_k(\lambda))\cdot\mathcal{I}_k(\boldsymbol{p}(\lambda))}{p_k(\lambda)}\\
&\quad \le \frac{\gamma_k(\hat{q}_k)^{1-\lambda}\cdot\gamma_k(\check{q}_k)^{\lambda}\cdot\mathcal{I}_k(\hat{\boldsymbol{p}})^{1-\lambda}\cdot\mathcal{I}_k(\check{\boldsymbol{p}})^{\lambda}}{p_k(\lambda)}\\
&\quad = \left(\frac{\gamma_k(\hat{q}_k)\cdot\mathcal{I}_k(\hat{\boldsymbol{p}})}{(\hat{p}_k)}\right)^{1-\lambda}\cdot\left(\frac{\gamma_k(\check{q}_k)\cdot\mathcal{I}_k(\check{\boldsymbol{p}})}{(\check{p}_k)}\right)^{\lambda}.
\end{aligned} \tag{112}$$

Exploiting (107) and (108), inequality (112) can be rewritten as

$$\frac{p_k(\lambda)}{\mathcal{I}_k(\boldsymbol{p}(\lambda))} \ge \gamma_k(q_k(\lambda)), \quad \forall k\in\mathcal{K}.$$

That is, for any $\lambda\in[0,1]$, the power vector $\boldsymbol{p}(\lambda)$ achieves the QoS targets $\boldsymbol{q}(\lambda)$. We know that $\boldsymbol{p}^{\min}(\boldsymbol{q}(\lambda))$, as defined by (19), achieves $\boldsymbol{q}(\lambda)$ with component-wise minimal power [3], thus

$$p_k^{\min}(\boldsymbol{q}(\lambda)) \le p_k(\lambda), \quad \forall\, k\in\mathcal{K}. \tag{113}$$

With (110) it can be concluded that

$$\begin{aligned}
p_k^{\min}(\boldsymbol{q}(\lambda)) &\le (\hat{p}_k)^{1-\lambda}\cdot(\check{p}_k)^{\lambda}\\
&= \left(p_k^{\min}(\hat{\boldsymbol{q}})\right)^{1-\lambda}\cdot\left(p_k^{\min}(\check{\boldsymbol{q}})\right)^{\lambda}, \quad \forall\,\lambda\in[0,1].
\end{aligned}$$

This shows that $p_k^{\min}(\boldsymbol{q})$ is log-convex on $\mathrm{int}\mathcal{Q}$ for all $k\in\mathcal{K}$. □

C. Proof of Theorem 1

Assume that $g(\mathrm{e}^x)$ is convex, then for any $\hat{x}, \check{x} \in \mathbb{R}$, with $x(\lambda) = (1-\lambda)\hat{x} + \lambda\check{x}$, we have

$$g\left(\mathrm{e}^{x(\lambda)}\right) \le (1-\lambda)g(\mathrm{e}^{\hat{x}}) + \lambda g(\mathrm{e}^{\check{x}}), \quad \forall\,\lambda\in[0,1]. \tag{114}$$

The function $c_k(\boldsymbol{s}) = \mathcal{I}_k(\mathrm{e}^{\boldsymbol{s}})/\mathrm{e}^{s_k}$ is log-convex for all k, i.e.,

$$c_k(\boldsymbol{s}(\lambda)) \le c_k(\hat{\boldsymbol{s}})^{1-\lambda}\cdot c_k(\check{\boldsymbol{s}})^{\lambda}, \quad \lambda\in[0,1] \tag{115}$$

where $\boldsymbol{s}(\lambda)$ is defined in (104). Exploiting (114), (115), and the monotonicity of g, we obtain

$$\begin{aligned}
g\left(\mathrm{e}^{\log c_k(\boldsymbol{s}(\lambda))}\right) &\le g(\exp\{(1-\lambda)\log c_k(\hat{\boldsymbol{s}}) + \lambda\log c_k(\check{\boldsymbol{s}})\})\\
&\le (1-\lambda)\cdot g(c_k(\hat{\boldsymbol{s}})) + \lambda\cdot g(c_k(\check{\boldsymbol{s}})).
\end{aligned}$$

The sum of convex functions is convex, thus the objective function in (22) is convex on $\mathbb{R}^K$.

Conversely, assume that (22) is convex. We want to show that this implies convexity of $g(\mathrm{e}^x)$. To this end, consider the set $\mathcal{G}$, which is the set of all g such that (22) is convex for *all* log-convex interference functions $\mathcal{I}$. Also, consider the set $\mathcal{G}_{\text{lin}}$, which is the set of all g such that (22) is convex for the specific linear interference functions $\mathcal{I}_1(\mathrm{e}^{\boldsymbol{s}}) = e^{s_2}$ and $\mathcal{I}_2(\mathrm{e}^{\boldsymbol{s}}) = e^{s_1}$. These functions are also log-convex, thus, $\mathcal{G} \subseteq \mathcal{G}_{\text{lin}}$. We now show that all $g \in \mathcal{G}_{\text{lin}}$ are convex. For an arbitrary $g \in \mathcal{G}_{\text{lin}}$, the function

$$F(\boldsymbol{s}, \alpha_1, \alpha_2) = \alpha_1 g(\mathrm{e}^{s_2 - s_1}) + \alpha_2 g(\mathrm{e}^{s_1 - s_2}) \tag{116}$$

is convex in $\boldsymbol{s}$ by assumption. Convexity is preserved when we set $s_1 = 0$. Let $\alpha_2 = 1 - \alpha_1$. A convergent series of convex functions is a convex function [37], thus

$$\lim_{\alpha_1 \to 1} F(\boldsymbol{s}, \alpha_1) = g(\mathrm{e}^{s_2}) \tag{117}$$

is convex, and therefore, $g(\mathrm{e}^{\boldsymbol{s}})$ is convex. It can be concluded that all $g \in \mathcal{G}$ are convex.

D. Proof of Theorem 4

For the Proof of Theorem 4 we will need the following result.

Lemma 12: Let $\boldsymbol{q}$ be the principal left-hand eigenvector of an irreducible stochastic $K \times K$ matrix $\boldsymbol{W}$, then the set $\mathcal{O}_q = \{\boldsymbol{z} \in \mathbb{R}^K : \boldsymbol{q}^T \boldsymbol{z} = 0\}$ equals the range of $(\boldsymbol{I} - \boldsymbol{W})$.

Proof: Every row stochastic $\boldsymbol{W}$ fulfills $\boldsymbol{W}\mathbf{1} = \mathbf{1}$, so $\mathbf{1}$ is an eigenvector of $\boldsymbol{W}$. Since $\boldsymbol{W}$ is irreducible by assumption, it follows from the Perron–Frobenius theorem (see, e.g., [11], [12]) that only the maximum eigenvalue, which equals the spectral radius $\rho(\boldsymbol{W})$, can be associated with a nonnegative eigenvector. Thus, $\boldsymbol{W}$ has a maximal eigenvalue $\rho(\boldsymbol{W}) = \rho(\boldsymbol{W}^T) = 1$. Because $\boldsymbol{W}^T$ is irreducible as well, the left-hand principal eigenvector $\boldsymbol{q} > 0$, is unique up to a scaling. We can assume $\|\boldsymbol{q}\|_1 = 1$ without loss of generality. We have $\boldsymbol{q}^T\boldsymbol{W} = \boldsymbol{q}^T$, or equivalently, $\boldsymbol{q}^T(\boldsymbol{I} - \boldsymbol{W}) = \mathbf{0}^T$. Thus

$$\boldsymbol{q}^T(\boldsymbol{I} - \boldsymbol{W})\boldsymbol{s} = 0, \quad \text{for all } \boldsymbol{s} \in \mathbb{R}^K. \tag{118}$$

Consider the range $\mathcal{R}(\boldsymbol{I} - \boldsymbol{W}) = (\boldsymbol{I} - \boldsymbol{W})\mathbb{R}^K$. For all $\boldsymbol{z} \in \mathcal{R}(\boldsymbol{I} - \boldsymbol{W})$, there exists a $\boldsymbol{s} \in \mathbb{R}^K$ with $(\boldsymbol{I} - \boldsymbol{W})\boldsymbol{s} = \boldsymbol{z}$. From (118) we know that $\boldsymbol{q}^T\boldsymbol{z} = 0$ thus, $\mathcal{R}(\boldsymbol{I}-\boldsymbol{W})$ lies in the $(K-1)$-dimensional hyperplane $\mathcal{O}_q$. That is,

$$\mathcal{R}(\boldsymbol{I} - \boldsymbol{W}) \subseteq \{\boldsymbol{z} \in \mathbb{R}^K : \boldsymbol{q}^T\boldsymbol{z} = 0\} = \mathcal{O}_q. \tag{119}$$

For vector spaces $\mathcal{M}$ and $\mathcal{N}$ such that $\mathcal{M} \subseteq \mathcal{N}$, it is known that $\dim \mathcal{M} = \dim \mathcal{N}$ implies $\mathcal{M} = \mathcal{N}$ (see, e.g., [13, p. 198]). From (119) we have $\dim \mathcal{R}(\boldsymbol{I} - \boldsymbol{W}) \le K - 1$. So in order to prove the lemma, it remains to show $\dim \mathcal{R}(\boldsymbol{I} - \boldsymbol{W}) \ge K - 1$, thus implying $\dim \mathcal{R}(\boldsymbol{I} - \boldsymbol{W}) = K - 1$.

Because $\boldsymbol{W}$ is irreducible and stochastic by assumption, there exists a decomposition $\boldsymbol{W} = \boldsymbol{B} + \mathbf{1}\boldsymbol{q}^T$ such that $\boldsymbol{I} - \boldsymbol{B}$ is nonsingular [50]. For any $\boldsymbol{z} \in \mathcal{O}_q$, we have $\boldsymbol{W}\boldsymbol{z} = \boldsymbol{B}\boldsymbol{z} + \mathbf{1}\boldsymbol{q}^T\boldsymbol{z} = \boldsymbol{B}\boldsymbol{z}$. Thus

$$(\boldsymbol{I} - \boldsymbol{B})\mathcal{O}_q = (\boldsymbol{I} - \boldsymbol{W})\mathcal{O}_q. \tag{120}$$

The hyperplane $\mathcal{O}_q$ has dimension $K - 1$. Since $(\boldsymbol{I} - \boldsymbol{B})$ is nonsingular, we have $\dim(\boldsymbol{I} - \boldsymbol{B})\mathcal{O}_q = K - 1$, and with (120) we have $\dim(\boldsymbol{I} - \boldsymbol{W})\mathcal{O}_q = K - 1$. Also, $(\boldsymbol{I} - \boldsymbol{W})\mathbb{R}^K \supset (\boldsymbol{I} - \boldsymbol{W})\mathcal{O}_q$ implies

$$\dim \mathcal{R}(\boldsymbol{I} - \boldsymbol{W}) \ge \dim(\boldsymbol{I} - \boldsymbol{W})\,\mathcal{O}_q = K - 1$$

which concludes the proof. □

We will now use Lemmas 12 and 9 to prove Theorem 4.

The matrix $\boldsymbol{W}$ is irreducible, so Lemma 12 implies $(\boldsymbol{I} - \boldsymbol{W})\mathbb{R}^K = \mathcal{O}_q$, where $\mathcal{O}_q = \{\boldsymbol{z} \in \mathbb{R}^K : \boldsymbol{q}^T\boldsymbol{z} = 0\}$. That is, for every $\boldsymbol{z} \in \mathcal{O}_q$, there exists an $\boldsymbol{s} \in \mathbb{R}^K$, such that $(\boldsymbol{I} - \boldsymbol{W})\boldsymbol{s} = \boldsymbol{z}$. Consider the special choice $\boldsymbol{z}^* = \log \boldsymbol{t} - \mathcal{C}'\mathbf{1}$, with $\mathcal{C}' = \boldsymbol{q}^T \log \boldsymbol{t}$. It can be verified that $\boldsymbol{q}^T\boldsymbol{z}^* = 0$, thus, $\boldsymbol{z}^* \in \mathcal{O}_q$. The associated vector $\boldsymbol{s}^*$ solves

$$(\boldsymbol{I} - \boldsymbol{W})\boldsymbol{s}^* = \log \boldsymbol{t} - \mathcal{C}'\mathbf{1}. \tag{121}$$

From Lemma 9 we know that with the substitutions $C' = \exp\{\mathcal{C}'\}$ and $\boldsymbol{p}^* = \exp\{\boldsymbol{s}^*\}$, we have

$$C'\boldsymbol{p}^* = \boldsymbol{\Gamma}\boldsymbol{\mathcal{I}}(\boldsymbol{p}^*, \boldsymbol{W}). \tag{122}$$

The vector $\boldsymbol{p}^* > 0$ is a fixed point of $\boldsymbol{\Gamma}\boldsymbol{\mathcal{I}}(\boldsymbol{p}, \boldsymbol{W})/C'$. It was shown in [10] (see also Lemma 8) that this implies $C' = C(\boldsymbol{\gamma}, \boldsymbol{W})$. Thus, $\boldsymbol{p}^*$ is a solution of the fixed point (67), for given $\boldsymbol{W}$.

It remains to prove uniqueness. Suppose that there are two vectors $\boldsymbol{p}^{(1)}$ and $\boldsymbol{p}^{(2)}$, with substitute variables $\boldsymbol{s}^{(1)}$ and $\boldsymbol{s}^{(2)}$, respectively, which fulfill

$$(\boldsymbol{I} - \boldsymbol{W})\boldsymbol{s}^{(1)} = \log \boldsymbol{t} - \mathcal{C}\mathbf{1} = (\boldsymbol{I} - \boldsymbol{W})\boldsymbol{s}^{(2)}.$$

Then

$$\boldsymbol{W}\left(\boldsymbol{s}^{(1)} - \boldsymbol{s}^{(2)}\right) = \left(\boldsymbol{s}^{(1)} - \boldsymbol{s}^{(2)}\right).$$

Since the power vectors can be scaled arbitrarily without affecting the optimum (66), we can assume $(\boldsymbol{s}^{(1)} - \boldsymbol{s}^{(2)}) > 0$ without loss of generality. Since $\boldsymbol{W}$ is a stochastic irreducible matrix, there is only one possible positive eigenvector $(\boldsymbol{s}^{(1)} - \boldsymbol{s}^{(2)}) = \mu\mathbf{1}$, thus

$$\boldsymbol{p}^{(1)} = \mathrm{e}^{\mu} \cdot \boldsymbol{p}^{(2)}.$$

This shows uniqueness up to a scaling.

E. Proof of Lemma 10

Consider the isolated blocks $\boldsymbol{W}^{(n)}$, $1 \le n \le i$, which are irreducible by definition. We know from Theorem 4 that each of these isolated subsystems is characterized by a fixed-point equation of the form (67), where all quantities are confined to the respective subsystem, with a unique (up to a scaling) power vector $\boldsymbol{p}^{(n)} \in \mathbb{R}^{K_n}_{++}$ and a min-max level $C(\boldsymbol{\gamma}^{(n)}, \boldsymbol{W}^{(n)})$, as defined by (70). Exploiting that the users $\mathcal{K}_n$ do not depend on powers of other blocks, we can use $\mathcal{I}_k(\boldsymbol{p}, \boldsymbol{W})$ instead of $\mathcal{I}_k(\boldsymbol{p}^{(n)}, \boldsymbol{W}^{(n)})$ for all $k \in \mathcal{K}_n$, as in (70). So for all isolated blocks n, with $1 \le n \le i$, we have

$$\gamma_k \mathcal{I}_k(\boldsymbol{p}, \boldsymbol{W}) = C\left(\boldsymbol{\gamma}^{(n)}, \boldsymbol{W}^{(n)}\right) \cdot p_k, \quad \forall k \in \mathcal{K}_n. \tag{123}$$

The K-dimensional power vector of the complete system is

$$\boldsymbol{p} = \left[\left(\boldsymbol{p}^{(1)}\right)^T, \ldots, \left(\boldsymbol{p}^{(i)}\right)^T, \left(\boldsymbol{p}^{(i+1)}\right)^T, \ldots, \left(\boldsymbol{p}^{(N)}\right)^T \right]^T. \tag{124}$$

With (123), the first i vectors $\boldsymbol{p}^{(1)}, \ldots, \boldsymbol{p}^{(i)}$ are determined up to a scaling. For all users belonging to the isolated blocks, we have

$$\frac{\gamma_k \mathcal{I}_k(\boldsymbol{p}, \boldsymbol{W})}{p_k} \leq \max_{1 \leq n \leq i} C\left(\boldsymbol{\gamma}^{(n)}, \boldsymbol{W}^{(n)}\right), \quad \forall\, k \in \cup_{1 \leq n \leq i} \mathcal{K}_n. \tag{125}$$

Next, consider the first nonisolated block $i+1$. From the structure of the matrix $\boldsymbol{W}$, it is clear that the interference $\mathcal{I}_k(\boldsymbol{p}, \boldsymbol{W})$, for any $k \in \mathcal{K}_{i+1}$, can only depend on the power vectors $\boldsymbol{p}^{(1)}, \ldots, \boldsymbol{p}^{(i+1)}$. The vectors $\boldsymbol{p}^{(1)}, \ldots, \boldsymbol{p}^{(i)}$ have already been determined. It will now be shown that for an arbitrary $\mu_{i+1} \in \mathbb{R}_{++}$ there is a unique power vector $\boldsymbol{p}^{(i+1)}$ such that

$$\gamma_k \mathcal{I}_k(\boldsymbol{p}, \boldsymbol{W}) = \mu_{i+1} \cdot p_k, \quad \forall\, k \in \mathcal{K}_{i+1}. \tag{126}$$

Here, $\boldsymbol{p}$ is defined as by (124). The last components $i+2, \ldots, N$ can be chosen arbitrarily because (126) does not depend on them. They will be constructed later.

Taking the logarithm of both sides of (126) and using $\boldsymbol{s}^{(n)} = \log \boldsymbol{p}^{(n)}$, we obtain (see Lemma 9)

$$\left(\boldsymbol{I} - \boldsymbol{W}^{(i+1)}\right) \boldsymbol{s}^{(i+1)} = -\log \mu_{i+1} + \log \boldsymbol{t}^{(i+1)} + \sum_{n=1}^{i} \boldsymbol{W}^{(i+1,n)} \boldsymbol{s}^{(n)}. \tag{127}$$

Since $\rho(\boldsymbol{W}^{(i+1)}) < 1$, the matrix $(\boldsymbol{I} - \boldsymbol{W}^{(i+1)})$ is invertible, so we can solve (127) for $\boldsymbol{s}^{(i+1)}$. For given $\boldsymbol{s}^{(1)}, \ldots, \boldsymbol{s}^{(i)}$ and μ_{i+1}, the power vector $\boldsymbol{p}^{(i+1)} = \exp \boldsymbol{s}^{(i+1)}$ is unique and it achieves the targets $\boldsymbol{\gamma}^{(i+1)}$ with equality.

By induction, it follows that unique vectors $\boldsymbol{s}^{(n)}$ are obtained for all nonisolated blocks $n = i+2, \ldots, n$. This is ensured because $\rho(\boldsymbol{W}^{(n)}) < 1$ for all nonisolated blocks. Arbitrary levels $\mu_{i+1}, \ldots, \mu_N$ can be achieved. We can choose $\mu_{i+1}, \ldots, \mu_N$ such that the resulting vector $\boldsymbol{p} > 0$ fulfills

$$\frac{\gamma_k \mathcal{I}_k(\boldsymbol{p}, \boldsymbol{W})}{p_k} \leq \max_{1 \leq n \leq i} C\left(\boldsymbol{\gamma}^{(n)}, \boldsymbol{W}^{(n)}\right), \quad \text{for all } k \in \mathcal{K}.$$

Hence

$$C(\boldsymbol{\gamma}, \boldsymbol{W}) = \inf_{\tilde{\boldsymbol{p}} > 0} \left(\max_{k \in \mathcal{K}} \frac{\gamma_k \mathcal{I}_k(\tilde{\boldsymbol{p}}, \boldsymbol{W})}{\tilde{p}_k} \right) \leq \max_{k \in \mathcal{K}} \frac{\gamma_k \mathcal{I}_k(\boldsymbol{p}, \boldsymbol{W})}{p_k} \leq \max_{1 \leq n \leq i} C\left(\boldsymbol{\gamma}^{(n)}, \boldsymbol{W}^{(n)}\right). \tag{128}$$

With (71), we can conclude that this is fulfilled with equality.

F. Proof of Theorem 9

For any $\boldsymbol{W} \in \mathcal{W}$ and $k \in \mathcal{K}$ we have

$$\gamma_k \mathcal{I}_k(\boldsymbol{p}, \boldsymbol{W}) \leq \gamma_k \max_{\boldsymbol{W} \in \mathcal{W}} \mathcal{I}_k(\boldsymbol{p}, \boldsymbol{W}) = \gamma_k \mathcal{I}_k(\boldsymbol{p})$$

thus

$$C(\boldsymbol{\gamma}, \boldsymbol{W}) \leq C(\boldsymbol{\gamma}), \quad \text{for all } \boldsymbol{W} \in \mathcal{W}.$$

The set $\mathcal{W}$ is compact by definition and the function $C(\boldsymbol{\gamma}, \boldsymbol{W})$ is continuous with respect to $\boldsymbol{W}$. So there exists a $\hat{\boldsymbol{W}} \in \mathcal{W}$ such that

$$C(\boldsymbol{\gamma}, \hat{\boldsymbol{W}}) = \max_{\boldsymbol{W} \in \mathcal{W}} C(\boldsymbol{\gamma}, \boldsymbol{W}).$$

Because $\hat{\boldsymbol{W}}$ is irreducible by assumption, we know from Theorem 4 that there is a $\hat{\boldsymbol{p}} > 0$ such that

$$\boldsymbol{\Gamma} \boldsymbol{\mathcal{I}}(\hat{\boldsymbol{p}}, \hat{\boldsymbol{W}}) = C(\boldsymbol{\gamma}, \hat{\boldsymbol{W}}) \hat{\boldsymbol{p}}. \tag{129}$$

The proof is by contradiction. Suppose $C(\boldsymbol{\gamma}, \hat{\boldsymbol{W}}) < C(\boldsymbol{\gamma})$. The vector $\hat{\boldsymbol{p}} > 0$ fulfills (129). Because of uniqueness (Lemma 8, part 2), $\hat{\boldsymbol{p}} > 0$ cannot be a fixed point of $\boldsymbol{\Gamma} \boldsymbol{\mathcal{I}}(\boldsymbol{p}, \hat{\boldsymbol{W}})/C(\boldsymbol{\gamma})$. There is an index k_0 such that

$$\mathcal{I}_{k_0}(\hat{\boldsymbol{p}}, \hat{\boldsymbol{W}}) < \max_{\boldsymbol{W} \in \mathcal{W}} \mathcal{I}_{k_0}(\hat{\boldsymbol{p}}, \boldsymbol{W}). \tag{130}$$

The maximization in (130) would lead to another stochastic matrix $\tilde{\boldsymbol{W}} \in \mathcal{W}$ with a balanced level

$$C(\boldsymbol{\gamma}, \tilde{\boldsymbol{W}}) > C(\boldsymbol{\gamma}, \hat{\boldsymbol{W}}) = \max_{\boldsymbol{W} \in \mathcal{W}} C(\boldsymbol{\gamma}, \boldsymbol{W}).$$

This is a contradiction, thus, $C(\boldsymbol{\gamma}, \hat{\boldsymbol{W}}) = C(\boldsymbol{\gamma})$ and $\hat{\boldsymbol{p}}$ fulfills $\boldsymbol{\Gamma} \boldsymbol{\mathcal{I}}(\hat{\boldsymbol{p}}) = C(\boldsymbol{\gamma}) \hat{\boldsymbol{p}}$.

G. Proof of Theorem 7

A simple way to prove this result is based on Theorem 9, which shows that there is a $\boldsymbol{p}^* > 0$ such that

$$c(\boldsymbol{\gamma}) = \sup_{\boldsymbol{p} > 0} \min_{k \in \mathcal{K}} \frac{\gamma_k \mathcal{I}_k(\boldsymbol{p})}{p_k} \geq \min_{k \in \mathcal{K}} \frac{\gamma_k \mathcal{I}_k(\boldsymbol{p}^*)}{p_k^*} = C(\boldsymbol{\gamma}).$$

With (77) we have $c(\boldsymbol{\gamma}) = C(\boldsymbol{\gamma})$.

H. Proof of Theorem 10

Consider arbitrary $k, l \in \mathcal{K}$ such that $[\boldsymbol{D}_{\mathcal{I}}]_{kl} = 1$. Then there exists a $\hat{\boldsymbol{p}} > 0$ and $\hat{\delta} > 0$ such that

$$\mathcal{I}_k(\hat{\boldsymbol{p}}) < \mathcal{I}_k(\hat{\boldsymbol{p}} + \hat{\delta} \boldsymbol{e}_l). \tag{131}$$

Now, consider an arbitrary δ such that $\delta > \hat{\delta}$. We have $\hat{p}_l < \hat{p}_l + \hat{\delta} < \hat{p}_l + \delta$, so there is a $\lambda = \lambda(\delta) \in (0, 1)$ such that

$$\log(\hat{p}_l + \hat{\delta}) = (1 - \lambda) \log \hat{p}_l + \lambda \log(\hat{p}_l + \delta). \tag{132}$$

That is, we have

$$\hat{p}_l + \hat{\delta} = (\hat{p}_l)^{1-\lambda} \cdot (\hat{p}_l + \delta)^{\lambda}. \tag{133}$$

The value λ for which (132) holds is given by

$$\frac{1}{\lambda} = \frac{\log\left(1 + \frac{\delta}{\hat{p}_l}\right)}{\log\left(1 + \frac{\hat{\delta}}{\hat{p}_l}\right)}. \tag{134}$$

Because $\mathcal{I}_k$ is log-convex, (9) is fulfilled. With (133) we have

$$\mathcal{I}_k(\hat{\boldsymbol{p}}+\hat{\delta}\boldsymbol{e}_l) \le (\mathcal{I}_k(\hat{\boldsymbol{p}}))^{1-\lambda}\cdot(\mathcal{I}_k(\hat{\boldsymbol{p}}+\delta\boldsymbol{e}_l))^{\lambda}.$$

This can be rewritten as

$$\frac{\mathcal{I}_k(\hat{\boldsymbol{p}}+\hat{\delta}\boldsymbol{e}_l)}{\mathcal{I}_k(\hat{\boldsymbol{p}})} \le \left(\frac{\mathcal{I}_k(\hat{\boldsymbol{p}}+\delta\boldsymbol{e}_l)}{\mathcal{I}_k(\hat{\boldsymbol{p}})}\right)^{\lambda}.$$

Thus, there is a constant $C_1=\mathcal{I}_k(\hat{\boldsymbol{p}}+\hat{\delta}\boldsymbol{e}_l)/\mathcal{I}_k(\hat{\boldsymbol{p}})>1$ such that

$$\mathcal{I}_k(\hat{\boldsymbol{p}}+\delta\boldsymbol{e}_l) \ge C_1^{1/\lambda}\cdot\mathcal{I}_k(\hat{\boldsymbol{p}}). \qquad (135)$$

Combining (134) and (135) we can conclude that

$$\lim_{\delta\to\infty}\mathcal{I}_k(\hat{\boldsymbol{p}}+\delta\boldsymbol{e}_l)=+\infty$$

which implies $[\boldsymbol{A}_{\mathcal{I}}]_{kl}=1$. The converse proof follows immediately from the definition.

I. Proof of Theorem 11

Assume that there is an irreducible $\boldsymbol{W}\in\mathcal{W}_{\mathcal{I}}$ such that (93) holds. We need to show that $\boldsymbol{A}_{\mathcal{I}}$ is irreducible. For all $k,l\in\mathcal{K}$ such that $w_{kl}>0$, we have

$$\lim_{\delta\to\infty}\mathcal{I}_k(\boldsymbol{p}+\delta\boldsymbol{e}_l)=+\infty, \quad \forall\boldsymbol{p}>0. \qquad (136)$$

Thus, every nonzero entry in $\boldsymbol{W}$ translates to a nonzero entry in $\boldsymbol{A}_{\mathcal{I}}$. Because $\boldsymbol{W}$ is irreducible by assumption, $\boldsymbol{A}_{\mathcal{I}}$ is irreducible as well.

Conversely, assume that $\boldsymbol{A}_{\mathcal{I}}$ is irreducible. For any $k\in\mathcal{K}$ we define an index set

$$\mathcal{A}_k=\{l\in\mathcal{K}:[\boldsymbol{A}_{\mathcal{I}}]_{kl}=1\}.$$

For all $l\in\mathcal{A}_k$ (136) is fulfilled. This is a consequence of definition (90) and Lemma 11. The matrix $\boldsymbol{A}_{\mathcal{I}}$ is irreducible by assumption. Thus, $\mathcal{A}_k$ is nonempty. The set $\mathcal{L}(\mathcal{I}_k)$ is also nonempty because the trivial case $\mathcal{I}_k(\boldsymbol{p})=0$, $\forall\boldsymbol{p}>0$, is ruled out by (136) and the assumption of irreducibility.

Next, consider an arbitrary index $k\in\mathcal{K}$. For some arbitrary $l\in\mathcal{A}_k$ we show by contradiction that there is a $\hat{\boldsymbol{w}}\in\mathcal{L}(\mathcal{I}_k)$ with $\hat{w}_{kl}>0$. Suppose that there is no such vector, then for all $\boldsymbol{p}>0$ and $\delta>0$, we would have

$$\begin{aligned}\mathcal{I}_k(\boldsymbol{p}+\delta\boldsymbol{e}_l)&=\max_{\boldsymbol{w}_k\in\mathcal{L}(\mathcal{I}_k)}\left(f_{\mathcal{I}_k}(\boldsymbol{w}_k)\cdot(p_l+\delta)^{w_{kl}}\cdot\prod_{r\neq l}(p_r)^{w_{kr}}\right)\\&=\max_{\boldsymbol{w}_k\in\mathcal{L}(\mathcal{I}_k)}\left(f_{\mathcal{I}_k}(\boldsymbol{w}_k)\prod_{r\neq l}(p_r)^{w_{kr}}\right)=M_1(\boldsymbol{p})\end{aligned}$$

where $M_1(\boldsymbol{p})>0$ is some constant independent of δ. Thus, $\lim_{\delta\to\infty}\mathcal{I}_k(\boldsymbol{p}+\delta\boldsymbol{e}_l)$ would be bounded, which contradicts the assumption $l\in\mathcal{A}_k$. It can be concluded that for all $l\in\mathcal{A}_k$ there is a $\hat{\boldsymbol{w}}_k^{(l)}\in\mathcal{L}(\mathcal{I}_k)$ such that $[\hat{\boldsymbol{w}}_k^{(l)}]_l>0$. From Lemma 5, we know that $\mathcal{L}(\mathcal{I}_k)$ is a convex set, so any convex combination

$$\tilde{\boldsymbol{w}}_k=(1-\lambda)\hat{\boldsymbol{w}}_k^{(l_1)}+\lambda\hat{\boldsymbol{w}}_k^{(l_2)},\ l_1,l_2\in\mathcal{A}_k, \quad 1<\lambda<1$$

is also contained in $\mathcal{L}(\mathcal{I}_k)$. This way, we can construct a $\tilde{\boldsymbol{w}}_k\in\mathcal{L}(\mathcal{I}_k)$ such that $\tilde{w}_{kl}>0$ for all $l\in\mathcal{A}_k$. This holds for any $k\in\mathcal{K}$, so there is a matrix $\tilde{\boldsymbol{W}}=[\tilde{\boldsymbol{w}}_1,\dots,\tilde{\boldsymbol{w}}_K]^T\in\mathcal{W}_{\mathcal{I}}$ having nonzero entries at the same positions as $\boldsymbol{A}_{\mathcal{I}}$. Because $\boldsymbol{A}_{\mathcal{I}}$ is irreducible by assumption, $\tilde{\boldsymbol{W}}$ is irreducible as well. Also

$$\begin{aligned}\mathcal{I}_k(\boldsymbol{p})&=\max_{\boldsymbol{w}_k\in\mathcal{L}(\mathcal{I}_k)}\left(f_{\mathcal{I}_k}(\boldsymbol{w}_k)\prod_{l\in\mathcal{K}}(p_l)^{w_{kl}}\right)\\&\ge f_{\mathcal{I}_k}(\tilde{\boldsymbol{w}}_k)\prod_{l\in\mathcal{K}}(p_l)^{\tilde{w}_{kl}}\end{aligned}$$

where $f_{\mathcal{I}_k}(\tilde{\boldsymbol{w}}_k)>0$ because $\tilde{\boldsymbol{W}}\in\mathcal{W}_{\mathcal{I}}$. Hence, (93) is fulfilled.

J. Proof of Theorem 12

Consider the set

$$\mathcal{S}(M,\boldsymbol{W})=\Bigg\{\boldsymbol{p}>0:\|\boldsymbol{p}\|_\infty=1,\ \gamma_k f_{\mathcal{I}_k}(\boldsymbol{w}_k)\prod_{l\in\mathcal{K}}(p_l)^{w_{kl}}\le M\cdot p_k,\forall k\Bigg\}. \qquad (137)$$

For the Proof of Theorem 12 we will need the following result.

Lemma 13: Let $\boldsymbol{W}\in\mathcal{W}_{\mathcal{I}}$ be a fixed irreducible stochastic matrix, and $M>0$ a fixed constant. If the set $\mathcal{S}:=\mathcal{S}(M,\boldsymbol{W})$ is nonempty, then there exists a constant $\underline{C}:=\underline{C}(M,\boldsymbol{W})>0$ such that

$$\min_{k\in\mathcal{K}}p_k\ge\underline{C}>0, \quad \text{for all } \boldsymbol{p}\in\mathcal{S}. \qquad (138)$$

Proof: Consider an arbitrary $\boldsymbol{p}\in\mathcal{S}$. Defining $C_k:=M/(\gamma_k f_{\mathcal{I}_k}(\boldsymbol{w}_k))$, we have

$$\prod_{l\in\mathcal{K}}(p_l)^{w_{kl}}\le C_k p_k, \quad k\in\mathcal{K}. \qquad (139)$$

For an arbitrary fixed $k\in\mathcal{K}$ we define a dependency set

$$L(k)=\{l\in\mathcal{K}:w_{kl}>0\} \qquad (140)$$

and bounds

$$\underline{p}(k)=\min_{l\in L(k)}p_l \qquad (141)$$

$$\bar{p}(k)=\max_{l\in L(k)}p_l. \qquad (142)$$

Consider an index $\bar{l}(k)\in L(k)$, for which $\bar{p}(k)=p_{\bar{l}(k)}$. We have

$$\begin{aligned}\prod_{l\in\mathcal{K}}(p_l)^{w_{kl}}&=\prod_{l\in L(k)}(p_l)^{w_{kl}}\\&\ge(\bar{p}(k))^{w_{k\bar{l}(k)}}\cdot(\underline{p}(k))^{\sum_{l\in L(k)\setminus\bar{l}(k)}w_{kl}}.\end{aligned}$$

Defining $\alpha_k=w_{k\bar{l}(k)}$ and exploiting $\sum_{l\in L(k)}w_{kl}=1$ and (139), we have

$$(\bar{p}(k))^{\alpha_k}\cdot(\underline{p}(k))^{1-\alpha_k}\le C_k\cdot p_k, \quad \forall\, k\in\mathcal{K}. \qquad (143)$$

Because $\boldsymbol{W}$ is irreducible by assumption, every user causes interference to at least one other user, which means that every index is contained in at least one dependency set. Thus

$$\underline{p}=\min_{k\in\mathcal{K}}\underline{p}(k)=\min_{k\in\mathcal{K}}p_k$$

$$\bar{p}=\max_{k\in\mathcal{K}}\bar{p}(k)=\max_{k\in\mathcal{K}}p_k.$$

Let k_1 be an index such that $p_{k_1} = \underline{p}$. Using $(\underline{p})^{1-\alpha_k} \leq (\underline{p}(k))^{1-\alpha_k}$, inequality (143) leads to

$$\bar{p}(k_1) \leq (C_{k_1})^{1/\alpha_{k_1}} \underline{p}. \tag{144}$$

We define the set

$$L_1 = \{k \in \mathcal{K} : p_k \leq \bar{p}(k_1)\}. \tag{145}$$

For all $k \in L_1$ we have

$$(\bar{p}(k))^{\alpha_k} (\underline{p}(k))^{1-\alpha_k} \leq C_k \cdot \bar{p}(k_1) \leq C_k \cdot (C_{k_1})^{1/\alpha_{k_1}} \underline{p} \tag{146}$$

where the first inequality follows from (143) and the second from (144). Again, using $(\underline{p})^{1-\alpha_k} \leq (\underline{p}(k))^{1-\alpha_k}$, inequality (146) leads to

$$\bar{p}(k) \leq (C_k)^{1/\alpha_k} \cdot (C_{k_1})^{1/(\alpha_k \alpha_{k_1})} \underline{p}, \quad \forall\, k \in L_1. \tag{147}$$

There exists a $k_2 \in L_1$ such that

$$\bar{p}(k_2) = \max_{k \in L_1} \bar{p}(k) \geq \bar{p}(k_1). \tag{148}$$

Here we have exploited $k_1 \in L_1$. Inequality (148) implies $L_1 \subseteq L_2$. With the index k_2 we define the set

$$L_2 = \{k \in \mathcal{K} : p_k \leq \bar{p}(k_2)\}. \tag{149}$$

Similar to the derivation of (146), we can use (143) and (147) to show that for all $k \in L_2$

$$(\bar{p}(k))^{\alpha_k} (\underline{p}(k))^{1-\alpha_k} \leq C_k \cdot (C_{k_2})^{1/\alpha_{k_2}} \cdot (C_{k_1})^{1/(\alpha_{k_1}\alpha_{k_2})} \cdot \underline{p}.$$

Using $(\underline{p})^{1-\alpha_k} \leq (\underline{p}(k))^{1-\alpha_k}$ we have for all $k \in L_2$

$$\bar{p}(k) \leq (C_k)^{1/\alpha_k} \cdot (C_{k_2})^{1/\alpha_{k_2}\alpha_k} \cdot (C_{k_1})^{1/(\alpha_{k_1}\alpha_{k_2}\alpha_k)} \cdot \underline{p}.$$

If L_2 is nonempty, then there is a $k_3 \in L_2$ such that

$$\bar{p}(k_3) = \max_{k \in L_2} \bar{p}(k) \geq \bar{p}(k_2). \tag{150}$$

The inequality in (150) follows from $L_1 \subseteq L_2$. With k_3 we define the set

$$L_3 = \{k \in \mathcal{K} : p_k \leq \bar{p}(k_3)\}. \tag{151}$$

Inequality (150) implies $L_2 \subseteq L_3$.

The above steps are repeated until there is an $N \in \mathbb{N}$ such that $L_N = \emptyset$. Then we have

$$L_1 \subseteq L_2 \subseteq L_3 \subseteq \cdots \subseteq L_{N-1} \tag{152}$$

and

$$\bar{p}(k_N) \leq (C_{k_N})^{1/\alpha_{k_N}} \cdot (C_{k_{N-1}})^{1/\alpha_{k_N}\alpha_{k_{N-1}}} \times \dots \times (C_{k_1})^{1/(\alpha_{k_1}\alpha_{k_2}\cdots\alpha_{k_N})} \cdot \underline{p}. \tag{153}$$

By assumption, the powers are upper-bounded by $\bar{p} = 1$ so we have $\bar{p}(k_N) \leq \bar{p}$. We now show by contradition that $\bar{p}(k_N) = \bar{p}$. Suppose that this is not true, i.e., $\bar{p}(k_N) < \bar{p}$, then the set L_{N-1} cannot contain all indices $\mathcal{K}$, because otherwise $\bar{p}(k_N) = \max_{k \in L_{N-1}} \bar{p}(k) = \bar{p}$. Thus, there is a nonempty set

$$G_1 = [1, \dots, K] \backslash L_{N-1}. \tag{154}$$

For any $\bar{k} \in G_1$ and any $k \in L_{N-1}$ we always have

$$p_{\bar{k}} > \bar{p}(k_N) \tag{155}$$

because otherwise $p_{\bar{k}} \in L_N$ which would contradict $L_N = \emptyset$. We now show by contradiction that inequality (155) implies $[\boldsymbol{W}]_{k\bar{k}} = 0$. Suppose that this is not true, then $\bar{k} \in L(k)$, thus $\bar{p}(k) = \max_{s \in L(k)} p_s \geq p_{\bar{k}}$. With (155) we would have

$$\bar{p}(k) > \bar{p}(k_N) = \max_{t \in L_{N-1}} \bar{p}(t) \geq \bar{p}(k).$$

This contradiction shows that $[\boldsymbol{W}]_{k\bar{k}} = 0$ for arbitrary $\bar{k} \in G_1$ and $k \in L_{N-1}$. That is, the directed graph of $\boldsymbol{W}$ has no paths between nodes from the nonintersecting sets G_1 and L_{N-1}. Thus, $\boldsymbol{W}$ would be reducible, which contradicts the assumption that $\boldsymbol{W}$ is irreducible. Hence, $\bar{p}(k_N) = \bar{p}$ holds.

Setting $\bar{p}(k_N) = \bar{p} = 1$ in (153) we obtain

$$\min_{k \in \mathcal{K}} p_k = \underline{p} \geq \underline{C} \tag{156}$$

with a constant $\underline{C} > 0$. □

The Proof of Lemma 13 characterizes the constant

$$\underline{C}(M, \boldsymbol{W}) = \inf_{\boldsymbol{p} \in \mathcal{S}(M, \boldsymbol{W})} (\min_{k \in \mathcal{K}} p_k).$$

Now, we will use this result to prove Theorem 12. To this end, consider an arbitrary $\epsilon > 0$. From (16) it can be observed that there always exists a vector $\boldsymbol{p}(\epsilon) > 0$, with $\max_k p_k(\epsilon) = 1$ (because $\boldsymbol{p}$ can be scaled arbitrarily) and

$$\gamma_k \mathcal{I}_k(\boldsymbol{p}(\epsilon)) \leq M_\epsilon \cdot p_k(\epsilon), \quad \forall\, k \in \mathcal{K} \tag{157}$$

where $M_\epsilon = (C(\boldsymbol{\gamma}) + \epsilon)$.

For arbitrary $\boldsymbol{W} \in \mathcal{W}_\mathcal{I}$ we define

$$C_\mathcal{I}(\boldsymbol{\gamma}, \boldsymbol{W}) = \inf_{\boldsymbol{p} > 0} \left(\max_{k \in \mathcal{K}} \frac{\gamma_k f_{\mathcal{I}_k}(\boldsymbol{w}_k) \cdot \prod_{l \in \mathcal{K}} (p_l)^{w_{kl}}}{p_k} \right).$$

We have

$$\begin{aligned}
&\max_{\boldsymbol{W} \in \mathcal{W}_\mathcal{I}} C_\mathcal{I}(\boldsymbol{\gamma}, \boldsymbol{W}) \\
&\quad = \max_{\boldsymbol{W} \in \mathcal{W}_\mathcal{I}} \inf_{\boldsymbol{p} > 0} \left(\max_{k \in \mathcal{K}} \frac{\gamma_k f_{\mathcal{I}_k}(\boldsymbol{w}_k) \cdot \prod_{l \in \mathcal{K}} (p_l)^{w_{kl}}}{p_k} \right) \\
&\quad \leq \inf_{\boldsymbol{p} > 0} \max_{\boldsymbol{W} \in \mathcal{W}_\mathcal{I}} \left(\max_{k \in \mathcal{K}} \frac{\gamma_k f_{\mathcal{I}_k}(\boldsymbol{w}_k) \cdot \prod_{l \in \mathcal{K}} (p_l)^{w_{kl}}}{p_k} \right) \\
&\quad = \inf_{\boldsymbol{p} > 0} \max_{k \in \mathcal{K}} \left(\max_{\boldsymbol{w}_k \in \mathcal{L}(\mathcal{I}_k)} \frac{\gamma_k f_{\mathcal{I}_k}(\boldsymbol{w}_k) \cdot \prod_{l \in \mathcal{K}} (p_l)^{w_{kl}}}{p_k} \right) \\
&\quad = \inf_{\boldsymbol{p} > 0} \max_{k \in \mathcal{K}} \left(\frac{\gamma_k \mathcal{I}_k(\boldsymbol{p})}{p_k} \right) = C(\boldsymbol{\gamma}).
\end{aligned}$$

Thus, $C_\mathcal{I}(\boldsymbol{\gamma}, \boldsymbol{W}) \leq C(\boldsymbol{\gamma})$ for all $\boldsymbol{W} \in \mathcal{W}_\mathcal{I}$. By assumption, there exists an irreducible $\hat{\boldsymbol{W}} \in \mathcal{W}_\mathcal{I}$. We have

$$M_\epsilon = (C(\boldsymbol{\gamma}) + \epsilon) \geq C_\mathcal{I}(\boldsymbol{\gamma}, \hat{\boldsymbol{W}}).$$

Consider the set (137). We have $\mathcal{S}(M_\epsilon, \hat{\boldsymbol{W}}) \neq \emptyset$. This follows from the irreducibility of $\hat{\boldsymbol{W}}$, which implies the existence of a $\tilde{\boldsymbol{p}} > 0$ such that $\gamma_k f_{\mathcal{I}_k}(\hat{\boldsymbol{w}}_k) \prod_l (\tilde{p}_l)^{\hat{w}_{kl}} = C_\mathcal{I}(\boldsymbol{\gamma}, \hat{\boldsymbol{W}}) \tilde{p}_l$ (see Theorem 4). Thus, the set $\mathcal{S}(C_\mathcal{I}(\boldsymbol{\gamma}, \hat{\boldsymbol{W}}), \hat{\boldsymbol{W}})$ is nonempty, and because $M_\epsilon \geq C_{\mathcal{I}}(\boldsymbol{\gamma}, \hat{\boldsymbol{W}})$, the set $\mathcal{S}(M_\epsilon, \hat{\boldsymbol{W}})$ is nonempty as well.

Lemma 13 implies the existence of a constant $\underline{C}(M_\epsilon, \hat{\boldsymbol{W}})$ such that

$$\min_{k \in \mathcal{K}} p_k \geq \underline{C}(M_\epsilon, \hat{\boldsymbol{W}}) > 0, \quad \forall\, \boldsymbol{p} \in \mathcal{S}(M_\epsilon, \hat{\boldsymbol{W}}). \tag{158}$$

The bound $\underline{C}(M_\epsilon, \hat{\boldsymbol{W}})$ is monotonically decreasing in M_ϵ because the set $\mathcal{S}(M_\epsilon, \hat{\boldsymbol{W}})$ is enlarged by increasing M_ϵ. Thus

$$0 < \underline{C}(M_1, \hat{\boldsymbol{W}}) \leq \underline{C}(M_\epsilon, \hat{\boldsymbol{W}}), \quad 0 < \epsilon \leq 1. \tag{159}$$

Because of (34) (representation theorem), we have $\mathcal{I}_k(\boldsymbol{p}(\epsilon)) \geq f_{\mathcal{I}_k}(\hat{\boldsymbol{w}}_k) \prod_l (p_l(\epsilon))^{\hat{w}_{kl}}$. With (157) we know that $\boldsymbol{p}(\epsilon) \in \mathcal{S}(M_\epsilon, \hat{\boldsymbol{W}})$. Combining (158) and (159), we have

$$\min_{k \in \mathcal{K}} p_k(\epsilon) \geq \underline{C}(M_1, \hat{\boldsymbol{W}}) > 0, \quad 0 < \epsilon \leq 1. \tag{160}$$

The family of vectors $\boldsymbol{p}(\epsilon)$ is bounded. There exists a zero sequence $\{\epsilon_n\}$ and a vector $\hat{\boldsymbol{p}}$ from the compact set $\{\boldsymbol{p} > 0 : \|\boldsymbol{p}\|_\infty \leq 1$ such that $\hat{\boldsymbol{p}} = \lim_{n\to\infty} \boldsymbol{p}(\epsilon_n)$. With (160) we have

$$\hat{\boldsymbol{p}} = \lim_{n\to\infty} \boldsymbol{p}(\epsilon_n) \geq \underline{C}(M_1, \hat{\boldsymbol{W}}) > 0.$$

It was shown in [10] that every interference function is continuous on $\mathbb{R}^K_{++}$, so

$$\gamma_k \mathcal{I}_k(\hat{\boldsymbol{p}}) = \lim_{n\to\infty} \gamma_k \mathcal{I}_k(\boldsymbol{p}(\epsilon_n)) \leq C(\boldsymbol{\gamma}) \hat{p}_k, \quad \forall\, k \in \mathcal{K} \tag{161}$$

where the inequality follows from (157). Defininig $\tilde{\mathcal{I}}_k(\boldsymbol{p}) = \frac{1}{C(\boldsymbol{\gamma})} \mathcal{I}_k(\boldsymbol{p})$, we have

$$\gamma_k \tilde{\mathcal{I}}_k(\hat{\boldsymbol{p}}) \leq \hat{p}_k, \quad \forall\, k \in \mathcal{K}. \tag{162}$$

Next, consider the set

$$E = \{\boldsymbol{p} \in \mathbb{R}^K_{++} : p_k \geq \gamma_k \tilde{\mathcal{I}}_k(\boldsymbol{p}), \ \forall\, k \in \mathcal{K}\}. \tag{163}$$

With (162) we know that E is nonempty. Consider an arbitrary $\boldsymbol{p} \in E$. We define the index set

$$G(\boldsymbol{p}) = \{k \in \mathcal{K} : p_k = \gamma_k \tilde{\mathcal{I}}_k(\boldsymbol{p})\} \tag{164}$$

and its complement

$$U(\boldsymbol{p}) = \mathcal{K} \backslash G(\boldsymbol{p}). \tag{165}$$

The set $G(\boldsymbol{p})$ is nonempty. In order to show this, suppose that there is a $\boldsymbol{p}' \in E$ with $G(\boldsymbol{p}') = \emptyset$, i.e., $p'_k > \gamma_k \tilde{\mathcal{I}}_k(\boldsymbol{p}')$ for all $k \in \mathcal{K}$. This would imply the contradiction

$$1 = C(\boldsymbol{\gamma}, \tilde{\mathcal{I}}) = \inf_{\boldsymbol{p} > 0} \left(\max_{k \in \mathcal{K}} \frac{\gamma_k \tilde{\mathcal{I}}_k(\boldsymbol{p})}{p_k} \right) \leq \frac{\gamma_k \tilde{\mathcal{I}}_k(\boldsymbol{p}')}{p'_k} < 1 \tag{166}$$

where $C(\boldsymbol{\gamma}, \tilde{\boldsymbol{\mathcal{I}}})$ is the min-max optimum for the normalized interference functions $\tilde{\mathcal{I}}_1, \ldots, \tilde{\mathcal{I}}_K$.

From (162) we know that $\hat{\boldsymbol{p}} \in E$. Let $\hat{\boldsymbol{p}}^{(1)}$ be the vector with components $\hat{p}^{(1)}_k = \gamma_k \tilde{\mathcal{I}}_k(\hat{\boldsymbol{p}}) \leq \hat{p}_k$, $k \in \mathcal{K}$. If $\hat{\boldsymbol{p}}^{(1)} = \hat{\boldsymbol{p}}$, then $\hat{\boldsymbol{p}}$ is a fixed point fulfilling (94). In this case, the proof is completed. Otherwise, axiom A3 yields $\gamma_k \tilde{\mathcal{I}}_k(\hat{\boldsymbol{p}}^{(1)}) \leq \gamma_k \tilde{\mathcal{I}}_k(\hat{\boldsymbol{p}}) = \hat{p}^{(1)}_k$, thus $\hat{\boldsymbol{p}}^{(1)} \in E$. That is, the set E has at least two elements. In what follows, we will show that there always exists a $\boldsymbol{p} \in E$ such that $G(\boldsymbol{p}) = \mathcal{K}$.

Consider two arbitrary vectors $\hat{\boldsymbol{p}}, \check{\boldsymbol{p}} \in E$ and $\boldsymbol{p}(\lambda) = \hat{\boldsymbol{p}}^{1-\lambda} \cdot \check{\boldsymbol{p}}^{\lambda}$ (component-wise), with $0 < \lambda < 1$. For any $k \in \mathcal{K}$ we have

$$1 \geq \gamma_k^{1-\lambda} \cdot \gamma_k^{\lambda} \cdot \frac{(\tilde{\mathcal{I}}_k(\hat{\boldsymbol{p}}))^{1-\lambda}}{(\hat{p}_k)^{1-\lambda}} \cdot \frac{(\tilde{\mathcal{I}}_k(\check{\boldsymbol{p}}))^{\lambda}}{(\check{p}_k)^{\lambda}} \geq \gamma_k \frac{\tilde{\mathcal{I}}_k(\boldsymbol{p}(\lambda))}{p_k(\lambda)}. \tag{167}$$

The first inequality follows from $C(\boldsymbol{\gamma}, \tilde{\mathcal{I}}) = 1$ and $\hat{\boldsymbol{p}}, \check{\boldsymbol{p}} \in E$, similar to (166). The second inequality follows because $\tilde{\mathcal{I}}_k$ is log-convex by assumption. From (167) we know that $\boldsymbol{p}(\lambda) \in E$. For any $k \in U(\hat{\boldsymbol{p}}) \cup U(\check{\boldsymbol{p}})$, at least one of the factors in (167) is strictly less than one thus, $p_k(\lambda) > \gamma_k \tilde{\mathcal{I}}_k(\boldsymbol{p}(\lambda))$, which implies $k \in U(\boldsymbol{p}(\lambda))$. Therefore

$$U(\hat{\boldsymbol{p}}) \cup U(\check{\boldsymbol{p}}) \subseteq U(\boldsymbol{p}(\lambda)). \tag{168}$$

Note that we have assumed $U(\boldsymbol{p}) \neq \emptyset$ for all vectors $\boldsymbol{p}$ under consideration. Because $U(\boldsymbol{p}) = \emptyset$ would mean that $\boldsymbol{p}$ is a fixed point, in which case the proof would be completed.

Next, let $\bar{U}$ denote the set of all $k \in \mathcal{K}$ such that there is a vector $\boldsymbol{p}^{(k)} \in E$ with $k \in U(\boldsymbol{p}^{(k)})$, that is, $p^{(k)}_k > \gamma_k \tilde{\mathcal{I}}_k(\boldsymbol{p}^{(k)})$. With (168) we can construct a vector $\bar{\boldsymbol{p}} \in E$ such that $\bar{U} = U(\bar{\boldsymbol{p}})$. Thus, for all vectors $\boldsymbol{p} \in E$ we have $U(\boldsymbol{p}) \subseteq U(\bar{\boldsymbol{p}})$.

Next, consider the fixed-point iteration

$$\bar{p}^{(n+1)}_k = \gamma_k \tilde{\mathcal{I}}_k\left([\bar{\boldsymbol{p}}]^{(n)}\right), \quad \text{with } \bar{p}^{(0)}_k = \bar{p}_k, \ \forall\, k \in \mathcal{K} \tag{169}$$

where the superscript n, with $n \geq 0$, denotes the nth iteration step. Because $\bar{\boldsymbol{p}} \in E$ we have $\bar{p}^{(1)}_k = \gamma_k \tilde{\mathcal{I}}_k(\bar{\boldsymbol{p}}^{(0)}) \leq \bar{p}^{(0)}_k$ for all $k \in \mathcal{K}$. Exploiting A3, this leads to

$$\bar{p}^{(2)}_k = \gamma_k \tilde{\mathcal{I}}_k\left(\bar{\boldsymbol{p}}^{(1)}\right) \leq \gamma_k \tilde{\mathcal{I}}_k\left(\bar{\boldsymbol{p}}^{(0)}\right) = \bar{p}^{(1)}_k, \quad \forall\, k \in \mathcal{K}.$$

Thus, $\bar{\boldsymbol{p}}^{(1)} \in E$. We also have $U(\bar{\boldsymbol{p}}^{(1)}) \subseteq U(\bar{\boldsymbol{p}})$. This follows by contradiction: suppose that there exists a $k \in U(\bar{\boldsymbol{p}}^{(1)})$ and k is not contained in $U(\bar{\boldsymbol{p}}) = \bar{U}$. This would imply $\bar{p}^{(1)}_k > \gamma_k \tilde{\mathcal{I}}_k(\bar{\boldsymbol{p}}^{(1)})$, thus leading to the contradiction $k \in \bar{U}$. For the complementary sets, this implies

$$G\left(\bar{\boldsymbol{p}}^{(1)}\right) \supseteq G\left(\bar{\boldsymbol{p}}^{(0)}\right) = G(\bar{\boldsymbol{p}}).$$

For any $k \in G(\bar{\boldsymbol{p}})$ we have

$$\bar{p}^{(1)}_k = \gamma_k \tilde{\mathcal{I}}_k\left(\bar{\boldsymbol{p}}^{(0)}\right) = \bar{p}^{(0)}_k.$$

Thus, $\bar{p}^{(1)}_k = \bar{p}^{(0)}_k$ for all $k \in G(\bar{\boldsymbol{p}})$.

In a similar way, we show $\bar{\boldsymbol{p}}^{(n)} \in E$, which implies $G(\bar{\boldsymbol{p}}^{(n)}) \supseteq G(\bar{\boldsymbol{p}}^{(0)})$. Thus, any $k \in G(\bar{\boldsymbol{p}})$ is contained in $G(\bar{\boldsymbol{p}}^{(n)})$. This implies $k \in G(\bar{\boldsymbol{p}}^{(n-1)})$. By induction, we have for all $n \in \mathbb{N}$

$$\bar{p}^{(n)}_k = \gamma_k \tilde{\mathcal{I}}_k\left(\bar{\boldsymbol{p}}^{(n-1)}\right) = \bar{p}^{(n-1)}_k = \bar{p}^{(n-2)}_k = \cdots = \bar{p}^{(0)}_k.$$

Thus, for any $k \in G(\bar{\boldsymbol{p}}^{(0)})$ we have

$$\bar{p}^{(n)}_k = \bar{p}^{(0)}_k, \quad \text{for all } n \in \mathbb{N}. \tag{170}$$

The fixed-point iteration (169) converges to a limit

$$\bar{\boldsymbol{p}}^* = \lim_{n\to\infty} \bar{\boldsymbol{p}}^{(n)}.$$

A finite limit exists because the sequence $\bar{\boldsymbol{p}}^{(n)}$ is monotonic decreasing and $\bar{\boldsymbol{p}}^{(n)} > 0$ for all $n \in \mathbb{N}$. Independent of the choice of n, we have

$$\left\|\bar{\boldsymbol{p}}^{(n)}\right\|_\infty \geq \max_{k \in G(\bar{\boldsymbol{p}}^{(n)})} \bar{p}_k^{(n)} \geq \max_{k \in G(\bar{\boldsymbol{p}}^{(0)})} \bar{p}_k^{(n)} = \max_{k \in G(\bar{\boldsymbol{p}}^{(0)})} \bar{p}_k^{(0)} = C_1 > 0$$

where C_1 is constant. The sequence $\|\bar{\boldsymbol{p}}^{(n)}\|_\infty$ converges as well, so there is another constant C_2 such that

$$C_2 = \lim_{n\to\infty} \left\|\bar{\boldsymbol{p}}^{(n)}\right\|_\infty \geq C_1 > 0. \quad (171)$$

Because of the monotonic convergence of $\bar{\boldsymbol{p}}^{(n)}$ we have

$$\left\|\bar{\boldsymbol{p}}^{(n)}\right\|_\infty \geq C_2 > C_1 > 0, \quad \text{for all } n \in \mathbb{N}.$$

The ratio of two convergent sequences is convergent if the denominator sequence has a nonzero limit, so

$$\hat{\boldsymbol{p}}^{(n)} = \frac{1}{\|\bar{\boldsymbol{p}}^{(n)}\|_\infty} \bar{\boldsymbol{p}}^{(n)}, \quad n \in \mathbb{N}$$

is convergent as well. We have $\|\hat{\boldsymbol{p}}^{(n)}\|_\infty = 1$. Also, we have shown $\hat{p}_k^{(n)} \geq \gamma_k \tilde{\mathcal{I}}_k(\boldsymbol{p}^{(n)})$ for all n and k. We have

$$\hat{p}_k^{(n)} \geq \frac{\gamma_k}{C(\boldsymbol{\gamma})} \mathcal{I}_k\left(\boldsymbol{p}^{(n)}\right) \geq \frac{\gamma_k}{C(\boldsymbol{\gamma})} f_{\mathcal{I}_k}(\hat{\boldsymbol{w}}_k) \prod_{l\in\mathcal{K}} (p_l)^{\hat{w}_{kl}}.$$

Thus, there is a constant $M_1 > 0$ such that $\hat{\boldsymbol{p}}^{(n)} \in \mathcal{S}(M_1, \hat{\boldsymbol{W}})$, as defined by (137). With Lemma 13 we know that all $\hat{\boldsymbol{p}}^{(n)}$ fulfill

$$\hat{p}_k^{(n)} \geq \underline{C}(M_1, \hat{\boldsymbol{W}}) > 0, \quad \text{for all } n \in \mathbb{N} \text{ and } k \in \mathcal{K}. \quad (172)$$

Next, consider the limit

$$\boldsymbol{p}^* = \lim_{n\to\infty} \hat{\boldsymbol{p}}^{(n)}.$$

Because of (172) we have $\boldsymbol{p}^* > 0$. For all $k \in \mathcal{K}$ we have

$$\gamma_k \tilde{\mathcal{I}}_k(\hat{\boldsymbol{p}}^{(n)}) = \frac{1}{\|\bar{\boldsymbol{p}}^{(n)}\|_\infty} \cdot \gamma_k \tilde{\mathcal{I}}_k(\bar{\boldsymbol{p}}^{(n)}) = \frac{1}{\|\bar{\boldsymbol{p}}^{(n)}\|_\infty} \cdot \bar{p}_k^{(n+1)} = \frac{\|\bar{\boldsymbol{p}}^{(n+1)}\|_\infty}{\|\bar{\boldsymbol{p}}^{(n)}\|_\infty} \cdot \hat{p}_k^{(n+1)}.$$

Because of $\lim_{n\to\infty} \|\bar{\boldsymbol{p}}^{(n+1)}\|_\infty / \|\bar{\boldsymbol{p}}^{(n)}\|_\infty = 1$, we have

$$\gamma_k \tilde{\mathcal{I}}_k(\boldsymbol{p}^*) = \lim_{n\to\infty} \gamma_k \tilde{\mathcal{I}}_k\left(\hat{\boldsymbol{p}}^{(n)}\right) = \lim_{n\to\infty} \hat{p}_k^{(n+1)} = p_k^*, \quad \forall\, k \in \mathcal{K}.$$

That is, $\boldsymbol{p}^* > 0$ fulfills $p_k^* C(\boldsymbol{\gamma}) = \gamma_k \mathcal{I}_k(\boldsymbol{p}^*)$ for all $k \in \mathcal{K}$.

K. Proof of Theorem 13

The proof is by contradiction. Suppose that for any $\boldsymbol{\gamma} > 0$ there exists a $\hat{\boldsymbol{p}} > 0$ such that

$$C(\boldsymbol{\gamma})\hat{p}_k = \gamma_k \mathcal{I}_k(\hat{\boldsymbol{p}}), \quad \text{for all } k \in \mathcal{K} \quad (173)$$

where $C(\boldsymbol{\gamma})$ is defined as by (16).

In order to simplify the discussion, we assume that $\boldsymbol{A}_{\mathcal{I}}$ has a single isolated block $\boldsymbol{A}_{\mathcal{I}}^{(1)}$ on its main diagonal. The proof for several isolated blocks is similar. The block $\boldsymbol{A}_{\mathcal{I}}^{(1)}$ is associated with users $1, \ldots, l_1$. The superscript $(\,\cdot\,)^{(1)}$ will be used in the following to indicate that the respective quantity belongs to the first block. The interference functions $\mathcal{I}_1, \ldots, \mathcal{I}_{l_1}$ and powers $p_1, \ldots p_{l_1}$ are collected in vectors $\boldsymbol{\mathcal{I}}^{(1)}(\boldsymbol{p})$ and $\boldsymbol{p}^{(1)}$, respectively.

For arbitrary $\boldsymbol{\gamma} > 0$ we define

$$\underline{C}(\boldsymbol{\gamma}) = \inf_{\boldsymbol{p}^{(1)}>0} \left(\max_{1\leq k\leq l_1} \frac{\gamma_k \mathcal{I}_k^{(1)}(\boldsymbol{p}^{(1)})}{p_k^{(1)}} \right) = \inf_{\boldsymbol{p}>0} \left(\max_{1\leq k\leq l_1} \frac{\gamma_k \mathcal{I}_k(\boldsymbol{p})}{p_k} \right) \leq C(\boldsymbol{\gamma}). \quad (174)$$

The last inequality holds because the maximum is restricted to the indices $k \leq l_1$. Also, $\mathcal{I}_k^{(1)}(\boldsymbol{p}^{(1)}) = \mathcal{I}_k(\boldsymbol{p})$ because k belongs to an isolated block.

We will now show that $\underline{C}(\boldsymbol{\gamma}) = C(\boldsymbol{\gamma})$. To this end, suppose that $\underline{C}(\boldsymbol{\gamma}) < C(\boldsymbol{\gamma})$. Because $\boldsymbol{A}_{\mathcal{I}}^{(1)}$ is irreducible, Corollary 5 implies the existence of a $\tilde{\boldsymbol{p}}^{(1)} > 0$ such that

$$\underline{C}(\boldsymbol{\gamma})\tilde{p}_k^{(1)} = \gamma_k \mathcal{I}_k^{(1)}\left(\tilde{\boldsymbol{p}}^{(1)}\right), \quad 1 \leq k \leq l_1. \quad (175)$$

This is compared with (173). We focus on the indices $k \leq l_1$. These users belong to the isolated block, so $\hat{\boldsymbol{p}}$ can be replaced by the vector $\hat{\boldsymbol{p}}^{(1)}$, which is the subvector of $\hat{\boldsymbol{p}}$ consisting of the first l_1 components. That is,

$$C(\boldsymbol{\gamma})\hat{p}_k^{(1)} = \gamma_k \mathcal{I}_k^{(1)}\left(\hat{\boldsymbol{p}}^{(1)}\right), \quad 1 \leq k \leq l_1. \quad (176)$$

Comparing (175) and (176), and using Lemma 8 (part 2), it can be concluded that $C(\boldsymbol{\gamma}) = \underline{C}(\boldsymbol{\gamma})$. The same can be shown for any isolated block.

For arbitrary $\boldsymbol{\gamma} > 0$, we define SIR targets

$$\gamma_k(\lambda) = \begin{cases} \lambda \cdot \gamma_k, & k \leq l_1 \\ \gamma_k, & k > l_1 \end{cases} \quad \lambda > 0 \quad (177)$$

which are collected in a vector $\boldsymbol{\gamma}(\lambda) = [\gamma_1(\lambda), \ldots, \gamma_K(\lambda)]^T$. The l_1-dimensional vector $\boldsymbol{\gamma}^{(1)}(\lambda) > 0$ contains the targets associated with the users of the first block $\boldsymbol{A}_{\mathcal{I}}^{(1)}$.

From (173) we know that for any $\boldsymbol{\gamma}(\lambda) > 0$ there is a $\boldsymbol{p}(\lambda) > 0$ such that

$$C(\boldsymbol{\gamma}(\lambda))p_k(\lambda) = \gamma_k(\lambda)\mathcal{I}_k(\boldsymbol{p}(\lambda)), \quad \text{for all } k \in \mathcal{K}. \quad (178)$$

Introducing a subvector $\boldsymbol{p}^{(1)}(\lambda)$, defined by

$$p_k^{(1)}(\lambda) = p_k(\lambda), \quad 1 \leq k \leq l_1,$$

the first l_1 components of (178) can be written as

$$C(\boldsymbol{\gamma}(\lambda)) \cdot p_k^{(1)}(\lambda) = \lambda \cdot \gamma_k \cdot \mathcal{I}_k^{(1)}\left(\boldsymbol{p}^{(1)}(\lambda)\right), \quad 1 \leq k \leq l_1.$$

For arbitrary $\lambda > 0$, we have

$$C(\boldsymbol{\gamma}(\lambda)) = \underline{C}(\boldsymbol{\gamma}(\lambda)) = \inf_{\boldsymbol{p}^{(1)}>0} \max_{1\leq k\leq l_1} \frac{\gamma_k(\lambda) \cdot \mathcal{I}_k^{(1)}\left(\boldsymbol{p}^{(1)}\right)}{p_k^{(1)}} = \lambda \cdot \inf_{\boldsymbol{p}^{(1)}>0} \max_{1\leq k\leq l_1} \frac{\gamma_k \cdot \mathcal{I}_k^{(1)}\left(\boldsymbol{p}^{(1)}\right)}{p_k^{(1)}} = \lambda \cdot \underline{C}(\boldsymbol{\gamma}) = \lambda \cdot C(\boldsymbol{\gamma}). \quad (179)$$

By assumption (97), we have $\underline{C}_1(\boldsymbol{\gamma}) > 0$, so

$$C(\boldsymbol{\gamma}(\lambda)) = \inf_{\boldsymbol{p}>0} \max_{k\in\mathcal{K}} \frac{\gamma_k(\lambda) \cdot \mathcal{I}_k(\boldsymbol{p})}{p_k} \geq \inf_{\boldsymbol{p}>0} \max_{k>l_1} \frac{\gamma_k \cdot \mathcal{I}_k(\boldsymbol{p})}{p_k} = \underline{C}_1(\boldsymbol{\gamma}) > 0. \quad (180)$$

Here we have exploited that $\gamma_k(\lambda) = \gamma_k$ for $k > l_1$. Combining (179) and (180) we obtain

$$\lambda \cdot C(\boldsymbol{\gamma}) \geq \underline{C}_1(\boldsymbol{\gamma}) > 0.$$

This inequality holds for all $\lambda > 0$. By letting $\lambda \to 0$, we obtain a contradiction, thus concluding the proof.

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