The Structure of General Interference Functions and Applications

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***Abstract*—This paper provides a theoretical framework for the analysis of interference-coupled multiuser systems. The fundamental behavior of such a system is modeled by interference functions, defined by axioms "nonnegativity, "scale-invariance," and "monotonicity." It is shown that every interference function has an interpretation as the optimum of a min-max problem, where the optimization is over a closed comprehensive positive coefficient set. This provides new insight into the structure of general interference functions and its elementary building blocks. Conversely, it is shown that every closed comprehensive positive set can be expressed as a level set of an interference function. This shows a close connection between the analysis of interference functions and multiuser performance regions, which are typically closed comprehensive. The generality of this framework allows for a wide range of potential applications. As an example, we analyze the problem of interference balancing.**

***Index Terms*—Feasible set, interference functions, multiuser interference, power control, wireless communications.**

I. Introduction

THE analysis of wireless multiuser systems is complicated by interference between communication links. The achievable performance of one link can depend on the communication strategies of other links. Due to these additional degrees of freedom, established principles and results for point-to-point links are not always transferable to multiuser systems.

A useful concept for analyzing interference-coupled systems is the set of jointly achievable link performances, often referred to as the *achievable region* or *feasible set*. Here, the term "performance" is used in a general sense, to describe different utility or cost measures. An example is the capacity region of the Gaussian multiple-input multiple-output (MIMO) multiple-access channel (MAC), and the dual broadcast channel (BC) [1]–[4]. Another example, is the region of signal-to-interference ratios (SIR), which is often studied in a power control and beamforming context (see, e.g., [5]–[12] and the references therein). There are many other possible definitions of feasible regions, depending on the chosen performance measure and possible systems constraints.

A thorough understanding of the underlying interference tradeoffs and the resulting feasible region often provides guidelines for the development of multiuser algorithms. For example, the analysis of the aforementioned MIMO broadcast region was accompanied by a search for optimum transmission strategies. A similar development could be observed in the power control and beamforming literature, where efficient algorithms for joint beamforming and resource allocation were derived.

Algorithms that are derived for a particular system layout have the advantage of being efficient and well-adapted to the system-specific structure. However, they are also quite specific in scope, and it is often difficult to transfer the results to other scenarios. For example, the results [1]–[4] hold for a Gaussian MIMO broadcast channel with dirty paper coding, but not necessarily for a other MIMO systems with practical system constraints, like imperfect channel knowledge or linear precoding at the transmitter.

It therefore makes sense to also analyze multiuser performance tradeoffs in a more abstract setting, by focusing on some core properties. One such property is *comprehensiveness*, which can be interpreted as "free disposability of utility" (cf. Definition 3 in Section II-B). Most resource allocation strategies implicitly or explicitly assume that the underlying feasible region is comprehensive. Many examples exist in the context of cooperative game theory [13]–[15].

In this paper, we analyze comprehensive feasible sets from the perspective of a wireless communication system. A connection is established between comprehensive feasible sets and an axiomatic framework of interference functions [16]. This framework is closely connected with the concept of general interference functions, which was introduced in [7] and extended in [16]–[18].

In Section II, we analyze the elementary structure of interference functions. It is shown that every interference function can be expressed as a min-max optimum, where the optimization variable is from a comprehensive set.

In Section III, we study the reverse approach: Certain comprehensive sets can be expressed as a sublevel set of an interference function. The results show a one-to-one correspondence between interference functions and certain comprehensive feasible sets.

In Section V, we show how the results can be applied to the problem of interference balancing.

Some notational conventions are as follows.

- $\mathcal{K} = \{1, 2, \ldots, K\}$ is the set of users (communication links).

Manuscript received December 27, 2007; revised January 7, 2008. Current version published October 22, 2008. This work was supported in part by the STREP project IST-026905 (MASCOT) within the sixth framework programme of the European Commission.

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Communicated by A. J. Goldsmith, Associate Editor for Communications.

Digital Object Identifier 10.1109/TIT.2008.929962

- The sets of nonnegative reals and positive reals are denoted by $\mathbb{R}_+$ and $\mathbb{R}_{++}$, respectively.
- Matrices and vectors are denoted by bold capital letters and bold lowercase letters, respectively. Let $\boldsymbol{y}$ be a vector, then $y_l = [\boldsymbol{y}]_l$ is the lth component.
- A vector inequality $\boldsymbol{x} > \boldsymbol{y}$ means $x_k > y_k$, for all k. The same definition applies to $\boldsymbol{x} \geq \boldsymbol{y}$ and the reverse directions. Also, $\boldsymbol{y} > 0$ means component-wise greater zero.

A. Axiomatic Framework of Interference Functions

The term "interference function" was introduced by Yates [7] in order to model interference in a power-controlled multiuser wireless system. He introduced a framework of axioms (positivity, scalability, monotonicity) in order to model how interference depends on the transmission powers.

Analyzing the basic building blocks of a theoretical model often provides valuable new insight into its underlying structure. Axiomatic characterizations also have a long-standing tradition in information theory. A famous example is the axiomatic characterization of the Shannon entropy by Khinchin [19] and Faddeev [20] (see, e.g., [21]).

In this paper, we build on the axiomatic framework [16], which can be regarded as a generalization of the theory of standard interference functions [7].

Definition 1: We say that $\mathcal{I} : \mathbb{R}_+^K \mapsto \mathbb{R}_+$ is an *interference function* if the following axioms are fulfilled:

A1 (nonnegativity) $\mathcal{I}(\boldsymbol{p}) \geq 0$

A2 (scale invariance) $\mathcal{I}(\alpha\boldsymbol{p}) = \alpha\mathcal{I}(\boldsymbol{p}), \quad \forall\, \alpha \in \mathbb{R}_+$

A3 (monotonicity) $\mathcal{I}(\boldsymbol{p}) \geq \mathcal{I}(\boldsymbol{p}'), \quad \text{if } \boldsymbol{p} \geq \boldsymbol{p}'$.

These properties are quite intuitive when we think of $of \boldsymbol{p} = [p_1, \ldots, p_K]^T$ as a vector of transmission powers, and $\mathcal{I}(\boldsymbol{p})$ as the resulting interference. However, other interpretations are possible, as will be seen later.

Note, that A1 (nonnegativity) is a direct consequence of A2 and A3. In order to rule out the trivial case $\mathcal{I}(\boldsymbol{p}) = 0$, we make an additional assumption

$$\text{There exists a } \boldsymbol{p}' > 0 \text{ such that } \mathcal{I}(\boldsymbol{p}') > 0. \tag{1}$$

It was shown in [16] that (1) implies $\mathcal{I}(\boldsymbol{p}) > 0$ for *all* $\boldsymbol{p} \in \mathbb{R}_{++}^K$. This is needed in some parts of the paper, e.g., to ensure that the SIR $p_k/\mathcal{I}_k(\boldsymbol{p})$ is well-defined. This additional requirement is not much of a restriction, since it is natural for most practical interference scenarios. Examples will be given in Section I-B.

The model A1–A3 differs slightly from the standard interference function introduced in [7], where *scalability* was required instead of *scale invariance*. Scalability is motivated by constant noise power adding to the interference. This property was required in [7], because of the specific power control problem under investigation.

In this paper, however, we are interested in some more general aspects of interference coupling. Namely, we show how interference functions can be used for the analysis of feasible regions. In this context, the model [7] is generally not appropriate. This will become clear in Section II.

B. Examples of Interference Functions

Now, we discuss some examples of interference functions which fulfill the axioms A1–A3 in Definition 1.

a) Linear Interference Function: A classical model from power control theory (see, e.g., [5], [6], [8]) is

$$\mathcal{I}_k(\boldsymbol{p}) = \boldsymbol{p}^T\boldsymbol{v}_k, \quad k \in \mathcal{K} \tag{2}$$

where $\boldsymbol{v}_k \in \mathbb{R}_+^K$ contains the interference coupling coefficients of the kth user. It can be observed that the function (2) fulfills A1–A3.

b) Linear Interference Function With Noise: The model (2) can be extended by a constant noise power σ^2. To this end, we introduce the extended power vector $\underline{\boldsymbol{p}} = [p_1, \ldots, p_K, \sigma^2]^T$. The interference-plus-noise power is

$$\mathcal{I}_k(\underline{\boldsymbol{p}}) = \underline{\boldsymbol{p}}^T \cdot \begin{bmatrix} \boldsymbol{v}_k \\ 1 \end{bmatrix} = \boldsymbol{p}^T\boldsymbol{v}_k + \sigma^2. \tag{3}$$

If we only consider the dependency on the powers $\boldsymbol{p}$, with constant σ^2, then we obtain a standard interference function in the sense of [7]. Whereas, if we define $\mathcal{I}$ as a function of the extended power vector $\underline{\boldsymbol{p}}$, as in (3), then we obtain an interference function in the sense of Definition 1.

A more detailed discussion on the aspects of noise can be found in [16], [22]. For the results of this paper, it is sufficient to know that the framework A1–A3 is general enough to incorporate noise.

c) Spectral Radius: Consider again the linear interference functions (2). The coupling coefficients are collected in a matrix

$$\boldsymbol{V} = [\boldsymbol{v}_1, \ldots, \boldsymbol{v}_K] \tag{4}$$

which is nonnegative and irreducible by assumption.

The SIR of user k is $\gamma_k(\boldsymbol{p}) = p_k/\mathcal{I}_k(\boldsymbol{p})$. The SIR feasible region is (see, e.g., [6], [8], [16])

$$\mathcal{S} = \{\boldsymbol{\gamma} > 0 : \rho_V(\boldsymbol{\gamma}) \leq 1\} \tag{5}$$

where $\rho_V(\boldsymbol{\gamma})$ is the spectral radius (here: maximum eigenvalue) of the matrix $(\text{diag}\{\boldsymbol{\gamma}\} \cdot \boldsymbol{V})$. The function $\rho_V(\boldsymbol{\gamma})$ is an indicator for the feasibility of an SIR vector $\boldsymbol{\gamma} = [\gamma_1, \ldots, \gamma_K]^T$. It fulfills the axioms A1–A3, so it is an *interference function*.

Hence, the SIR region (5) is a sublevel set of an interference function. The structure of the region $\mathcal{S}$ is intimately connected with the properties of $\rho_V(\boldsymbol{\gamma})$. This was already exploited in [23]–[25], where it was shown that $\mathcal{S}$ is convex on a logarithmic scale.

This is an example where the framework [7] is not appropriate. The function $\rho_V(\boldsymbol{\gamma})$ is *scale invariant* but not *scalable* as required in [7].

d) Min-Max Balancing: The previous example can be generalized to arbitrary interference functions characterized by A1–A3. In this case, the SIR region is

$$\mathcal{S} = \{\boldsymbol{\gamma} > 0 : C(\boldsymbol{\gamma}) \leq 1\} \tag{6}$$

where

$$C(\boldsymbol{\gamma}) = \inf_{\boldsymbol{p}>0} \left(\max_{k\in\mathcal{K}} \frac{\gamma_k \cdot \mathcal{I}_k(\boldsymbol{p})}{p_k} \right) \tag{7}$$

is an indicator function for the feasibility of SIR values $\boldsymbol{\gamma}$ (see, e.g., [16] for details). The function $C(\boldsymbol{\gamma})$ fulfills the axioms A1–A3, so it is an *interference function*.

e) Robustness: Linear interference functions (2) can be generalized by introducing parameter-dependent coupling coefficients $\boldsymbol{v}_k(c_k)$. Assume that the parameter c_k stands for an *uncertainty* chosen from a compact *uncertainty region* $\mathcal{C}_k$. A typical source of uncertainty are channel estimation errors. Then, the worst case interference is given by

$$\mathcal{I}_k(\boldsymbol{p}) = \max_{c_k \in \mathcal{C}_k} \boldsymbol{p}^T \boldsymbol{v}_k(c_k), \quad k \in \mathcal{K}. \tag{8}$$

Performing power allocation with respect to the interference functions (8) guarantees a certain degree of robustness. Robust power allocation was studied, e.g., in [26], [27].

f) Adaptive Receivers: The linear model (2) can also be extended in the following way:

$$\mathcal{I}_k(\boldsymbol{p}) = \min_{z_k \in \mathcal{Z}_k} \boldsymbol{p}^T \boldsymbol{v}_k(z_k), \quad \forall\, k \in \mathcal{K}. \tag{9}$$

Here, a parameter z_k is chosen from a compact set $\mathcal{Z}_k$ such that interference is minimized. Since this is a typical aim of a receiver, we refer to z_k as a *receive strategy*. See [16] for a more detailed discussion and an extension to *transmit strategies*.

g) Norms: Let $\alpha > 0$ be arbitrary and $w_k > 0, \forall\, k \in \mathcal{K}$, then

$$\mathcal{I}_{w,\alpha}(\boldsymbol{p}) = \left(\sum_{k=1}^{K} w_k \cdot (p_k)^\alpha \right)^{1/\alpha} \tag{10}$$

is an interference function.

h) Elementary Building Blocks of Convex Interference Functions: It was shown in [28] that every convex interference function can be expressed as

$$\mathcal{I}(\boldsymbol{p}) = \max_{\boldsymbol{v} \in \mathcal{V}} \sum_{k \in \mathcal{K}} v_k \cdot p_k \tag{11}$$

where $\mathcal{V}$ is a compact convex set depending on $\mathcal{I}$. A similar representation exists for concave interference functions, where $\max$ is replaced by $\min$.

Notice, that the representation (11) has the same structure as the robust interference function (8). That is, every convex interference function has an interpretation as a worst case optimization over a set of possible interference values. Likewise, every concave interference function can be interpreted as a minimum over receive strategies, as in (9).

i) Elementary Building Blocks of Log-Convex Interference Functions: We say that $\mathcal{I}(\boldsymbol{p})$ is a log-convex interference function if $\log \mathcal{I}(\exp \boldsymbol{s})$ is convex on $\mathbb{R}^K$ after a change of variable $\boldsymbol{p} = \exp \boldsymbol{s}$ (component-wise).

It was shown in [29] that an arbitrary log-convex interference function $\mathcal{I}(\boldsymbol{p})$ can be expressed as

$$\mathcal{I}(\boldsymbol{p}) = \max_{\boldsymbol{w} \in \mathcal{L}(\mathcal{I})} \left(f_{\mathcal{I}}(\boldsymbol{w}) \cdot \prod_{l=1}^{K} (p_l)^{w_l} \right) \tag{12}$$

$$\text{where} \quad \mathcal{L}(\mathcal{I}) = \left\{ \boldsymbol{w} \in \mathbb{R}^K_+ : f_{\mathcal{I}}(\boldsymbol{w}) > 0 \right\}$$

$$f_{\mathcal{I}}(\boldsymbol{w}) = \inf_{\boldsymbol{p} > 0} \frac{\mathcal{I}(\boldsymbol{p})}{\prod_{l=1}^{K} (p_l)^{w_l}}. \tag{13}$$

This result will be used later in Section IV.

II. Analysis of Interference Functions

The examples in the previous section show that interference functions occur in many different contexts. The framework is not restricted to power control problems. It is also observed that certain elementary operations on interference functions lead to new interference functions. For example, the geometric mean of interference functions is an interference function. Any sum of interference functions is an interference function. The maximum/minimum of interference functions is an interference function. This rich mathematical structure justifies the name "interference calculus" used in the title of our paper.

The representations (11) and (12) are of particular interest. They show that every (log-)convex or concave interference function can be expressed as a maximum over elementary interference functions, where the optimization is over certain coefficient sets. This specific structure has proved useful for the analysis and development of resource allocation algorithms in [28], [29]. It therefore makes sense to ask whether a similar structure can be shown for general interference functions, which are not necessarily (log-)convex or concave. In the remainder of this section it will be shown that *every* interference function has a min-max and max-min representation.

Another interesting aspect is observed from (6). The interference function $C(\boldsymbol{\gamma})$ characterizes the structure of the SIR region $\mathcal{S}$. In the literature on resource allocation and game theory, for example, it is common to assume that the achievable region is convex. In wireless communications, however, we are often dealing with nonconvex achievable regions, like the SIR region $\mathcal{S}$. By analyzing general interference functions, we will also gain deeper insight into the structure of general (comprehensive) achievable regions.

A. Representation Theorem

We start with a simple but fundamental property.

Lemma 1: Let $\mathcal{I}$ be an arbitrary interference function characterized by A1–A3. For arbitrary $\boldsymbol{p}, \hat{\boldsymbol{p}} > 0$, we have

$$\left(\min_{k \in \mathcal{K}} \frac{p_k}{\hat{p}_k} \right) \cdot \mathcal{I}(\hat{\boldsymbol{p}}) \le \mathcal{I}(\boldsymbol{p}) \le \left(\max_{k \in \mathcal{K}} \frac{p_k}{\hat{p}_k} \right) \cdot \mathcal{I}(\hat{\boldsymbol{p}}). \tag{14}$$

Proof: Defining $\bar{\lambda} = \max_k(p_k/\hat{p}_k)$, we have $\boldsymbol{p} \le \bar{\lambda}\hat{\boldsymbol{p}}$. With A3, we have $\mathcal{I}(\boldsymbol{p}) \le \bar{\lambda}\mathcal{I}(\hat{\boldsymbol{p}})$, which proves the right-hand inequality (14). The left-hand inequality is shown analogously. □

Another fundamental property is continuity [16].

Lemma 2: $\mathcal{I}(\boldsymbol{p})$ is continuous on $\mathbb{R}^K_{++}$.

Now, we show that every interference function satisfying A1–A3 has a min-max representation. To this end, we introduce level sets

$$\underline{L}(\mathcal{I}) = \{\hat{\boldsymbol{p}} > 0 : \mathcal{I}(\hat{\boldsymbol{p}}) \le 1\} \tag{15}$$

$$\overline{L}(\mathcal{I}) = \{\hat{\boldsymbol{p}} > 0 : \mathcal{I}(\hat{\boldsymbol{p}}) \ge 1\} \tag{16}$$

$$B(\mathcal{I}) = \{\hat{\boldsymbol{p}} > 0 : \mathcal{I}(\hat{\boldsymbol{p}}) = 1\}. \tag{17}$$

Definition 2: A set $\mathcal{V} \subset \mathbb{R}^K_{++}$ is said to be *relatively closed* in $\mathbb{R}^K_{++}$ if there exists a closed set $\mathcal{A} \subset \mathbb{R}^K$ such that $\mathcal{V} =$

$\mathcal{A}\bigcap\mathbb{R}^K_{++}$. For the sake of simplicity we will refer to such sets as *closed* in the following.

With Lemma 2, we know that the sets $\underline{L}(\mathcal{I})$, $B(\mathcal{I})$, and $\overline{L}(\mathcal{I})$ are relatively closed in $\mathbb{R}^K_{++}$. This leads to our first theorem, which will serve as a basis for most of the following results.

Theorem 1: Let $\mathcal{I}$ be an arbitrary interference function. For any $\boldsymbol{p} \in \mathbb{R}^K_{++}$, we have

$$\mathcal{I}(\boldsymbol{p}) = \min_{\hat{\boldsymbol{p}}\in\underline{L}(\mathcal{I})} \left(\max_{k\in\mathcal{K}} \frac{p_k}{\hat{p}_k}\right) \tag{18}$$

$$= \max_{\hat{\boldsymbol{p}}\in\overline{L}(\mathcal{I})} \left(\min_{k\in\mathcal{K}} \frac{p_k}{\hat{p}_k}\right). \tag{19}$$

Proof: We first show (18). Consider an arbitrary fixed $\boldsymbol{p} > 0$ and $\hat{\boldsymbol{p}} \in \underline{L}(\mathcal{I})$. With Lemma 1 we have

$$\mathcal{I}(\boldsymbol{p}) \leq \left(\max_{k\in\mathcal{K}} \frac{p_k}{\hat{p}_k}\right) \cdot \mathcal{I}(\hat{\boldsymbol{p}}) \leq \max_{k\in\mathcal{K}} \frac{p_k}{\hat{p}_k} \tag{20}$$

where the last inequality follows from the definition (15). This holds for arbitrary $\hat{\boldsymbol{p}} \in \underline{L}(\mathcal{I})$, thus

$$\mathcal{I}(\boldsymbol{p}) \leq \inf_{\hat{\boldsymbol{p}}\in\underline{L}(\mathcal{I})} \max_{k\in\mathcal{K}} \frac{p_k}{\hat{p}_k}. \tag{21}$$

Now, we choose $\hat{\boldsymbol{p}}'$ with $\hat{p}'_k = p_k/\mathcal{I}(\boldsymbol{p})$, $\forall\, k$. With A2 we have $\mathcal{I}(\hat{\boldsymbol{p}}') = 1$, so $\hat{\boldsymbol{p}}' \in \underline{L}(\mathcal{I})$. This particular choice fulfills $\max_{k\in\mathcal{K}} (p_k/\hat{p}'_k) = \mathcal{I}(\boldsymbol{p})$. Thus, $\hat{\boldsymbol{p}}'$ achieves the infimum (21) and (18) holds. The second equality is shown analogously: With Lemma 1, we have

$$\mathcal{I}(\boldsymbol{p}) \geq \left(\min_{k\in\mathcal{K}} \frac{p_k}{\hat{p}_k}\right) \cdot \mathcal{I}(\hat{\boldsymbol{p}}) \geq \min_{k\in\mathcal{K}} \frac{p_k}{\hat{p}_k} \tag{22}$$

for all $\boldsymbol{p} > 0$ and $\hat{\boldsymbol{p}} \in \overline{L}(\mathcal{I})$. In analogy to the first case, it can be observed that (22) is fulfilled with equality for $\hat{\boldsymbol{p}}' = \boldsymbol{p}/\mathcal{I}(\boldsymbol{p})$, with $\hat{\boldsymbol{p}}' \in \overline{L}(\mathcal{I})$. Thus, (19) is fulfilled. □

Theorem 1 says that every $\mathcal{I}(\boldsymbol{p})$ can be represented as an optimum over elementary building blocks

$$\overline{\mathcal{I}}(\boldsymbol{p},\hat{\boldsymbol{p}}) = \max_{k\in\mathcal{K}} \frac{p_k}{\hat{p}_k} \tag{23}$$

$$\underline{\mathcal{I}}(\boldsymbol{p},\hat{\boldsymbol{p}}) = \min_{k\in\mathcal{K}} \frac{p_k}{\hat{p}_k}. \tag{24}$$

Assume that $\hat{\boldsymbol{p}}$ is an arbitrary fixed parameter, so (23) and (24) are functions in $\boldsymbol{p}$. Both $\overline{\mathcal{I}}$ and $\underline{\mathcal{I}}$ fulfill the axioms A1–A3, so they can be considered as elementary interference functions.

Note, that the existence of an optimizer $\hat{\boldsymbol{p}}$ in (18) is ensured by the additional assumption (1). This rules out $\mathcal{I}(\boldsymbol{p}) = 0$, so $\underline{L}(\mathcal{I}) = \mathbb{R}^K_{++}$ cannot occur.

Next, consider the set $B(\mathcal{I})$, as defined by (17). In the proof of Theorem 1 it was shown that $\hat{\boldsymbol{p}}' \in \overline{L}(\mathcal{I}) \cap \underline{L}(\mathcal{I}) = B(\mathcal{I})$. That is, we can restrict the optimization to the boundary $B(\mathcal{I})$.

Corollary 1: Let $\mathcal{I}$ be an arbitrary interference function. For any $\boldsymbol{p} \in \mathbb{R}^K_{++}$, we have

$$\mathcal{I}(\boldsymbol{p}) = \min_{\hat{\boldsymbol{p}}\in B(\mathcal{I})} \overline{\mathcal{I}}(\boldsymbol{p},\hat{\boldsymbol{p}}) \tag{25}$$

$$= \max_{\hat{\boldsymbol{p}}\in B(\mathcal{I})} \underline{\mathcal{I}}(\boldsymbol{p},\hat{\boldsymbol{p}}). \tag{26}$$

Note, that the optimization domain $\overline{L}(\mathcal{I})$ in (19) cannot be replaced by $\underline{L}(\mathcal{I})$. Since $B(\mathcal{I}) \subseteq \underline{L}(\mathcal{I})$, relation (26) implies

$$\mathcal{I}(\boldsymbol{p}) \leq \sup_{\hat{\boldsymbol{p}}\in\underline{L}(\mathcal{I})} \underline{\mathcal{I}}(\boldsymbol{p},\hat{\boldsymbol{p}}) = +\infty.$$

Likewise, $B(\mathcal{I}) \subseteq \overline{L}(\mathcal{I})$ and (25) implies

$$\mathcal{I}(\boldsymbol{p}) \geq \inf_{\hat{\boldsymbol{p}}\in\overline{L}(\mathcal{I})} \overline{\mathcal{I}}(\boldsymbol{p},\hat{\boldsymbol{p}}) = 0.$$

So by exchanging the respective optimization domain, we only obtain trivial bounds.

B. Elementary Sets and Interference Functions

In this subsection, we will analyze the elementary interference functions $\overline{\mathcal{I}}(\boldsymbol{p},\hat{\boldsymbol{p}})$ and $\underline{\mathcal{I}}(\boldsymbol{p},\hat{\boldsymbol{p}})$ for an arbitrary and fixed parameter $\hat{\boldsymbol{p}} \in \mathbb{R}^K_{++}$. This approach helps to better understand the structure of interference functions and corresponding level sets.

Definition 3: A set $\mathcal{V} \subset \mathbb{R}^K_{++}$ is said to be upward-comprehensive if for all $\boldsymbol{w} \in \mathcal{V}$ and $\boldsymbol{w}' \in \mathbb{R}^K_{++}$

$$\boldsymbol{w}' \geq \boldsymbol{w} \Rightarrow \boldsymbol{w}' \in \mathcal{V}. \tag{27}$$

A set $\mathcal{V} \subset \mathbb{R}^K_{++}$ is said to be downward-comprehensive if for all $\boldsymbol{w} \in \mathcal{V}$ and $\boldsymbol{w}' \in \mathbb{R}^K_{++}$

$$\boldsymbol{w}' \leq \boldsymbol{w} \Rightarrow \boldsymbol{w}' \in \mathcal{V}. \tag{28}$$

We start by showing convexity.

Lemma 3: Let $\hat{\boldsymbol{p}} > 0$ be arbitrary and fixed. The function $\overline{\mathcal{I}}(\boldsymbol{p},\hat{\boldsymbol{p}})$ is convex on $\mathbb{R}^K_{+}$. The function $\underline{\mathcal{I}}(\boldsymbol{p},\hat{\boldsymbol{p}})$ is concave on $\mathbb{R}^K_{+}$.

Proof: The maximum over convex functions is convex. The minimum over concave functions is concave. □

As an immediate consequence of Theorem 1, every interference function $\mathcal{I}$ can be expressed as a minimum over elementary convex interference functions $\overline{\mathcal{I}}(\boldsymbol{p},\hat{\boldsymbol{p}})$ with $\hat{\boldsymbol{p}} \in \underline{L}(\mathcal{I})$. Alternatively, $\mathcal{I}$ can be expressed as a maximum over concave interference functions. Note that this behavior is due to the properties A1–A3 and cannot be generalized to arbitrary functions. Also, the resulting function $\mathcal{I}$ need neither be convex nor concave. The special case of convex/concave interference functions is studied separately in [28]. Some aspects will also be discussed later in Section IV.

A sublevel set of a convex function is convex, so the set

$$\underline{L}(\overline{\mathcal{I}}) = \{\boldsymbol{p} > 0 : \overline{\mathcal{I}}(\boldsymbol{p},\hat{\boldsymbol{p}}) \leq 1\} \tag{29}$$

is convex. We have $\overline{\mathcal{I}}(\hat{\boldsymbol{p}},\hat{\boldsymbol{p}}) = 1$, and $\overline{\mathcal{I}}(\boldsymbol{p},\hat{\boldsymbol{p}}) = \max_{k\in\mathcal{K}} p_k/\hat{p}_k \leq 1$ for all $\boldsymbol{p} \in \underline{L}(\overline{\mathcal{I}})$. Thus

$$p_k \leq \hat{p}_k, \quad \forall\, k \in \mathcal{K}. \tag{30}$$

The concave function $\underline{\mathcal{I}}(\boldsymbol{p},\hat{\boldsymbol{p}})$ is associated with a convex superlevel set

$$\overline{L}(\underline{\mathcal{I}}) = \{\boldsymbol{p} > 0 : \underline{\mathcal{I}}(\boldsymbol{p},\hat{\boldsymbol{p}}) \geq 1\}. \tag{31}$$

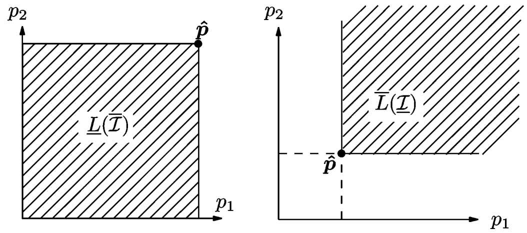

Fig. 1. Illustration of the convex comprehensive sets $\underline{L}(\overline{\mathcal{I}})$ and $\overline{L}(\underline{\mathcal{I}})$, as defined by (29) and (31), respectively.

Every $\boldsymbol{p} \in \overline{L}(\underline{\mathcal{I}})$ fulfills

$$p_k \geq \hat{p}_k, \quad \forall\, k \in \mathcal{K}. \tag{32}$$

Both sets $\underline{L}(\overline{\mathcal{I}})$ and $\overline{L}(\underline{\mathcal{I}})$ are illustrated in Fig. 1.

Let us summarize the results. Starting from an interference function $\mathcal{I}$, we obtain the sublevel set $\underline{L}(\mathcal{I}) \subset \mathbb{R}^K_{++}$, as defined by (15). For any $\hat{\boldsymbol{p}} \in \underline{L}(\mathcal{I})$, there exists a sublevel set of the form (29), which is contained in $\underline{L}(\mathcal{I})$. So the region $\underline{L}(\mathcal{I})$ is the union over convex downward-comprehensive sets. Therefore, $\underline{L}(\mathcal{I})$ is downward-comprehensive (this also follows from (15) with A3). However, $\underline{L}(\mathcal{I})$ is not necessarily convex. From Theorem 1 we know that we can use (18) to get back the original interference function $\mathcal{I}$.

There are analogous statements for the superlevel set $\overline{L}(\mathcal{I})$.

Corollary 2: Let $\mathcal{I}$ be an arbitrary interference function. The sublevel set $\underline{L}(\mathcal{I})$, as defined by (15), is closed and downward-comprehensive. The superlevel set $\overline{L}(\mathcal{I})$, as defined by (16), is closed and upward-comprehensive.

For any $\hat{\boldsymbol{p}} > 0$, there is a set of interference functions $I_{\hat{\boldsymbol{p}}} = \{\mathcal{I} : \mathcal{I}(\hat{\boldsymbol{p}}) = 1\}$. The following theorem shows the special role of the interference function $\underline{\mathcal{I}}(\boldsymbol{p}, \hat{\boldsymbol{p}}) \in I_{\hat{\boldsymbol{p}}}$.

Theorem 2: Consider an arbitrary $\hat{\boldsymbol{p}} > 0$ and an interference function $\mathcal{I}$, with $\mathcal{I}(\hat{\boldsymbol{p}}) = 1$, such that

$$\mathcal{I}(\boldsymbol{p}) \leq \underline{\mathcal{I}}(\boldsymbol{p}, \hat{\boldsymbol{p}}), \quad \forall\, \boldsymbol{p} > 0, \tag{33}$$

then this can only be satisfied with equality.

Proof: Inequality (33) implies $\overline{L}(\mathcal{I}) \subseteq \overline{L}(\underline{\mathcal{I}})$, or in other words, every $\boldsymbol{p} \in \{\boldsymbol{p} : \mathcal{I}(\boldsymbol{p}) \geq 1\}$ fulfills $\underline{\mathcal{I}}(\boldsymbol{p}, \hat{\boldsymbol{p}}) \geq 1$. This can be written as $\min_k p_k/\hat{p}_k \geq 1$, or equivalently $\boldsymbol{p} \geq \hat{\boldsymbol{p}}$. With $\mathcal{I}(\hat{\boldsymbol{p}}) = 1$ and A3, it follows that $\mathcal{I}(\boldsymbol{p}) \geq \mathcal{I}(\hat{\boldsymbol{p}}) = 1$. Thus, the set $\overline{L}(\underline{\mathcal{I}}) = \{\boldsymbol{p} : \boldsymbol{p} \geq \hat{\boldsymbol{p}}\}$ also belongs to $\overline{L}(\mathcal{I})$. Consequently, $\overline{L}(\mathcal{I}) = \overline{L}(\underline{\mathcal{I}})$. With Theorem 1 we can conclude that $\mathcal{I}(\boldsymbol{p}) = \underline{\mathcal{I}}(\boldsymbol{p}, \hat{\boldsymbol{p}})$ for all $\boldsymbol{p} > 0$. □

Theorem 2 shows that $\underline{\mathcal{I}}(\boldsymbol{p}, \hat{\boldsymbol{p}})$ is the *smallest* interference function from the set $I_{\hat{\boldsymbol{p}}}$. Here "smallest" is used in the sense of a relation $\mathcal{I}_1 \leq \mathcal{I}_2$, which means that $\mathcal{I}_1(\boldsymbol{p}) \leq \mathcal{I}_2(\boldsymbol{p})$ for all $\boldsymbol{p} > 0$.

Analogously, we show the following result.

Theorem 3: Consider $\hat{\boldsymbol{p}} > 0$ and an interference function $\mathcal{I}$, with $\mathcal{I}(\hat{\boldsymbol{p}}) = 1$, such that

$$\mathcal{I}(\boldsymbol{p}) \geq \overline{\mathcal{I}}(\boldsymbol{p}, \hat{\boldsymbol{p}}), \quad \forall\, \boldsymbol{p} > 0 \tag{34}$$

then this can only be satisfied with equality.

The interference function $\overline{\mathcal{I}}(\boldsymbol{p}, \hat{\boldsymbol{p}})$ is the greatest interference function from the set $I_{\hat{\boldsymbol{p}}}$.

Theorems 2 and 3 show that only the interference functions $\overline{\mathcal{I}}(\boldsymbol{p}, \hat{\boldsymbol{p}})$ and $\underline{\mathcal{I}}(\boldsymbol{p}, \hat{\boldsymbol{p}})$ provide majorants[1] and minorants for arbitrary interference functions. This is a property by which general interference functions are characterized.

III. Synthesis of Interference Functions

In the previous section, we have analyzed the basic building blocks of an interference function $\mathcal{I}$, and its connection with level sets. Now, we study the converse approach, i.e., the synthesis of an interference function for a given set $\mathcal{V}$.

A. Interference Functions and Comprehensive Sets

We start by showing that for any closed downward-comprehensive set $\mathcal{V} \subset \mathbb{R}^K_{++}$, we can synthesize an interference function $\mathcal{I}_{\mathcal{V}}(\boldsymbol{p})$. By constructing the sublevel set $\underline{L}(\mathcal{I}_{\mathcal{V}})$ we get back the original set.

Theorem 4: For any nonempty, closed, and downward-comprehensive set $\mathcal{V} \subset \mathbb{R}^K_{++}$, $\mathcal{V} \neq \mathbb{R}^K_{++}$, there exists an interference function

$$\mathcal{I}_{\mathcal{V}}(\boldsymbol{p}) := \inf_{\hat{\boldsymbol{p}} \in \mathcal{V}} \max_{k \in \mathcal{K}} \frac{p_k}{\hat{p}_k} = \min_{\hat{\boldsymbol{p}} \in \mathcal{V}} \max_{k \in \mathcal{K}} \frac{p_k}{\hat{p}_k} \tag{35}$$

and $\underline{L}(\mathcal{I}_{\mathcal{V}}) = \mathcal{V}$.

Proof: For any nonempty set $\mathcal{V} \subset \mathbb{R}^K_{++}$, the function $\mathcal{I}_{\mathcal{V}}$ fulfills properties A1–A3. With the additional assumption $\mathcal{V} \neq \mathbb{R}^K_{++}$, we know that there exists a $\hat{\boldsymbol{p}} > 0$ such that $\mathcal{I}_{\mathcal{V}}(\hat{\boldsymbol{p}}) > 0$. Therefore, $\mathcal{I}_{\mathcal{V}}(\boldsymbol{p}) > 0$ for all $\boldsymbol{p} > 0$. We only need to show $\underline{L}(\mathcal{I}_{\mathcal{V}}) = \mathcal{V}$, then it follows from Theorem 1 that the infimum is attained, i.e., the right-hand equality in (35) holds.

Consider an arbitrary $\boldsymbol{p} \in \underline{L}(\mathcal{I}_{\mathcal{V}})$, i.e., $\mathcal{I}_{\mathcal{V}}(\boldsymbol{p}) \leq 1$. Defining $\boldsymbol{p}(\lambda) = \lambda \boldsymbol{p}$, with $0 < \lambda < 1$, we have $\mathcal{I}_{\mathcal{V}}\big(\boldsymbol{p}(\lambda)\big) = \lambda \mathcal{I}_{\mathcal{V}}(\boldsymbol{p}) < 1$. According to the definition (35), there exists a $\hat{\boldsymbol{p}} \in \mathcal{V}$ such that

$$\max_{k \in \mathcal{K}} \frac{p_k(\lambda)}{\hat{p}_k} < 1. \tag{36}$$

Comprehensiveness implies $\boldsymbol{p}(\lambda) < \hat{\boldsymbol{p}}$ and, therefore, $\boldsymbol{p}(\lambda) \in \mathcal{V}$. Since $\mathcal{V}$ is closed, $\lim_{\lambda \to 1} \boldsymbol{p}(\lambda) = \boldsymbol{p}$ implies $\boldsymbol{p} \in \mathcal{V}$. Thus

$$\underline{L}(\mathcal{I}_{\mathcal{V}}) \subseteq \mathcal{V}. \tag{37}$$

Conversely, consider an arbitrary $\hat{\boldsymbol{p}} \in \mathcal{V}$, for which

$$\mathcal{I}_{\mathcal{V}}(\hat{\boldsymbol{p}}) = \inf_{\tilde{\boldsymbol{p}} \in \mathcal{V}} \max_{k \in \mathcal{K}} \frac{\hat{p}_k}{\tilde{p}_k} \leq \max_{k \in \mathcal{K}} \frac{\hat{p}_k}{\hat{p}_k} = 1. \tag{38}$$

This shows $\hat{\boldsymbol{p}} \in \underline{L}(\mathcal{I}_{\mathcal{V}})$ and therefore

$$\mathcal{V} \subseteq \underline{L}(\mathcal{I}_{\mathcal{V}}). \tag{39}$$

Combining (37) and (39), we have $\underline{L}(\mathcal{I}_{\mathcal{V}}) = \mathcal{V}$. □

It can be observed that the restriction $\mathcal{V} \neq \mathbb{R}^K_{++}$ is closely linked with the assumption (1). In particular, there exists a $\boldsymbol{p} > 0$ such that $\mathcal{I}_{\mathcal{V}}(\boldsymbol{p}) > 0$ if and only if the corresponding set $\mathcal{V}$ fulfills $\mathcal{V} \neq \mathbb{R}^K_{++}$.

Similar results exist for upward-comprehensive sets:

[1]An interference function $\mathcal{I}'(\boldsymbol{p})$ is said to be a *minorant* of $\mathcal{I}(\boldsymbol{p})$ if $\mathcal{I}'(\boldsymbol{p}) \leq \mathcal{I}(\boldsymbol{p})$ for all $\boldsymbol{p} > 0$. It is said to be a *majorant* if $\mathcal{I}'(\boldsymbol{p}) \geq \mathcal{I}(\boldsymbol{p})$ for all $\boldsymbol{p} > 0$.

Theorem 5: For any nonempty, closed, and upward-comprehensive set $\mathcal{V} \subset \mathbb{R}^K_{++}$, $\mathcal{V} \neq \mathbb{R}^K_{++}$, there exists an interference function

$$\mathcal{I}_\mathcal{V}(\boldsymbol{p}) := \sup_{\hat{\boldsymbol{p}} \in \mathcal{V}} \min_{k \in \mathcal{K}} \frac{p_k}{\hat{p}_k} = \max_{\hat{\boldsymbol{p}} \in \mathcal{V}} \min_{k \in \mathcal{K}} \frac{p_k}{\hat{p}_k} \tag{40}$$

and $\overline{L}(\mathcal{I}_\mathcal{V}) = \mathcal{V}$.

Proof: The proof is similar to the proof of Theorem 4. Every $\boldsymbol{p} \in \overline{L}(\mathcal{I}_\mathcal{V})$ is also contained in $\mathcal{V}$, thus implying $\overline{L}(\mathcal{I}_\mathcal{V}) \subseteq \mathcal{V}$. Conversely, it is shown that every $\hat{\boldsymbol{p}} \in \mathcal{V}$ is also contained in the set $\overline{L}(\mathcal{I}_\mathcal{V})$, thus, $\mathcal{V} \subseteq \overline{L}(\mathcal{I}_\mathcal{V})$. □

The following corollary is an immediate consequence.

Corollary 3: Let $\mathcal{V}_1, \mathcal{V}_2$ be two arbitrary closed comprehensive sets, as defined in the previous theorems. If $\mathcal{I}_{\mathcal{V}_1} = \mathcal{I}_{\mathcal{V}_2}$, then $\mathcal{V}_1 = \mathcal{V}_2$.

Proof: If the sets are downward-comprehensive, then this is a direct consequence of Theorem 4, because $\mathcal{V}_1 = \underline{L}(\mathcal{I}_{\mathcal{V}_1}) = \underline{L}(\mathcal{I}_{\mathcal{V}_2}) = \mathcal{V}_2$. For upward-comprehensive sets, the result follows from Theorem 5. □

B. Comprehensive Hull

Next, assume that $\mathcal{V} \subset \mathbb{R}^K_{++}$, $\mathcal{V} \neq \mathbb{R}^K_{++}$, is an arbitrary nonempty closed set which is not necessarily comprehensive. In this case, (35) still yields an interference function. However, the properties stated by Theorems 4 and 5 need not be fulfilled. That is, $\underline{L}(\mathcal{I}_\mathcal{V}) \neq \mathcal{V}$ and $\overline{L}(\mathcal{I}_\mathcal{V}) \neq \mathcal{V}$ in general.

The next theorem shows that the level sets $\underline{L}(\mathcal{I}_\mathcal{V})$ and $\overline{L}(\mathcal{I}_\mathcal{V})$ provide comprehensive hulls of the original set $\mathcal{V}$.

Theorem 6: Let $\mathcal{V}_0 \supseteq \mathcal{V}$ be the downward-comprehensive hull of $\mathcal{V}$, i.e., the smallest closed downward-comprehensive subset of $\mathbb{R}^K_{++}$ containing $\mathcal{V}$. Let $\mathcal{I}_\mathcal{V}(\boldsymbol{p})$ be defined by (35), then

$$\underline{L}(\mathcal{I}_\mathcal{V}) = \mathcal{V}_0. \tag{41}$$

Proof: From Corollary 2, we know that $\underline{L}(\mathcal{I}_\mathcal{V})$ is downward-comprehensive. By assumption, $\mathcal{V}_0$ is the smallest downward-comprehensive set containing $\mathcal{V}$, so together with (39) we have

$$\mathcal{V} \subseteq \mathcal{V}_0 \subseteq \underline{L}(\mathcal{I}_\mathcal{V}). \tag{42}$$

We also have

$$\begin{aligned} \mathcal{V}_0 \supseteq \mathcal{V} &\Rightarrow \mathcal{I}_{\mathcal{V}_0}(\boldsymbol{p}) \leq \mathcal{I}_\mathcal{V}(\boldsymbol{p}), \quad \forall\, \boldsymbol{p} \in \mathbb{R}^K_{++} \\ &\Rightarrow \underline{L}(\mathcal{I}_{\mathcal{V}_0}) \supseteq \underline{L}(\mathcal{I}_\mathcal{V}). \end{aligned} \tag{43}$$

From Theorem 4 we know that $\underline{L}(\mathcal{I}_{\mathcal{V}_0}) = \mathcal{V}_0$. Combining (42) and (43), the result (41) follows. □

To summarize, $\mathcal{V} \subseteq \underline{L}(\mathcal{I}_\mathcal{V})$ is fulfilled for any nonempty closed set $\mathcal{V} \subset \mathbb{R}^K_{++}$, $\mathcal{V} \neq \mathbb{R}^K_{++}$. The set $\underline{L}(\mathcal{I}_\mathcal{V})$ is the downward-comprehensive hull of $\mathcal{V}$. The set $\mathcal{V}$ is downward-comprehensive if and only if $\mathcal{V} = \underline{L}(\mathcal{I}_\mathcal{V})$. Examples are given in Fig. 2.

Likewise, an upward-comprehensive hull can be constructed for any nonempty closed set $\mathcal{V} \subset \mathbb{R}^K_{++}$, $\mathcal{V} \neq \mathbb{R}^K_{++}$.

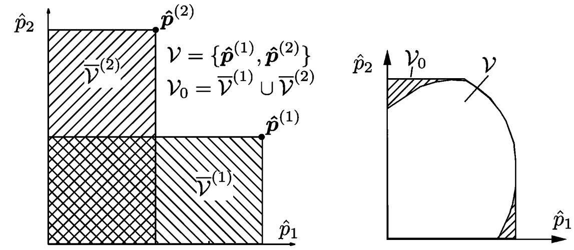

Fig. 2. Two examples illustrating Theorem 6: The set $\mathcal{V}_0 = \underline{L}(\mathcal{I}_\mathcal{V})$ is the comprehensive hull of an arbitrary noncomprehensive closed set $\mathcal{V} \subset \mathbb{R}^K_{++}$.

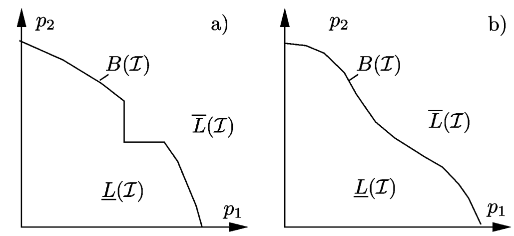

Fig. 3. Illustration of Theorem 8. Example a) leads to a nonstrictly monotone interference function, whereas example b) is associated with a strictly monotone interference function, i.e., no segment of the boundary is parallel to the coordinate axes.

Theorem 7: Let $\mathcal{V}_\infty \supseteq \mathcal{V}$ be the upward-comprehensive hull of $\mathcal{V}$, i.e., the smallest closed upward-comprehensive subset of $\mathbb{R}^K_{++}$ containing $\mathcal{V}$. Let $\mathcal{I}_\mathcal{V}(\boldsymbol{p})$ be defined by (40), then

$$\overline{L}(\mathcal{I}_\mathcal{V}) = \mathcal{V}_\infty. \tag{44}$$

Proof: This is shown analogously to Theorem 6. □

C. Strict Monotonicity

We now study interference functions with a special monotonicity property. To this end we need some definitions.

Definition 4: $\boldsymbol{p}^{(1)} \succ \boldsymbol{p}^{(2)}$ means $p^{(1)}_k \geq p^{(2)}_k$, $\forall\, k \in \mathcal{K}$, and there exists at least one component k_0 such that $p^{(1)}_{k_0} > p^{(2)}_{k_0}$.

Definition 5: An interference function $\mathcal{I}(\boldsymbol{p})$ is said to be *strictly monotone* if $\boldsymbol{p}^{(1)} \succ \boldsymbol{p}^{(2)}$ implies $\mathcal{I}(\boldsymbol{p}^{(1)}) > \mathcal{I}(\boldsymbol{p}^{(2)})$.

The next theorem shows that strict monotonicity of $\mathcal{I}(\boldsymbol{p})$ corresponds to certain properties of the associated level sets $\underline{L}(\mathcal{I})$ and $\overline{L}(\mathcal{I})$, whose boundary is $B(\mathcal{I})$.

Theorem 8: An interference function $\mathcal{I}(\boldsymbol{p})$ is strictly monotone if and only if no segment of the boundary $B(\mathcal{I})$, as defined by (17), is parallel to a coordinate axis.

Proof: Assume that $\mathcal{I}(\boldsymbol{p})$ is strictly monotone. We will show by contradiction that there is no parallel segment. To this end, suppose that a segment of the boundary $B(\mathcal{I})$ is parallel to a coordinate axis. On this line, consider two arbitrary points $\boldsymbol{p}^{(1)}$, $\boldsymbol{p}^{(2)}$ with $\boldsymbol{p}^{(1)} \succ \boldsymbol{p}^{(2)}$. We have $1 = \mathcal{I}(\boldsymbol{p}^{(1)}) = \mathcal{I}(\boldsymbol{p}^{(2)})$, i.e., $\mathcal{I}$ is not strictly monotone, which is a contradiction. Conversely, assume that there is no parallel segment. Consider a boundary point $\hat{\boldsymbol{p}}$ with $\mathcal{I}(\hat{\boldsymbol{p}}) = 1$. An arbitrary $\boldsymbol{p} \succ \hat{\boldsymbol{p}}$ does not belong to $B(\mathcal{I})$. That is, $\mathcal{I}(\boldsymbol{p}) > 1 = \mathcal{I}(\hat{\boldsymbol{p}})$, thus $\mathcal{I}$ is strictly monotone. □

This result is illustrated in Fig. 3.

IV. Connection With Convex Interference Functions

In this section, we assume that $\mathcal{I}(\boldsymbol{p})$ is a *convex* interference function. For this special case, different elementary interference functions can be used, as shown in [28]. In particular, there exists a compact downward-comprehensive convex set $\mathcal{V}$ such that $\mathcal{I}(\boldsymbol{p})$ has a representation (11). The function $\mathcal{I}$ is uniquely determined by the set $\mathcal{V}$ (see [28] for details).

Next, we show a different way of connecting convex comprehensive sets with convex or concave interference functions. This approach is less direct than the one in [28], but it allows for an interesting interpretation in terms of level sets.

A. Convex Interference Functions and Level Sets

Consider the convex interference function (11), generated from a nonempty convex compact downward-comprehensive set $\mathcal{V} \subset \mathbb{R}^K_{++}$, $\mathcal{V} \neq \mathbb{R}^K_{++}$. From Corollary 2, we know that the set $\underline{L}(\mathcal{I})$ is closed and downward-comprehensive. Exploiting the convexity of $\mathcal{I}$, it can be shown that $\underline{L}(\mathcal{I})$ is upper-bounded. The set $\underline{L}(\mathcal{I})$ is also convex, since it is a sublevel set of a convex function. However, $\underline{L}(\mathcal{I}) \neq \mathcal{V}$ in general. The result of Theorem 4 does not apply since $\mathcal{I}$ is constructed from $\mathcal{V}$ in a different way.

In order to express $\mathcal{V}$ as a sublevel set of a convex interference function, we need to introduce another interference function

$$\mathcal{I}_1(\boldsymbol{p}) = \max_{\boldsymbol{v} \in \underline{L}(\mathcal{I})} \sum_{k \in \mathcal{K}} v_k p_k. \tag{45}$$

Unlike $\mathcal{I}$, the new function $\mathcal{I}_1$ is constructed with the level set $\underline{L}(\mathcal{I})$, so it depends on the original set $\mathcal{V}$ only indirectly. The maximum (45) is guaranteed to exist since $\underline{L}(\mathcal{I})$ is a compact set (relatively in $\mathbb{R}^K_{++}$).

The function $\mathcal{I}_1$ is also a convex interference function. The next theorem shows that the sublevel set $\underline{L}(\mathcal{I}_1)$ equals the original set $\mathcal{V}$.

Theorem 9: Consider an arbitrary nonempty compact downward-comprehensive convex set $\mathcal{V} \subset \mathbb{R}^K_{++}$, $\mathcal{V} \neq \mathbb{R}^K_{++}$, from which we synthesize a convex interference function $\mathcal{I}$, as defined by (11). Let $\mathcal{I}_1$ be defined by (45), then

$$\mathcal{V} = \underline{L}(\mathcal{I}_1). \tag{46}$$

Proof: Let $\boldsymbol{v} \in \mathcal{V}$, then it can be observed from (11) that $\sum_k v_k p_k \leq 1$ for all $\boldsymbol{p} \in \underline{L}(\mathcal{I})$. Thus

$$1 \geq \max_{\boldsymbol{p} \in \underline{L}(\mathcal{I})} \sum_{k \in \mathcal{K}} v_k p_k = \mathcal{I}_1(\boldsymbol{v}).$$

That is, $\boldsymbol{v} \in \mathcal{V}$ is also contained in the sublevel set of $\mathcal{I}_1$, i.e., $\boldsymbol{v} \in \underline{L}(\mathcal{I}_1)$, thus implying $\mathcal{V} \subseteq \underline{L}(\mathcal{I}_1)$. It remains to show the converse, i.e., $\mathcal{V} \supseteq \underline{L}(\mathcal{I}_1)$. Consider an arbitrary $\boldsymbol{v} \in \underline{L}(\mathcal{I}_1)$. It can be observed from (45) that $\sum_k v_k p_k \leq 1$ for all $\boldsymbol{p} \in \mathbb{R}^K_{++}$ such that $\mathcal{I}(\boldsymbol{p}) \leq 1$. Now we choose $\boldsymbol{p} > 0$ such that $\mathcal{I}(\boldsymbol{p}) = 1$. This implies

$$\sum_{k \in \mathcal{K}} v_k p_k - \mathcal{I}(\boldsymbol{p}) \leq 1 - 1 = 0.$$

Thus

$$\sup_{\boldsymbol{p} > 0 : \mathcal{I}(\boldsymbol{p}) = 1} \left(\sum_k v_k p_k - \mathcal{I}(\boldsymbol{p}) \right) \leq 0. \tag{47}$$

Let $\hat{\boldsymbol{p}} > 0$ be arbitrary. Because of the properties of the set $\mathcal{V}$, we have $\mathcal{I}(\hat{\boldsymbol{p}}) > 0$ and $\hat{\lambda} := 1/\mathcal{I}(\hat{\boldsymbol{p}}) < +\infty$. Defining $\tilde{\boldsymbol{p}} = \hat{\lambda}\hat{\boldsymbol{p}}$ and exploiting A2, we have

$$\begin{aligned} \sum_{k \in \mathcal{K}} v_k \hat{p}_k - \mathcal{I}(\hat{\boldsymbol{p}}) &= \frac{1}{\hat{\lambda}} \cdot \hat{\lambda} \left(\sum_{k \in \mathcal{K}} v_k \hat{p}_k - \mathcal{I}(\hat{\boldsymbol{p}}) \right) \\ &= \frac{1}{\hat{\lambda}} \left(\sum_{k \in \mathcal{K}} v_k \tilde{p}_k - \mathcal{I}(\tilde{\boldsymbol{p}}) \right) \leq 0. \end{aligned} \tag{48}$$

The last inequality follows from $\mathcal{I}(\tilde{\boldsymbol{p}}) = 1$ and (47). Consequently

$$\overline{\mathcal{I}}^*(\boldsymbol{v}) := \sup_{\hat{\boldsymbol{p}} > 0} \left(\sum_{k \in \mathcal{K}} v_k \hat{p}_k - \mathcal{I}(\hat{\boldsymbol{p}}) \right) \leq 0.$$

The function $\overline{\mathcal{I}}^*(\boldsymbol{v})$ is the *conjugate* of $\mathcal{I}$. It was shown in [28] that $\overline{\mathcal{I}}^*(\boldsymbol{v}) < +\infty$ implies $\boldsymbol{v} \in \mathcal{V}$. That is, every $\boldsymbol{v} \in \underline{L}(\mathcal{I}_1)$ is also contained in $\mathcal{V}$, which concludes the proof. □

Theorem 9 shows that *any* convex compact downward-comprehensive set from $\mathbb{R}^K_{++}$ can be expressed as a sublevel set of a convex interference function. Conversely, it is clear from the results of Section II that any sublevel set of a convex interference function is compact downward-comprehensive convex.

Similar results can be derived for *concave* interference functions. Consider a nonempty convex closed upward-comprehensive set $\mathcal{V} \subset \mathbb{R}^K_{++}$, $\mathcal{V} \neq \mathbb{R}^K_{++}$. This set is associated with a concave interference function

$$\mathcal{I}(\boldsymbol{p}) = \min_{\boldsymbol{v} \in \mathcal{V}} \sum_{k \in \mathcal{K}} v_k p_k. \tag{49}$$

The superlevel set $\overline{L}(\mathcal{I})$ is closed upward-comprehensive convex. However, $\overline{L}(\mathcal{I}) \neq \mathcal{V}$ in general. In order to express $\mathcal{V}$ as a superlevel set, we need to introduce an additional interference function

$$\mathcal{I}_2(\boldsymbol{p}) = \min_{\boldsymbol{v} \in \overline{L}(\mathcal{I})} \sum_{k \in \mathcal{K}} v_k p_k. \tag{50}$$

We have the following result.

Theorem 10: Consider an arbitrary nonempty closed upward-comprehensive convex set $\mathcal{V} \subset \mathbb{R}^K_{++}$, $\mathcal{V} \neq \mathbb{R}^K_{++}$, from which we synthesize a concave interference function $\mathcal{I}$, as defined by (49). Let $\mathcal{I}_2$ be defined by (50), then

$$\mathcal{V} = \overline{L}(\mathcal{I}_2). \tag{51}$$

Proof: The proof is similar to the proof of Theorem 9. □

Theorem 10 shows that *every* closed upward-comprehensive convex set from $\mathbb{R}^K_{++}$ can be expressed as a superlevel set of a concave interference function. Conversely, every superlevel set of a concave interference function is closed downward-comprehensive convex.

Theorems 9 and 10 have an interesting interpretation in terms of resource allocation problems:

Every convex interference function has a representation (11). This can be interpreted as the maximum weighted total *network utility* from a utility set $\mathcal{V} = \{\boldsymbol{v} > 0 : \mathcal{I}_1(\boldsymbol{v}) \leq 1\}$. Here, the convex interference function $\mathcal{I}_1(\boldsymbol{v})$ can be seen as an indicator function measuring the feasibility of the utilities $\boldsymbol{v}$. Likewise, every concave interference function has a representation (49). This can be interpreted as the minimum weighted total *network cost* from a feasible set $\mathcal{V} = \{\boldsymbol{v} > 0 : \mathcal{I}_2(\boldsymbol{v}) \geq 1\}$. The concave interference function $\mathcal{I}_2(\boldsymbol{v})$ can be seen as an indicator function providing a single measure for the feasibility of a given cost vector $\boldsymbol{v}$.

The following example shows a possible application of these results.

Example 1: The *Nash bargaining* strategy from cooperative game theory (see, e.g., [14], [15]) is typically studied under the assumption of a convex comprehensive utility set $\mathcal{V}$, as specified in Theorem 9. Under this assumption, the Nash bargaining solution $\mathcal{N}(\boldsymbol{\alpha})$, as a function of weighting factors $\boldsymbol{\alpha} = [\alpha_1, \ldots, \alpha_K]^T$, with $\|\boldsymbol{\alpha}\|_1 = 1$, is given as

$$\mathcal{N}(\boldsymbol{\alpha}) = \max_{\boldsymbol{v} \in \mathcal{V}} \prod_{k \in \mathcal{K}} (v_k)^{\alpha_k}. \quad (52)$$

From Theorem 9 we know that there is a convex interference function $\mathcal{I}_1$ such that

$$\mathcal{V} = \{\boldsymbol{v} > 0 : \mathcal{I}_1(\boldsymbol{v}) \leq 1\}.$$

The bargaining solution (52) is attained on the boundary of $\mathcal{V}$ being characterized by $\mathcal{I}_1(\boldsymbol{v}) = 1$. Thus, (52) can be rewritten as [30]

$$\mathcal{N}(\boldsymbol{\alpha}) = \max_{\{\boldsymbol{v} > 0 : \mathcal{I}_1(\boldsymbol{v}) = 1\}} \prod_{l \in \mathcal{K}} (v_l)^{\alpha_l} = \sup_{\boldsymbol{v} > 0} \frac{\prod_{l \in \mathcal{K}} (v_l)^{\alpha_l}}{\mathcal{I}_1(\boldsymbol{v})}. \quad (53)$$

It was shown in [28] that any convex interference function is a log-convex interference function, so $\mathcal{I}_1$ can be expressed as (12). Comparing (53) with the function $f_{\mathcal{I}_1}$, as defined by (13), we have

$$\mathcal{N}(\boldsymbol{\alpha}) = \frac{1}{f_{\mathcal{I}_1}(\boldsymbol{\alpha})}. \quad (54)$$

This provides an interesting link between the Nash bargaining theory and the theory of (log-convex) interference functions. Problem (52) can also be interpreted as a *proportional fair* operating point [31] of a wireless system.

Note that there are other bargaining strategies which only rely on downward-comprehensive utility sets. Also in this case the set can be expressed as a sublevel set of an interference function, as shown by Theorem 4.

B. Convex/Concave Bounds

Now, we return to our basic interference model A1–A3, without requiring convexity or concavity. In this case, it is generally not possible to express $\mathcal{I}$ as an optimum over linear functions, as in (11). However, we can use the previous results to derive convex/concave bounds.

Introducing sets

$$\mathcal{V}^{(1)} = \{\tilde{\boldsymbol{p}} : \text{there exists a } \hat{\boldsymbol{p}} \in \underline{L}(\mathcal{I}) \text{ and } \hat{\boldsymbol{p}} = 1/\tilde{\boldsymbol{p}}\}$$
$$\mathcal{V}^{(2)} = \{\tilde{\boldsymbol{p}} : \text{there exists a } \hat{\boldsymbol{p}} \in \overline{L}(\mathcal{I}) \text{ and } \hat{\boldsymbol{p}} = 1/\tilde{\boldsymbol{p}}\}$$

and applying Theorem 1, an arbitrary interference function $\mathcal{I}(\boldsymbol{p})$ can be expressed as

$$\mathcal{I}(\boldsymbol{p}) = \min_{\tilde{\boldsymbol{p}} \in \mathcal{V}^{(1)}} \max_{k \in \mathcal{K}} (p_k \cdot \tilde{p}_k) = \max_{\tilde{\boldsymbol{p}} \in \mathcal{V}^{(2)}} \min_{k \in \mathcal{K}} (p_k \cdot \tilde{p}_k).$$

The set $\mathcal{V}^{(1)}$ can be rewritten as

$$\begin{aligned}\mathcal{V}^{(1)} &= \{\tilde{\boldsymbol{p}} > 0 : \mathcal{I}(1/\tilde{\boldsymbol{p}}) \leq 1\} \\ &= \{\tilde{\boldsymbol{p}} > 0 : 1 \leq 1/\mathcal{I}(1/\tilde{\boldsymbol{p}})\} = \overline{L}(\mathcal{I}_{\text{inv}})\end{aligned}$$

where we have used the definition

$$\mathcal{I}_{\text{inv}}(\boldsymbol{p}) = \frac{1}{\mathcal{I}(1/\boldsymbol{p})}, \quad \text{for } \boldsymbol{p} > 0. \quad (55)$$

It can be verified that $\mathcal{I}_{\text{inv}}$ is an interference function: Property A2 follows from

$$\mathcal{I}_{\text{inv}}(\lambda \boldsymbol{p}) = \frac{1}{\mathcal{I}(1/\lambda \boldsymbol{p})} = \frac{1}{\frac{1}{\lambda} \cdot \mathcal{I}(1/\boldsymbol{p})} = \lambda \mathcal{I}_{\text{inv}}(\boldsymbol{p}).$$

Properties A1 and A3 are easily shown as well.

Defining $\mathcal{W} = \{\boldsymbol{w} > 0 : \|\boldsymbol{w}\|_1 = 1\}$, we have

$$\max_{k \in \mathcal{K}} p_k = \sup_{w \in \mathcal{W}} \sum_{k \in \mathcal{K}} w_k p_k, \quad \text{for any } \boldsymbol{p} > 0.$$

Hence, an arbitrary $\mathcal{I}(\boldsymbol{p})$ can be represented as

$$\mathcal{I}(\boldsymbol{p}) = \min_{\tilde{\boldsymbol{p}} \in \overline{L}(\mathcal{I}_{\text{inv}})} \sup_{\boldsymbol{w} \in \mathcal{W}} \sum_{k \in \mathcal{K}} w_k \tilde{p}_k \cdot p_k \quad (56)$$

$$= \max_{\tilde{\boldsymbol{p}} \in \underline{L}(\mathcal{I}_{\text{inv}})} \inf_{\boldsymbol{w} \in \mathcal{W}} \sum_{k \in \mathcal{K}} w_k \tilde{p}_k \cdot p_k. \quad (57)$$

It can be observed from (57) that this representation has a similar form as the convex function (11). For any given $\boldsymbol{w}$, a linear function is maximized over parameters $\tilde{\boldsymbol{p}}$. However, the interference function (57) is generally not convex because of the additional optimization with respect to $\boldsymbol{w}$, so the combined weights $w_k \tilde{p}_k$ are contained in a more general set. By choosing an arbitrary fixed $\boldsymbol{w} \in \mathcal{W}$, we obtain a convex upper bound

$$\mathcal{I}(\boldsymbol{p}) \leq \sup_{\tilde{\boldsymbol{p}} \in \underline{L}(\mathcal{I}_{\text{inv}})} \sum_{k \in \mathcal{K}} w_k \tilde{p}_k \cdot p_k =: \overline{\mathcal{I}}_{\text{conv}}(\boldsymbol{p}, \boldsymbol{w}). \quad (58)$$

Note, that this convex upper bound can be trivial, i.e., the right-hand side of (58) can tend to infinity. Inequality (58) holds for all $\boldsymbol{w} \in \mathcal{W}$, so

$$\mathcal{I}(\boldsymbol{p}) \leq \inf_{\boldsymbol{w} \in \mathcal{W}} \overline{\mathcal{I}}_{\text{conv}}(\boldsymbol{p}, \boldsymbol{w}). \quad (59)$$

Similar results can be derived from (56), leading to a concave lower bound. This bound can also be trivial (i.e., zero).

Another interesting problem is the construction of a minorant $\hat{\mathcal{I}}$, such that $\hat{\mathcal{I}}(\hat{\boldsymbol{p}}) = \mathcal{I}(\hat{\boldsymbol{p}})$ for some point $\hat{\boldsymbol{p}}$, and $\hat{\mathcal{I}}(\boldsymbol{p}) \leq \mathcal{I}(\boldsymbol{p})$ for all $\boldsymbol{p} > 0$. For general interference functions $\mathcal{I}(\boldsymbol{p})$, such a minorant is provided by the elementary interference function $\underline{\mathcal{I}}(\boldsymbol{p}, \hat{\boldsymbol{p}})$, as shown in Section II-B. For the special case of convex interference functions (11), another minorant is obtained by choosing an arbitrary $\boldsymbol{v}' \in \mathcal{V}$, for which we have $\mathcal{I}(\boldsymbol{p}) \geq \sum_k v'_k p_k$. However, such a linear minorant does not always exist, as shown by the following example.

Example 2: Consider the log-convex interference function

$$\mathcal{I}(\boldsymbol{p}) = C_1 \prod_{l\in\mathcal{K}} (p_l)^{w_l}, \quad \|\boldsymbol{w}\|_1 = 1, \quad \boldsymbol{w} > 0, \ C_1 > 0. \quad (60)$$

We show by contradiction that no linear interference function can be a minorant of (60). Assume that there is a $\boldsymbol{w} > 0$ such that $\mathcal{I}(\boldsymbol{p}) \geq \sum_l p_l w_l$ for all $\boldsymbol{p} > 0$. Then we can construct a vector $\boldsymbol{p}(\rho) = (1, \ldots, 1, \rho, 1)$, where the rth component is set to some $\rho > 0$. The position r is chosen such that $w_r, v_r \neq 0$. By assumption, $\mathcal{I}(\boldsymbol{p}(\rho)) = C_1 \rho^{w_r} \geq v_r \rho + \sum_{l\neq r} v_l$. Dividing both sides by ρ we have $C_1 \rho^{w_r - 1} \geq v_r + \sum_{l\neq r} v_l/\rho$. Letting $\rho \to \infty$ leads to the contradiction $0 \geq v_r > 0$.

This discussion shows that in order to derive "good" minorants or majorants, it is important to exploit the structure of the interference function. Otherwise, trivial bounds can be obtained. For a more detailed analysis of convex and concave interference functions, the reader is referred to [28].

V. Interference Balancing

Thus far, we have studied a single interference function and its connection with level sets. Next, we study the interaction between K interference functions $\mathcal{I}_1, \ldots, \mathcal{I}_K$. We focus on the min-max balancing problem (7) discussed in the Introduction.

A. General Results

The min-max optimum $C(\boldsymbol{\gamma})$, as defined by (7), provides a measure for the feasibility of SIR values $\boldsymbol{\gamma} = [\gamma_1, \ldots, \gamma_K]^T$. The feasible SIR region is the sublevel set $\mathcal{S}$, as defined by (6). Since $C(\boldsymbol{\gamma})$ fulfills the axioms A1–A3, we can apply the results of Section II. The following corollary is a consequence of Theorem 1.

Corollary 4: The function $C(\boldsymbol{\gamma})$ can be represented as

$$C(\boldsymbol{\gamma}) = \min_{\hat{\boldsymbol{\gamma}}\in\mathcal{S}} \left(\max_{k\in\mathcal{K}} \frac{\gamma_k}{\hat{\gamma}_k} \right). \quad (61)$$

Note, that the optimization in (61) is over the feasible set $\mathcal{S}$ directly, whereas a parameterization with respect to the power allocation $\boldsymbol{p} > 0$ is used in (7). Analyzing the dependency on the transmission powers is of practical interest, since this is the way how the transmitter controls the quality of service (QoS) values.

If $C(\boldsymbol{\gamma}) < 1$, then the SIR point $\boldsymbol{\gamma}$ lies strictly in the interior of the SIR region $\mathcal{S}$. In this case, there always exists a vector $\boldsymbol{p}^{(1)} > 0$ such that

$$\max_{k\in\mathcal{K}} \frac{\gamma_k \cdot \mathcal{I}_k(\boldsymbol{p}^{(1)})}{p_k^{(1)}} \leq 1$$

or, equivalently, $\gamma_k \mathcal{I}_k(\boldsymbol{p}^{(1)}) \leq p_k^{(1)}$ for all $k \in \mathcal{K}$.

Now, consider a $\boldsymbol{\gamma} > 0$ on the boundary of the feasible set, i.e., $C(\boldsymbol{\gamma}) = 1$. Then for any $\epsilon > 0$ there exists a vector $\boldsymbol{p}(\epsilon)$ such that

$$\max_{k\in\mathcal{K}} \frac{\gamma_k \cdot \mathcal{I}_k(\boldsymbol{p}(\epsilon))}{p_k(\epsilon)} \leq 1 + \epsilon \quad (62)$$

or, equivalently, $\gamma_k/(1+\epsilon) \leq p_k(\epsilon)/\mathcal{I}_k(\boldsymbol{p}(\epsilon))$, for all $k \in \mathcal{K}$. That is, the point $\boldsymbol{\gamma}$ is achieved asymptotically for $\epsilon \to 0$. If (62) is fulfilled for $\epsilon = 0$, then we say that the boundary is *achievable*. For a more detailed discussion on achievability see, e.g., [16], [32]. For most practical scenarios, achievability is ensured by the presence of noise and limited transmission powers (see, e.g., [16]).

Using these results, the SIR feasible region $\mathcal{S}$ can be defined as

$$\mathcal{S} = \{\boldsymbol{\gamma} > 0 : \text{for every } \epsilon > 0 \text{ there exists a vector } \boldsymbol{p}(\epsilon) > 0 \text{ such that (62) is fulfilled}\}. \quad (63)$$

The interior of the set $\mathcal{S}$ can be parametrized by the variable $\boldsymbol{p} > 0$. However, this does not always hold for the boundary $\{\boldsymbol{\gamma} > 0 : C(\boldsymbol{\gamma}) = 1\}$ because a vector $\hat{\boldsymbol{p}} > 0$ fulfilling $\hat{\gamma}_k \mathcal{I}_k(\hat{\boldsymbol{p}})/\hat{p}_k = 1$ does not need to exist, but it can always be approximated arbitrarily close. Thus, it is generally not possible to replace the infimum in (7) by a minimum.

B. Max-min Fairness

The min-max optimization (7) is one possible approach to fairness. In this definition, the value $C(\boldsymbol{\gamma})$ is the infimum over the weighted inverse SIR $\gamma_k \mathcal{I}_k(\boldsymbol{p})/p_k$. Note, that $\inf \max_k \mathrm{SIR}_k^{-1} = (\sup \min_k \mathrm{SIR}_k)^{-1}$. This optimization strategy is also referred to as *max-min fairness*.

An alternative approach is *min-max fairness*. This can also be formulated in terms of weighted inverse SIR, as the max-min optimization problem

$$c(\boldsymbol{\gamma}) = \sup_{\boldsymbol{p}>0} \left(\min_{k\in\mathcal{K}} \frac{\gamma_k \mathcal{I}_k(\boldsymbol{p})}{p_k} \right). \quad (64)$$

It is not obvious whether the max-min optimum $c(\boldsymbol{\gamma})$ and the min-max optimum $C(\boldsymbol{\gamma})$, as defined by (7), are identical. Both strategies can be regarded as fair. Note that we do not only interchange the optimization order, but also the domain, so Fan's minimax inequality cannot be applied here. Both values do not necessarily coincide. The difference is sometimes referred to as the *fairness gap* [8].

Example 3: Consider the linear interference model introduced in Section I-B. If the coupling matrix $\boldsymbol{V}$ is irreducible, then $c(\boldsymbol{\gamma}) = C(\boldsymbol{\gamma})$ always holds. But this need not hold true for reducible coupling matrices. Consider the example

$$\boldsymbol{V} = \left[\begin{array}{cc|cc} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ \hline \multicolumn{2}{c|}{\multirow{2}{*}{$\boldsymbol{B}$}} & 0 & \mu \\ & & \mu & 0 \end{array}\right] \quad \text{with} \quad \begin{array}{l} \boldsymbol{B} \geq 0 \\ 0 < \mu < 1. \end{array} \quad (65)$$

We have $C(\boldsymbol{\gamma}) = 1$, but $c(\boldsymbol{\gamma}) \leq 1$. If $\boldsymbol{B} = 0$, then we have two isolated subsystems with spectral radius 1 and μ. In this case, $C(\boldsymbol{\gamma}) = 1 > \mu = c(\boldsymbol{\gamma})$, which demonstrates that min-max fairness and max-min fairness are generally not equivalent. But also a different behavior can occur. For example, $C(\boldsymbol{\gamma}) = c(\boldsymbol{\gamma})$ is fulfilled if $\boldsymbol{B} > 0$.

In order to better understand these effects, notice that the function $c(\boldsymbol{\gamma})$ fulfills the properties A1–A3. That is, $c(\boldsymbol{\gamma})$ is an interference function, so we can use Theorem 1 to analyze and compare both functions C and c.

The function C was already used in the definition (5) of the SIR region $\mathcal{S}$. With Theorem 1 it is clear that $\mathcal{S} = \underline{L}(C)$. Now, we will show some interesting analogies between $\underline{L}(C)$ and $\overline{L}(c)$, defined as

$$\overline{L}(c) = \{\boldsymbol{\gamma} > 0 : c(\boldsymbol{\gamma}) \geq 1\}. \tag{66}$$

From (64), we know that for every $\epsilon > 0$ there exists a $\boldsymbol{p}(\epsilon)$ such that

$$\min_{k\in\mathcal{K}} \frac{\gamma_k \mathcal{I}_k(\boldsymbol{p}(\epsilon))}{p_k(\epsilon)} \geq c(\boldsymbol{\gamma}) - \epsilon.$$

If $c(\boldsymbol{\gamma}) \geq 1$, then

$$\gamma_k \geq (1-\epsilon) \cdot \frac{p_k(\epsilon)}{\mathcal{I}_k(\boldsymbol{p}(\epsilon))}, \quad \forall\, k \in \mathcal{K}. \tag{67}$$

This can be used for the following characterization:

$$\overline{L}(c) = \{\boldsymbol{\gamma} > 0 : \text{for every } \epsilon > 0 \text{ there exists a vector } \boldsymbol{p}(\epsilon) > 0 \text{ such that (67) is fulfilled}\}.$$

With Theorem 1 we have

$$\sup_{\boldsymbol{p}>0}\left(\min_{k\in\mathcal{K}} \frac{\gamma_k \mathcal{I}_k(\boldsymbol{p})}{p_k}\right) = \max_{\hat{\boldsymbol{\gamma}}\in\overline{L}(c)}\left(\min_{k\in\mathcal{K}} \frac{\gamma_k}{\hat{\gamma}_k}\right) = c(\boldsymbol{\gamma}). \tag{68}$$

Again, we can generally not replace the supremum by a maximum since the boundary of $\overline{L}(c)$ cannot always be parametrized by $\boldsymbol{p} > 0$.

It was shown in [16] that $c(\boldsymbol{\gamma})$ is always smaller than $C(\boldsymbol{\gamma})$. As mentioned before, this result is due to the specific properties A1–A3, and does not follow from Fan's minimax inequality. Now, we can use the results of this paper to show this property with a different approach, based on level sets.

Theorem 11: $c(\boldsymbol{\gamma}) \leq C(\boldsymbol{\gamma})$ for all $\boldsymbol{\gamma} > 0$.

Proof: Consider an arbitrary $\tilde{\boldsymbol{\gamma}}$ from the interior of $\overline{L}(c)$, i.e., $c(\tilde{\boldsymbol{\gamma}}) \geq 1$. From (64), we know that there exists a $\tilde{\boldsymbol{p}} > 0$ satisfying

$$\frac{\tilde{\gamma}_k \mathcal{I}_k(\tilde{\boldsymbol{p}})}{\tilde{p}_k} > 1, \quad \forall\, k \in \mathcal{K}. \tag{69}$$

Now, we show that $\tilde{\boldsymbol{\gamma}}$ also lies in the interior of $\overline{L}(C)$. From the definition of $C(\boldsymbol{\gamma})$, it follows that for all $\epsilon > 0$ there exists a vector $\boldsymbol{p}(\epsilon) > 0$ such that

$$\frac{\tilde{\gamma}_k \mathcal{I}_k(\boldsymbol{p}(\epsilon))}{\tilde{p}_k(\epsilon)} \leq C(\tilde{\boldsymbol{\gamma}}) + \epsilon, \quad \forall\, k \in \mathcal{K}. \tag{70}$$

The ratio $\mathcal{I}(\boldsymbol{p})/p_k$ is invariant with respect to a scaling of $\boldsymbol{p}$, thus we can assume $\boldsymbol{p}(\epsilon) \geq \tilde{\boldsymbol{p}}$ without affecting (70). In addition, we can assume that there is an index $\hat{k}$ such that $p_{\hat{k}}(\epsilon) = \tilde{p}_{\hat{k}}$. With (69), (70), and property A3, we have

$$\begin{aligned} 1 < \frac{\tilde{\gamma}_{\hat{k}} \mathcal{I}_{\hat{k}}(\tilde{\boldsymbol{p}})}{\tilde{p}_{\hat{k}}} &= \frac{\tilde{\gamma}_{\hat{k}} \mathcal{I}_{\hat{k}}(\tilde{\boldsymbol{p}})}{p_{\hat{k}}(\epsilon)} \\ &\leq \frac{\tilde{\gamma}_{\hat{k}} \mathcal{I}_{\hat{k}}(\boldsymbol{p}(\epsilon))}{p_{\hat{k}}(\epsilon)} \leq C(\tilde{\boldsymbol{\gamma}}) + \epsilon. \end{aligned} \tag{71}$$

This inequality holds for all $\epsilon > 0$. Letting $\epsilon \to 0$, it follows that $C(\tilde{\boldsymbol{\gamma}}) > 1$, so $\tilde{\boldsymbol{\gamma}}$ is also contained in the interior of $\overline{L}(C)$. Therefore

$$c(\gamma) = \max_{\hat{\boldsymbol{\gamma}}\in\overline{L}(c)}\left(\min_{k\in\mathcal{K}} \frac{\gamma_k}{\hat{\gamma}_k}\right) \leq \max_{\hat{\boldsymbol{\gamma}}\in\overline{L}(C)}\left(\min_{k\in\mathcal{K}} \frac{\gamma_k}{\hat{\gamma}_k}\right) = C(\boldsymbol{\gamma}). \tag{72}$$

Example 3 shows that strict inequality $c(\boldsymbol{\gamma}) < C(\boldsymbol{\gamma})$ can actually occur. □

C. Fixed-Point Characterization

We will now study under which condition the infimum (7) can be attained. This question is closely connected with the achievability of the boundary of the SIR region $\mathcal{S}$, which was already discussed in Section V-A.

From Theorem 1 and [16, Theorem 2.14] we know that there exists a $\tilde{\boldsymbol{p}} \in \mathcal{S}$ such that the balanced level $C(\boldsymbol{\gamma})$ is achieved by all users, i.e.,

$$C(\boldsymbol{\gamma})\tilde{p}_k = \gamma_k \mathcal{I}_k(\tilde{\boldsymbol{p}}), \quad \forall\, k \in \mathcal{K} \tag{73}$$

if and only if there exists a $\mu > 0$ and a $\tilde{\boldsymbol{p}} > 0$ such that

$$\mu \cdot \tilde{p}_k = \gamma_k \cdot \max_{l\in\mathcal{K}} \frac{\tilde{p}_l}{\hat{p}_l^{(k)}}, \quad \forall\, k \in \mathcal{K} \tag{74}$$

where

$$\hat{\boldsymbol{p}}^{(k)} = \arg\min_{\hat{\boldsymbol{p}}\in\underline{L}(\mathcal{I}_k)} \max_l \frac{\tilde{p}_l}{\hat{p}_l}.$$

With Theorem 1 it is clear that (73) implies (74). Conversely, assume that (74) is fulfilled. By the uniqueness of the balanced optimum [16], $\mu = C(\boldsymbol{\gamma})$ can be concluded, so (73) is fulfilled.

For the special case of monotone interference functions, as studied in Section III-C, we have the following result.

Theorem 12: Let $\mathcal{I}_1, \ldots, \mathcal{I}_K$ be interference functions such that the boundaries of the corresponding sets $\underline{L}(\mathcal{I}_k)$ do not contain segments parallel to the coordinate axes, and there is no self-interference, then for any $\boldsymbol{\gamma} > 0$ there exists a vector $\boldsymbol{p} > 0$ such that

$$C(\boldsymbol{\gamma})p_k = \gamma_k \mathcal{I}_k(\boldsymbol{p}), \quad k \in \mathcal{K} \tag{75}$$

where $C(\boldsymbol{\gamma})$ is defined by (7).

Proof: This is a consequence of Theorem 8 and the result [16, Sec.2.5]. □

One practical example for which the achievability of the boundary is important is the aforementioned problem of combined beamforming and power allocation. Some algorithms, like the ones proposed in [10], [12], require that the signal-to-interference noise ratio (SINR) targets are feasible. This can be tested by solving the min-max balancing problem (7), which requires the existence of a fixed-point $\tilde{\boldsymbol{p}} > 0$ fulfilling (73).

A more general example is Yates' fixed-point iteration for power control [7], which also requires that the chosen SINR target $\boldsymbol{\gamma}$ lies in the interior of the SIR feasible region $\mathcal{S}$. That is, $C(\boldsymbol{\gamma}) < 1$ must be fulfilled, otherwise, the iteration diverges. By computing $C(\boldsymbol{\gamma})$, it can be checked whether this is fulfilled or not. Again, this requires the existence of a fixed-point $\tilde{\boldsymbol{p}} > 0$ fulfilling (73).

VI. Conclusion

Interference functions were originally introduced in the context of power control [7]. But their significance goes beyond this application. In this paper, we introduce a general *interference calculus* based on the axioms "scale invariance" and "monotonicity." The proposed theory provides an abstract framework for modeling and analyzing interference in multiuser systems.

We show that every interference function can be expressed as an optimum over elementary building blocks, optimized over a closed comprehensive set. Moreover, any closed comprehensive subset of $\mathbb{R}^K_{++}$ can be expressed as a sublevel/superlevel set of an interference function. This shows a direct connection between the theory of interference functions and the analysis of achievable regions. A special case is the SIR region with the indicator function $C(\boldsymbol{\gamma})$, as discussed in Section I-B. Regions of other performance measures, like Gaussian capacity, minimum mean-square error (MMSE), bit-error rate (BER), etc., can be derived as bijective mappings of the SIR region. This is a standard approach in the design of wireless systems.

The generality of the proposed framework makes it potentially useful for the analysis of various resource allocation problems. In Section V, we have discussed the example of interference balancing. Another example is cooperative game theory, as discussed in Section IV-A.

The focus of this paper is on elementary properties of interference functions and associated feasible sets. It should be noted that this is the basis for more specific interference functions, like the log-convex interference functions studied in [29], or the convex/concave interference functions [28]. In [28], [29], it is shown how convexity and the properties A1–A3 can be exploited to solve certain resource allocation problems. Other applications can be found in [33], [34]. These are all examples which show that the basic properties A1–A3 often lead to resource allocation problems with a nice analytical structure, and efficient algorithmic solutions.

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