Concave and Convex Interference Functions—General Characterizations and Applications

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***Abstract*—Many resource allocation problems can be studied within the framework of interference functions. Basic properties of interference functions are non-negativity, scale-invariance, and monotonicity. In this paper, we study interference functions with additional properties, namely convexity, concavity, and log-convexity. Such interference functions occur naturally in various contexts, e.g., adaptive receive strategies, robust power control, or resource allocation over convex utility sets. We show that every convex (resp. concave) interference function can be expressed as a maximum (resp. minimum) over a weighted sum of its arguments. This analytical insight provides a link between the axiomatic interference framework and conventional interference models that are based on the definition of a coupling matrix. We show how the results can be used to derive best-possible convex/concave approximations for general interference functions. The results have further application in the context of feasible sets of multiuser systems. Convex approximations for general feasible sets are derived. Finally, we show how convexity can be exploited to solve the problem of signal-to-interference-plus-noise ratio (SINR)-constrained power minimization with super-linear convergence.**

***Index Terms*—Adaptive receivers and transmitters, interference, power control, resource allocation, robustness.**

I. Introduction

THE performance of interference-coupled multiuser systems can be significantly improved by adaptive strategies, which are able to adjust to varying interference situations. Some examples are multiuser beamforming [1]–[6], code division multiple access (CDMA) receiver design [7], base station assignment [8]–[10], or robust reception under channel uncertainties [11]–[13]. All these examples have in common that the interference experienced by each user depends on the transmission powers in a nonlinear way. Adaptivity introduces additional degrees of freedom, which complicate the task of resource allocation.

An axiom-based framework for modeling interference under such nonlinear dependencies was proposed in [14], with extensions in [15]–[18]. In this paper, interference is only characterized by certain monotonicity and scalability axioms (as detailed in Section I-C). This model can be applied to various types of interference scenarios, and it was successfully used as a basis for the development of iterative resource allocation algorithms. For example, the algorithms [1], [2], [8]–[10] can be regarded as special cases of this framework. Many more application examples exist in the literature.

This axiomatic approach to interference modeling is quite attractive, since it helps to better understand the effects of interference coupling. The results can be applied to all interference scenarios falling under this framework. This is convenient since it means that many resource allocation including adaptive designs are globally solvable by the fixed-point iteration proposed in [14]. However, the generality of the axiomatic framework also means that more specific properties (if available) are not exploited. For example, it was shown in [19]–[25]) that the *convexity* of linear interference functions can be exploited in various ways for the development of efficient algorithmic solutions. However, these results are based on the specific linear interference model, which is the classical model in the context of power control (see e.g., [26]–[30]). Unlike the axiomatic framework, the linear model does not allow for adaptive designs.

The power control framework was extended to include adaptive receivers [1]–[8], [10], [11], [13], [24], [31]–[34]). These nonlinear interference models also allow for efficient algorithms. Most of these approaches exploit convexity, either explicitly [6], [24], [34], or implicitly [1]–[5], [8], [10], [13], [31], as will be seen in the following.

In this paper, we propose a new axiomatic interference framework that combines convexity with the properties of interference functions [17], [18]. The proposed framework also provides a link between the axiomatic approach [14]–[18] and other models based on the notion of a *interference coupling matrix*. The advantage of the axiomatic approach is its generality and wide range of applications, whereas efficient algorithms have been reported for the matrix-based approach (see, e.g., [1]–[7], [9]–[11], [13], [24], [31]–[33]). This paper provides a unifying framework for all these individual models.

Some notational conventions are as follows. Matrices and vectors are denoted by bold capital letters and bold lowercase letters, respectively. Let $\boldsymbol{y}$ be a vector, then $y_l = [\boldsymbol{y}]_l$ is the lth component. The notation $\boldsymbol{y} \geq 0$ means that $y_l \geq 0$ for all components l. Also, $\exp\{\boldsymbol{y}\}$ and $\log\{\boldsymbol{y}\}$ denotes component-wise exponential and logarithm, respectively. The set of non-negative reals is denoted as $\mathbb{R}_+$. The set of positive reals is denoted as $\mathbb{R}_{++}$.

In Section I-A, we will discuss examples of *concave interference functions*. Some examples for *convex interference functions* will be discussed in Section I-B.

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A. Example: Adaptive Receive or Transmit Strategies

An abstract interference model for adaptive receive or transmit strategies was proposed in [17] and [31]. The interference power experienced by the kth user from a user set $\mathcal{K} = \{1, 2, \ldots, K\}$ can be written as

$$\mathcal{I}_k(\boldsymbol{p}) = \min_{z_k \in \mathcal{Z}_k} \boldsymbol{p}^T \boldsymbol{v}_k(z_k) \tag{1}$$

where

- $\boldsymbol{p} = [p_1, \ldots, p_K]^T$ is a vector of transmission powers (the *power allocation*). In the presence of non-negligible noise power σ^2, an extended power vector

$$\underline{\boldsymbol{p}} = \begin{bmatrix} \boldsymbol{p} \\ \sigma^2 \end{bmatrix} = [p_1, \ldots, p_K, \sigma^2]^T \tag{2}$$

  can be used. In most parts of this paper, noise is an option but not an essential part of the model. In this way, the framework differs from related work [14]. This will be discussed later in Section I-C.
- $\boldsymbol{v}_k(z_k)$ is a non-negative vector of interference coupling coefficients, depending on a *receive strategy* z_k from a compact set $\mathcal{Z}_k$. If one component of $\boldsymbol{p}$ stands for noise power, then the respective component of $\boldsymbol{v}_k(z_k)$ is strictly positive.

This nonlinear interference model includes the interference scenarios [1]–[10] as special cases.

Notice, that all the results of this paper can also be applied to interference scenarios depending on *transmit strategies*. Similar to the uplink/downlink duality observed in the context of downlink beamforming [1]–[3], optimal transmit strategies can be found by optimizing an equivalent system with "virtual receive strategies," as discussed in [17] and [31].

In order to provide an example for a concrete realization of the interference function (1), we will now discuss the interference functions resulting from beamforming receivers. Consider an uplink beamforming system with K single-antenna transmitters and an M-element antenna array at the receiver. Independent signals $s_1, \ldots, s_K$ are transmitted over vector-valued channels $\boldsymbol{h}_1, \ldots, \boldsymbol{h}_K \in \mathbb{C}^M$, with spatial covariance matrices $\boldsymbol{R}_k = \mathrm{E}\left[\boldsymbol{h}_k \boldsymbol{h}_k^H\right]$. The superimposed signals at the array output are received by a bank of linear filters $\boldsymbol{u}_1, \ldots, \boldsymbol{u}_K$ (the "beamformers"). The output of the kth beamformer is

$$y_k = \boldsymbol{u}_k^H \left(\sum_{l \in \mathcal{K}} \boldsymbol{h}_l s_l + \boldsymbol{n} \right) \tag{3}$$

where $\boldsymbol{n} \in \mathbb{C}^M$ is an additive white Gaussian noise (AWGN) vector, with $\mathrm{E}[\boldsymbol{n}\boldsymbol{n}^H] = \sigma^2 \boldsymbol{I}$. The interference power coupling coefficients of the kth user are

$$[\boldsymbol{v}_k(\boldsymbol{u}_k)]_l = \begin{cases} \frac{\boldsymbol{u}_k^H \boldsymbol{R}_l \boldsymbol{u}_k}{\boldsymbol{u}_k^H \boldsymbol{R}_k \boldsymbol{u}_k}, & 1 \le l \le K, l \ne k \\ \frac{\|\boldsymbol{u}_k\|^2}{\boldsymbol{u}_k^H \boldsymbol{R}_k \boldsymbol{u}_k}, & l = K+1 \\ 0, & l = k. \end{cases} \tag{4}$$

With the commonly used normalization $\|\boldsymbol{u}_k\|_2 = 1$, the interference function for the beamforming case is

$$\begin{aligned} \mathcal{I}_k(\underline{\boldsymbol{p}}) &= \left[\max_{\|\boldsymbol{u}_k\|_2 = 1} \frac{\boldsymbol{u}_k^H \boldsymbol{R}_k \boldsymbol{u}_k}{\boldsymbol{u}_k^H \left(\sum_{l \ne k} p_l \boldsymbol{R}_l + \sigma^2 \boldsymbol{I} \right) \boldsymbol{u}_k} \right]^{-1} \\ &= \min_{\|\boldsymbol{u}_k\|_2 = 1} \underline{\boldsymbol{p}}^T \boldsymbol{v}_k(\boldsymbol{u}_k). \end{aligned} \tag{5}$$

It can be observed that the interference coupling is not constant. For any power vector $\boldsymbol{p} > 0$, the beamformer $\boldsymbol{u}_k$ adapts to the interference in such a way that the signal-to-interference-plus-noise ratio (SINR) is maximized. This optimization can be solved efficiently via an eigenvalue decomposition [35]. For deterministic channels $\boldsymbol{h}_1, \ldots, \boldsymbol{h}_K$, we have $\boldsymbol{R}_l = \boldsymbol{h}_l \boldsymbol{h}_l^H$, so the interference resulting from optimum beamformers is obtained in closed from [35]

$$\mathcal{I}_k(\underline{\boldsymbol{p}}) = \frac{1}{\boldsymbol{h}_k^H \left(\sigma^2 \boldsymbol{I} + \sum_{l \ne k} p_l \boldsymbol{h}_l \boldsymbol{h}_l^H \right)^{-1} \boldsymbol{h}_k}. \tag{6}$$

This interference function is a special case of the more general beamforming model (5). Both functions are special cases of the generic interference function (1), which in turn is a special case of the axiomatic framework of convex interference functions which will be introduced in this paper.

For interference functions of the type (1), efficient algorithmic solutions for max–min fairness [33] and SINR-constrained power minimization [31], [34] are known. In [34], it was shown that the power minimization problem can be solved with super-linear convergence rate. This behavior is due to the special structure of the interference model (1). For the more general axiom-based framework of "standard interference functions" only a fixed-point iteration with linear convergence is known [14], [16].

It should be noted that the interference function (1) can also be used to model adaptive *transmit strategies*. By exploiting the duality between receive and transmit strategies (see, e.g., the discussion in [17] and [31]), the parameter z_k can be interpreted as a *transmit strategy* in a "dual channel." This concept is closely related to the "virtual uplink" proposed in the beamforming context [1].

B. Example: Robust Designs

In (1), the parameter z_k was chosen such that interference is minimized. If we replace min by max, we obtain the *worst case interference* function

$$\mathcal{I}_k(\boldsymbol{p}) = \max_{z_k \in \mathcal{Z}_k} \boldsymbol{p}^T \boldsymbol{v}_k(z_k), \qquad k \in \mathcal{K}. \tag{7}$$

Here, the parameter z_k can be regarded as an *uncertainty* from a compact uncertainty region $\mathcal{Z}_k$. Optimizing the system with respect to the worst case interference provides a certain degree of robustness (see, e.g., [11], [13], and [24]).

The source of uncertainty can be system imperfections or channel estimation errors. As an example, consider again the downlink beamforming scenario discussed in the previous section. In the presence of imperfect channel estimation, the spatial covariance matrices can be modeled as $\boldsymbol{R}_k = \hat{\boldsymbol{R}}_k + \boldsymbol{\Delta}_k$, where $\hat{\boldsymbol{R}}_k$ is the estimated covariance, and $\boldsymbol{\Delta}_k \in \mathcal{Z}_k$ is the estimation error from a compact uncertainty region $\mathcal{Z}_k$. In order to improve the robustness, the system can be optimized with respect to the worst case interference functions

$$\mathcal{I}_k(\underline{\boldsymbol{p}}) = \max_{\boldsymbol{\Delta}_k \in \mathcal{Z}_k} \frac{\sum_{l \neq k} p_l \boldsymbol{u}_l^H (\hat{\boldsymbol{R}}_k + \boldsymbol{\Delta}_k) \boldsymbol{u}_l + \sigma^2}{\boldsymbol{u}_k^H (\hat{\boldsymbol{R}}_k + \boldsymbol{\Delta}_k) \boldsymbol{u}_k}. \tag{8}$$

Using a vector notation, as in (4), the interference function (8) can be rewritten in the canonical form (7). Other types of uncertainties, like *noise uncertainty* are straightforward extensions of this model.

Note that in related work [36], [37], an additional optimization with respect to the beamformers $\boldsymbol{u}_1, \ldots, \boldsymbol{u}_K$ is performed. This generally leads to a nonconvex max–min-type interference function, which differs from the model (8). This more general case is beyond the scope of this paper and will be studied elsewhere [18]. Here, we focus on the aspects of convexity.

The worst case interference function (7), and thus (8), are convex. Convex interference functions have some special properties which allow for efficient algorithmic solutions. For example, it was shown in [13] that the problem of robust power minimization subject to SINR constraints can be solved with super-linear convergence rate for arbitrary interference functions of the type (7), assuming a constant noise component. Again, the same problem can be solved by the fixed-point iteration [14]. However, the fixed-point iteration is based on a more general axiomatic framework and does not exploit the special structure (7). Therefore, only linear convergence is achieved [16]. This example provides further motivation for analyzing and exploiting special properties of interference functions, if available.

Another interesting property of the interference function (7) is *log-convexity* after a change of variable $\boldsymbol{p} = \exp \boldsymbol{s}$, where $\boldsymbol{s} \in \mathbb{R}^K$ [17]. A function $\mathcal{I}(\mathrm{e}^{\boldsymbol{s}})$ is said to be log-convex if the logarithm of the function is convex [38].

If multiuser interference can be modeled by log-convex interference functions $\mathcal{I}_1(\mathrm{e}^{\boldsymbol{s}}), \ldots, \mathcal{I}_K(\mathrm{e}^{\boldsymbol{s}})$, then the resulting log-SIR region (the set of jointly achievable signal-to-interference ratios (SIRs) on a logarithmic scale) is *convex*. This is of practical importance for resource allocation in a network since it allows the application of standard convex optimization strategies for finding an optimum on the boundary of the region. This result can be extended to any other SIR-dependent quality-of-service (QoS) measure with a log-convex inverse mapping $\mathrm{SIR} = \gamma(\mathrm{QoS})$ [17]. The change of variable $\boldsymbol{p} = \mathrm{e}^{\boldsymbol{s}}$ was already successfully used in the context of linear interference functions [19]–[25]. The linear case corresponds to the above model (7) with a *fixed* parameter z_k (nonadaptive interference coupling coefficients). In this paper, we will use it in the context of a more general axiomatic interference model, which will be introduced in the next section.

C. Axiomatic Approach to Interference Modeling

The examples in the previous sections show that interference under adaptive designs can often be expressed in canonical forms (1) or (7). That is, a linear function is either minimized or maximized over a set of "coupling coefficients." Interference functions of this kind are common in power control theory and related areas (see, e.g., [23] and [39] and references therein). By exploiting the special matrix structure of (1) and (7), efficient algorithmic solutions were found in the literature for various types of resource allocation problems, like SINR-constrained power minimization [13], [31] or max–min fairness [31]–[33].

A conceptually different approach was introduced by Yates [14], who proposed an axiomatic framework for characterizing the effects of interference. A slightly different version of this framework was proposed in [17]:

Definition 1: We say that $\mathcal{I} : \mathbb{R}_+^K \mapsto \mathbb{R}_+$ is an *interference function* if it fulfills the axioms

A1	(non-negativity)	$\mathcal{I}(\boldsymbol{p}) \geq 0$
A2	(scale invariance)	$\mathcal{I}(\alpha \boldsymbol{p}) = \alpha \mathcal{I}(\boldsymbol{p}) \quad \forall \alpha \in \mathbb{R}_+$.
A3	(monotonicity)	$\mathcal{I}(\boldsymbol{p}) \geq \mathcal{I}(\boldsymbol{p}')$ if $\boldsymbol{p} \geq \boldsymbol{p}'$

This framework differs from Yates' standard interference function [14] in the way noise is handled. While noise was implicitly assumed in [14], the model A1–A3 requires the definition of an extended power vector $\underline{\boldsymbol{p}}$, as defined by (2). This explains property A2, which differs from the "scalability" required in [14]. Thus, if noise is required as part of the model, then it must be explicitly included by considering the extended power vector $\underline{\boldsymbol{p}}$. In addition, $\mathcal{I}(\underline{\boldsymbol{p}})$ must be required to be strictly monotone with respect to the noise component. In this case, the model A1–A3 is equivalent to Yates' framework of standard interference functions (see [17] for more details).

The reason for choosing the axiomatic framework A1–A3 is the aim for a unifying canonical model, which allows the analysis and comparison of different kinds of interference functions, independent from possible noise requirements. Although noise is mostly a reasonable assumption, there is strong practical and theoretical motivation for interference functions without noise. For example, there is a rich history of resource allocation problems in the absence of noise, ranging from classical power control based on Perron–Frobenius theory [39], to beamforming [32], [40], [41], and max–min fair resource allocation [33], where interference functions of the form (1) were assumed.

A second motivation for the framework A1–A3 is the analysis of feasible utility regions, which can be expressed as sublevel sets of interference functions. In this context, noise is meaningless. This aspect of interference functions will be discussed in detail in Sections III-C, IV-D, and IV-E.

Finally, note that property A1 *non-negativity* turns into *positivity* under the additional requirement that there exists a $\boldsymbol{p}' > 0$ such that $\mathcal{I}(\boldsymbol{p}') > 0$. Then, positivity holds for *all* $\boldsymbol{p} > 0$, as shown in [17]. This basically means that the trivial case $\mathcal{I}(\boldsymbol{p}) = 0$ is ruled out. However, interference *can* be zero in some cases, so it is important not to require *positivity* from the outset.

D. Problem Formulation and Contributions

The axiomatic approach described in the previous section has proved useful for solving the problem of SINR-constrained power minimization [14]. Given K interference functions $\mathcal{I}_1(\underline{\boldsymbol{p}}), \ldots, \mathcal{I}_K(\underline{\boldsymbol{p}})$, which are strictly monotonic in the noise component, the unique global optimum is obtained by a fixed-point iteration with linear convergence [14], [16], [34].

It can be observed that the specific matrix-based interference functions discussed in Sections I-A and I-B are special cases of the axiomatic framework A1–A3. Hence, the fixed-point iteration can be applied.

However, a better convergence rate can be achieved by exploiting the special matrix structure of (1) and (7), as proposed in [13], [31]. Then, the same power minimization problem can be achieved with super-linear convergence [13], [34]. This behavior is due to the convex (resp. concave) structure of the matrix-based interference functions (1) and (7).

Another example is the SIR balancing problem [33], which was solved by exploiting the properties of the concave matrix-based interference functions (1).

Clearly, convexity/concavity is an important property which should be exploited whenever possible. Now, an interesting question is how convexity/concavity can be exploited when starting from an axiomatic interference model, i.e., requiring A1–A3 plus an additional convexity/concavity property. This interference model is not based on a coupling matrix. No theory exist for this general case.

In this paper, it will be shown that all the results discussed so far can indeed be generalized to an axiomatic interference model. This provides a unifying framework, which contains many of the previous results as special cases. For example, algorithms with super-linear convergence not only exist for the particular matrix-based interference functions (1) and (7), but for *all* convex and concave interference functions fulfilling A1–A3. More aspects and advantages of having an axiomatic framework for convex/concave interference functions will be discussed throughout this paper.

In Sections II and III, we will start by analyzing the structure of *concave* and *convex* interference functions, respectively. It was already observed that (5) is a concave interference function. Here, we show that *every* concave interference function can be represented as a minimum over linearly weighted powers, as in (5), and the coefficients $\boldsymbol{v}$ are chosen from a closed comprehensive convex set from $\mathbb{R}_+^K$. Similar results are shown for convex interference functions, except that the maximum is taken, as in (7). Thus, every convex/concave interference function can be interpreted as an optimum over elementary linear interference functions. In Section III-C, we study interference functions which are convex on a logarithmic scale (log-convex). Such a change of variable is often useful to reveal a "hidden convexity" (see, e.g., [19]–[25]). In this paper, we show that log-convex interference functions are an important generalization of convex interference functions. Every convex interference function is a log-convex interference function, but the converse is false.

In Section IV, we will discuss possible applications. Having analyzed the elementary building blocks of convex and concave interference functions, we now exploit this knowledge to derive best possible convex/concave approximations of *arbitrary* interference functions. Then, we will discuss the close connection between interference functions and SIR feasible regions, which are sublevel sets of a min–max indicator function. This indicator function is itself an interference function, so we can apply the previous results to the analysis of the structure of the SIR region. Under the certain conditions, the region is convex. If the region is not convex, then best-possible convex approximations can be given.

Finally, in Section IV-F, we discuss how the results can be applied to the problem of SINR-constrained power minimization [14]. If convexity or concavity is added to the axiomatic framework, then the representations (1) and (7) can be exploited for the design of an iterative algorithm that solves the problem with super-linear convergence. Without convexity, only the fixed-point iteration [14] is known, which is known to have linear convergence rate [16], [34].

II. Concave Interference Functions

In this section, we analyze the structure of arbitrary concave interference functions. Applications of the results will be discussed later in Section IV.

Definition 2: A function $\mathcal{I} : \mathbb{R}_+^K \mapsto \mathbb{R}_+$ is said to be a *concave interference function* if A1–A3 are fulfilled and in addition $\mathcal{I}$ is concave on $\mathbb{R}_+^K$.

Examples are the interference functions (1), (5), and (6).

A. Representation of Concave Interference Functions

A useful concept for analyzing general concave functions is the conjugate function (see, e.g., [38] and [42])

$$\underline{\mathcal{I}}^*(\boldsymbol{w}) = \inf_{\boldsymbol{p}>0} \left(\sum_{l=1}^K w_l p_l - \mathcal{I}(\boldsymbol{p}) \right), \qquad \boldsymbol{w} \in \mathbb{R}^K. \quad (9)$$

For the special problem at hand, we can exploit that $\mathcal{I}(\boldsymbol{p})$ is an interference function, i.e., properties A1–A3 are fulfilled in addition to convexity. This leads to the following observations.

Lemma 1: For any given $\boldsymbol{w} \in \mathbb{R}^K$, the conjugate (9) is either minus infinity or zero, i.e.,

$$\underline{\mathcal{I}}^*(\boldsymbol{w}) > -\infty \Leftrightarrow \underline{\mathcal{I}}^*(\boldsymbol{w}) = 0. \quad (10)$$

Proof: The norm of $\boldsymbol{p}$ in (9) is not constrained, thus for all $\mu > 0$

$$\begin{aligned}\underline{\mathcal{I}}^*(\boldsymbol{w}) &= \inf_{\boldsymbol{p}>0} \left(\sum_{l=1}^K w_l \cdot \mu p_l - \mathcal{I}(\mu \boldsymbol{p}) \right), \\ &= \mu \cdot \inf_{\boldsymbol{p}>0} \left(\sum_{l=1}^K w_l \cdot p_l - \mathcal{I}(\boldsymbol{p}) \right) = \mu \cdot \underline{\mathcal{I}}^*(\boldsymbol{w}). \quad (11)\end{aligned}$$

The second step follows from A2. Assume $\underline{\mathcal{I}}^*(\boldsymbol{w}) > -\infty$, then (11) can only hold for all $\mu > 0$ if $\underline{\mathcal{I}}^*(\boldsymbol{w}) = 0$.

Lemma 2: If $\boldsymbol{w}$ has a negative component, then $\underline{\mathcal{I}}^*(\boldsymbol{w}) = -\infty$. Proof: Assume $w_r < 0$ for some arbitrary index r. Introducing a power vector $\boldsymbol{p}(\lambda)$ with $p_l(\lambda) = 1$, $l \neq r$ and $p_l(\lambda) = \lambda$, $l = r$, where $\lambda \in \mathbb{R}_{++}$, we have

$$\begin{aligned}\underline{\mathcal{I}}^*(\boldsymbol{w}) &\leq \lambda \cdot w_r + \sum_{l \neq r} w_l - \mathcal{I}\left(\boldsymbol{p}(\lambda)\right), \\ &\leq \lambda \cdot w_r + \sum_{l \neq r} w_l = -\lambda \cdot |w_r| + \sum_{l \neq r} w_l.\end{aligned}$$

The first inequality follows from $\underline{\mathcal{I}}^*(\boldsymbol{w})$ being the infimum over all power vectors. The second inequality follows from axiom A1. Letting $\lambda \to \infty$, the right-hand side of the inequality tends to $-\infty$. ∎

From Lemmas 1 and 2, it can be concluded that the set of vectors $\boldsymbol{w}$ leading to a finite conjugate $\underline{\mathcal{I}}^*(\boldsymbol{w}) > -\infty$ is

$$\mathcal{N}_0(\mathcal{I}) = \{\boldsymbol{w} \in \mathbb{R}_+^K : \underline{\mathcal{I}}^*(\boldsymbol{w}) = 0\}. \tag{12}$$

Next, it is shown that every $\boldsymbol{w} \in \mathcal{N}_0(\mathcal{I})$ is associated with a hyperplane upper-bounding the interference function.

Lemma 3: For any $\boldsymbol{w} \in \mathcal{N}_0(\mathcal{I})$, we have

$$\mathcal{I}(\boldsymbol{p}) \le \sum_{l\in\mathcal{K}} w_l p_l, \quad \forall \boldsymbol{p} > 0. \tag{13}$$

Proof: With definition (12) we have

$$0 = \underline{\mathcal{I}}^*(\boldsymbol{w}) = \inf_{\hat{\boldsymbol{p}}>0}\left(\sum_{l=1}^{K} w_l \cdot \hat{p}_l - \mathcal{I}(\hat{\boldsymbol{p}})\right) \le \sum_{k\in\mathcal{K}} w_l \cdot p_l - \mathcal{I}(\boldsymbol{p})$$

for all $\boldsymbol{p} > 0$, thus (13) holds. ∎

This leads to our first main result, which shows that every concave interference function is characterized as a minimum over a sum of weighted powers.

Theorem 1: Let $\mathcal{I}$ be an arbitrary concave interference function, then

$$\mathcal{I}(\boldsymbol{p}) = \min_{\boldsymbol{w}\in\mathcal{N}_0(\mathcal{I})} \sum_{k\in\mathcal{K}} w_k p_k, \text{ for all } \boldsymbol{p} > 0. \tag{14}$$

Proof: Consider an arbitrary fixed $\boldsymbol{p} > 0$. Since $\mathcal{I}(\boldsymbol{p})$ is concave, we know that (see, e.g., [38] and [42]), there exists a vector $\tilde{\boldsymbol{w}} \in \mathbb{R}^K$ such that

$$\tilde{\boldsymbol{w}}^T\hat{\boldsymbol{p}} - \mathcal{I}(\hat{\boldsymbol{p}}) \ge \tilde{\boldsymbol{w}}^T\boldsymbol{p} - \mathcal{I}(\boldsymbol{p}) \text{ for all } \hat{\boldsymbol{p}} > 0. \tag{15}$$

The vector $\tilde{\boldsymbol{w}}$ must be non-negative, otherwise (15) cannot be fulfilled for all $\hat{\boldsymbol{p}} > 0$. This can be shown by contradiction. Suppose that $\tilde{w}_r < 0$ for some index r, and we choose $\hat{\boldsymbol{p}}_\epsilon$ such that $[\hat{\boldsymbol{p}}_\epsilon]_l = p_l, l \ne r$, and $[\hat{\boldsymbol{p}}_\epsilon]_r = p_r + \epsilon$, with $\epsilon > 0$. With A3 (monotonicity), we know that $\hat{\boldsymbol{p}}_\epsilon \ge \boldsymbol{p}$ implies $\mathcal{I}(\hat{\boldsymbol{p}}_\epsilon) \ge \mathcal{I}(\boldsymbol{p})$. Thus, (15) leads to $0 \le \tilde{\boldsymbol{w}}^T(\hat{\boldsymbol{p}}_\epsilon - \boldsymbol{p}) = \epsilon \cdot \tilde{w}_r$. This contradicts the assumption $\tilde{w}_r < 0$. It was shown in [17] that the function $\mathcal{I}(\boldsymbol{p})$ is continuous on $\mathbb{R}_{++}^K$, thus $\boldsymbol{p} < +\infty$ implies $\mathcal{I}(\boldsymbol{p}) < +\infty$. Therefore

$$\tilde{\boldsymbol{w}}^T\boldsymbol{p} - \mathcal{I}(\boldsymbol{p}) > -\infty. \tag{16}$$

Inequality (15) holds for all $\hat{\boldsymbol{p}} > 0$. Taking the infimum and using (16), we have

$$\inf_{\hat{\boldsymbol{p}}>0}\left(\sum_{l\in\mathcal{K}} \tilde{w}_l \cdot \hat{p}_l - \mathcal{I}(\hat{\boldsymbol{p}})\right) \ge \tilde{\boldsymbol{w}}^T\boldsymbol{p} - \mathcal{I}(\boldsymbol{p}) > -\infty. \tag{17}$$

Comparison with (9) shows that $\underline{\mathcal{I}}^*(\tilde{\boldsymbol{w}}) > -\infty$ and therefore $\tilde{\boldsymbol{w}} \in \mathcal{N}_0(\mathcal{I})$. Lemma 3 implies

$$\mathcal{I}(\boldsymbol{p}) \le \sum_{l\in\mathcal{K}} \tilde{w}_l p_l \text{ for all } \boldsymbol{p} > 0. \tag{18}$$

Now, (15) holds for all $\hat{\boldsymbol{p}} > 0$, so it holds as well for $\lambda\hat{\boldsymbol{p}}$, with $\lambda > 0$. Because of property A2, we have $\mathcal{I}(\lambda\hat{\boldsymbol{p}}) = \lambda\mathcal{I}(\hat{\boldsymbol{p}})$, and thus

$$0 = \lim_{\lambda\to 0}\left(\lambda\tilde{\boldsymbol{w}}^T\hat{\boldsymbol{p}} - \lambda\mathcal{I}(\hat{\boldsymbol{p}})\right) \ge \tilde{\boldsymbol{w}}^T\boldsymbol{p} - \mathcal{I}(\boldsymbol{p}). \tag{19}$$

Thus, $\mathcal{I}(\boldsymbol{p}) \ge \tilde{\boldsymbol{w}}^T\boldsymbol{p}$. Comparison with (18) shows that this inequality can only be fulfilled with equality. It can be concluded that for any $\boldsymbol{p} > 0$, there exists a $\tilde{\boldsymbol{w}} \in \mathcal{N}_0(\mathcal{I})$ which minimizes $\boldsymbol{w}^T\boldsymbol{p}$, such that the lower bound $\mathcal{I}(\boldsymbol{p})$ is achieved. Hence, (14) holds. ∎

The proof shows that every $\tilde{\boldsymbol{w}}$ fulfilling (15) for a given point $\boldsymbol{p}$, is a minimizer of (14). Conversely, any $\tilde{\boldsymbol{w}} \in \mathcal{N}_0(\mathcal{I})$ which fulfills

$$\mathcal{I}(\boldsymbol{p}) = \min_{\boldsymbol{w}\in\mathcal{N}_0(\mathcal{I})} \sum_{l\in\mathcal{K}} w_l p_l = \sum_{l\in\mathcal{K}} \tilde{w}_l p_l \tag{20}$$

also fulfills the inequality (15). This is a consequence of Lemma 3, which leads to

$$\begin{aligned}\mathcal{I}(\hat{\boldsymbol{p}}) - \mathcal{I}(\boldsymbol{p}) &= \mathcal{I}(\hat{\boldsymbol{p}}) - \sum_{l\in\mathcal{K}} \tilde{w}_l p_l \\ &\le \sum_{l\in\mathcal{K}} \tilde{w}_l(\hat{p} - p_l) \text{ for all } \hat{\boldsymbol{p}} > 0.\end{aligned}$$

Thus, for any given $\boldsymbol{p} > 0$, the set of optimal coefficients $\tilde{\boldsymbol{w}}$ achieving the minimum (20), is identical to the set of $\tilde{\boldsymbol{w}} \in \mathcal{N}_0(\mathcal{I})$ for which (15) is fulfilled.

B. *Properties of the Set* $\mathcal{N}_0(\mathcal{I})$

Theorem 1 shows that an arbitrary concave interference function $\mathcal{I}$ can be characterized as the minimum of a weighted sum of powers, optimized over the set $\mathcal{N}_0(\mathcal{I})$. In this section, we will further analyze the relationship between $\mathcal{I}$ and $\mathcal{N}_0(\mathcal{I})$. The results will be needed later, e.g., in Section IV-B.

Definition 3: A set $\mathcal{V} \subseteq \mathbb{R}_+^K$ is said to be *upward-comprehensive* if for all $\boldsymbol{w} \in \mathcal{V}$ and $\boldsymbol{w}' \in \mathbb{R}_+^K$

$$\boldsymbol{w}' \ge \boldsymbol{w} \Rightarrow \boldsymbol{w}' \in \mathcal{V}. \tag{21}$$

Definition 4: A set is said to be UCCC if it is upward-comprehensive closed convex.

Lemma 4: Let $\mathcal{I}$ be a concave interference function, then $\mathcal{N}_0(\mathcal{I}) \subseteq \mathbb{R}_+^K$, as defined by (12), is a nonempty UCCC set.

Proof: From the proof of Theorem 1 it is clear that $\mathcal{N}_0(\mathcal{I})$ is nonempty. This is a consequence of the concavity of $\mathcal{I}$.

Now, we show convexity of $\mathcal{N}_0(\mathcal{I})$. Let $\hat{\boldsymbol{w}}, \check{\boldsymbol{w}} \in \mathcal{N}_0(\mathcal{I})$ and $\boldsymbol{w}(\lambda) = (1-\lambda)\hat{\boldsymbol{w}} + \lambda\check{\boldsymbol{w}}$. Using $\mathcal{I}(\boldsymbol{p}) = (1-\lambda)\mathcal{I}(\boldsymbol{p}) + \lambda\mathcal{I}(\boldsymbol{p})$, we have

$$\begin{aligned}\underline{\mathcal{I}}^*\left(\boldsymbol{w}(\lambda)\right) &= \inf_{\boldsymbol{p}>0}\left((1-\lambda)\sum_{l\in\mathcal{K}} \hat{w}_l p_l + \lambda\sum_{l\in\mathcal{K}} \check{w}_l p_l - \mathcal{I}(\boldsymbol{p})\right) \\ &\ge (1-\lambda)\inf_{\boldsymbol{p}>0}\left(\sum_{l\in\mathcal{K}} \hat{w}_l p_l - \mathcal{I}(\boldsymbol{p})\right) + \end{aligned} \tag{22}$$

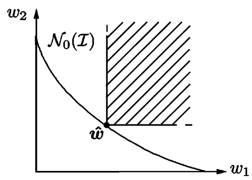

Fig. 1. Illustration of Lemma 4: the coefficient set $\mathcal{N}_0(\mathcal{I})$ is upward-comprehensive closed convex (UCCC). For any $\hat{\boldsymbol{w}} \in \mathcal{N}_0(\mathcal{I})$, all points $\boldsymbol{w} \geq \hat{\boldsymbol{w}}$ (shaded box) are also contained in $\mathcal{N}_0(\mathcal{I})$.

$$+ \lambda \inf_{\boldsymbol{p}>0}\left(\sum_{l\in\mathcal{K}} \check{w}_l p_l - \mathcal{I}(\boldsymbol{p})\right)$$
$$= (1-\lambda)\underline{\mathcal{I}}^*(\hat{\boldsymbol{w}}) + \lambda\underline{\mathcal{I}}^*(\check{\boldsymbol{w}}) > -\infty. \tag{23}$$

Thus, $\boldsymbol{w}(\lambda) \in \mathcal{N}_0(\mathcal{I})$, which proves convexity.

Now, we show that $\mathcal{N}_0(\mathcal{I})$ is closed. Let $\boldsymbol{w}^{(n)}$ be an arbitrary convergent Cauchy sequence in $\mathcal{N}_0(\mathcal{I})$, i.e., there exists a $\boldsymbol{w}^*$ such that $\lim_{n\to\infty} w_k^{(n)} = w_k^*$ for all components $k \in \mathcal{K}$. We need to show that the limit $\boldsymbol{w}^*$ is also contained in $\mathcal{N}_0(\mathcal{I})$.

Since $\boldsymbol{w}^{(n)} \in \mathbb{R}_+^K$, also $\boldsymbol{w}^* \in \mathbb{R}_+^K$. For an arbitrary fixed $\boldsymbol{p} > 0$, we have

$$\sum_{k\in\mathcal{K}} w_k^* p_k - \mathcal{I}(\boldsymbol{p}) = \lim_{n\to\infty}\left(\sum_{k\in\mathcal{K}} w_k^{(n)} p_k - \mathcal{I}(\boldsymbol{p})\right)$$
$$\geq \liminf_{n\to\infty}\left(\inf_{\tilde{\boldsymbol{p}}>0}\left(\sum_{k\in\mathcal{K}} w_k^{(n)}\tilde{p}_k - \mathcal{I}(\tilde{\boldsymbol{p}})\right)\right)$$
$$= \liminf_{n\to\infty}\left(\underline{\mathcal{I}}^*(\boldsymbol{w}^{(n)})\right) = 0. \tag{24}$$

The last step follows from $\boldsymbol{w}^{(n)} \in \mathcal{N}_0(\mathcal{I})$, which implies $\underline{\mathcal{I}}^*(\boldsymbol{w}^{(n)}) = 0$ for all n. Since inequality (24) holds for all $\boldsymbol{p} > 0$, we have

$$\underline{\mathcal{I}}^*(\boldsymbol{w}^*) = \inf_{\boldsymbol{p}>0}\left(\sum_{l\in\mathcal{K}} w_l^* p_l - \mathcal{I}(\boldsymbol{p})\right) \geq 0 > -\infty. \tag{25}$$

Thus, $\boldsymbol{w}^* \in \mathcal{N}_0(\mathcal{I})$, which proves that $\mathcal{N}_0(\mathcal{I})$ is closed. It remains to show upward-comprehensiveness. Consider an arbitrary $\hat{\boldsymbol{w}} \in \mathcal{N}_0(\mathcal{I})$. If $\boldsymbol{w} \geq \hat{\boldsymbol{w}}$ then

$$\sum_{l\in\mathcal{K}} p_l w_l - \mathcal{I}(\boldsymbol{p}) \geq \sum_{l\in\mathcal{K}} p_l \hat{w}_l - \mathcal{I}(\boldsymbol{p}) \geq \underline{\mathcal{I}}^*(\hat{\boldsymbol{w}}) > -\infty$$

for all $\boldsymbol{p} > 0$. Thus, $\boldsymbol{w} \in \mathcal{N}_0(\mathcal{I})$. ∎

Remark: The proof of Lemma 4 does not rely on concavity, except for the comment on non-emptiness. Thus, $\mathcal{N}_0(\mathcal{I})$ is a UCCC set for any interference function fulfilling A1–A3.

Thus far, we have analyzed the elementary building blocks of concave interference functions. Lemma 4 shows that any concave interference function $\mathcal{I}$ is associated with a UCCC coefficient set $\mathcal{N}_0(\mathcal{I})$, as illustrated in Fig. 1. This approach, and the resulting representation (14), can be referred to as *analysis*.

Next, we study the converse approach, namely the *synthesis* of a concave interference function. Starting from an arbitrary nonempty UCCC set $\mathcal{V} \subseteq \mathbb{R}_+^K$, we can construct a function

$$\mathcal{I}_{\mathcal{V}}(\boldsymbol{p}) = \min_{\boldsymbol{w}\in\mathcal{V}} \sum_{l\in\mathcal{K}} w_l p_l. \tag{26}$$

It can be verified that $\mathcal{I}_{\mathcal{V}}$ is concave and fulfills the properties A1–A3. Thus, every UCCC set leads to a concave interference function.

The next theorem shows that the operations *analysis* and *synthesis* are reversible.

Theorem 2: For any nonempty UCCC set $\mathcal{V} \subseteq \mathbb{R}_+^K$ we have

$$\mathcal{V} = \mathcal{N}_0(\mathcal{I}_{\mathcal{V}}). \tag{27}$$

Proof: Consider an arbitrary $\boldsymbol{v} \in \mathcal{V}$. With (26), we have

$$\underline{\mathcal{I}}^*(\boldsymbol{v}) = \inf_{\boldsymbol{p}>0}\left(\sum_{l\in\mathcal{K}} v_l p_l - \mathcal{I}_{\mathcal{V}}(\boldsymbol{p})\right)$$
$$\geq \inf_{\boldsymbol{p}>0}\left(\sum_{l\in\mathcal{K}} v_l p_l - \sum_{l\in\mathcal{K}} v_l p_l\right) = 0. \tag{28}$$

Thus, $\boldsymbol{v} \in \mathcal{N}_0(\mathcal{I}_{\mathcal{V}})$, and consequently $\mathcal{V} \subseteq \mathcal{N}_0(\mathcal{I}_{\mathcal{V}})$. Next, equality is shown by contradiction. Hence, we suppose $\mathcal{V} \neq \mathcal{N}_0(\mathcal{I}_{\mathcal{V}})$. This implies the existence of a $\hat{\boldsymbol{w}} > 0$ with $\hat{\boldsymbol{w}} \notin \mathcal{V}$ and $\hat{\boldsymbol{w}} \in \mathcal{N}_0(\mathcal{I}_{\mathcal{V}})$. Note, that $\hat{\boldsymbol{w}}$ can be assumed to be strictly positive since $\mathbb{R}_{++}^K \cap \mathcal{V} \neq \mathbb{R}_{++}^K \cap \mathcal{N}_0(\mathcal{I}_{\mathcal{V}})$, otherwise we would have the contradiction

$$\mathcal{V} = \overline{\mathbb{R}_{++}^K \cap \mathcal{V}} = \overline{\mathbb{R}_{++}^K \cap \mathcal{N}_0(\mathcal{I}_{\mathcal{V}})} = \mathcal{N}_0(\mathcal{I}_{\mathcal{V}}).$$

Now, we can exploit that the set $\mathcal{V}$ is convex and its intersection with $\mathbb{R}_{++}^K$ is nonempty (this follows from comprehensiveness). From the separating hyperplanes theorem (see, e.g., [38] or [42, Theorem 4.1.1, p. 51]), we known that there is a $\hat{\boldsymbol{p}} > 0$ such that

$$\mathcal{I}_{\mathcal{V}}(\hat{\boldsymbol{p}}) = \min_{\boldsymbol{v}\in\mathcal{V}} \sum_{l\in\mathcal{K}} v_l \hat{p}_l$$
$$> \sum_{l\in\mathcal{K}} \hat{w}_l \hat{p}_l$$
$$\geq \min_{\boldsymbol{w}\in\mathcal{N}_0(\mathcal{I}_{\mathcal{V}})} \sum_{l\in\mathcal{K}} w_l \hat{p}_l = \mathcal{I}_{\mathcal{V}}(\hat{\boldsymbol{p}}) \tag{29}$$

where the last equality follows from Theorem 1. This is a contradiction, thus $\mathcal{V} = \mathcal{N}_0(\mathcal{I}_{\mathcal{V}})$. ∎

The next corollary shows a one-to-one correspondence between any concave interference function $\mathcal{I}$ and its UCCC set $\mathcal{N}_0(\mathcal{I})$.

Corollary 1: Let $\mathcal{N}^{(1)}$ and $\mathcal{N}^{(2)}$ be two arbitrary UCCC sets from $\mathbb{R}_+^K$. If $\mathcal{I}_{\mathcal{N}^{(1)}}(\boldsymbol{p}) = \mathcal{I}_{\mathcal{N}^{(2)}}(\boldsymbol{p})$ for all $\boldsymbol{p} > 0$, then $\mathcal{N}^{(1)} = \mathcal{N}^{(2)}$.

Proof: The assumption implies $\mathcal{N}_0(\mathcal{I}_{\mathcal{N}^{(1)}}) = \mathcal{N}_0(\mathcal{I}_{\mathcal{N}^{(2)}})$. The result follows with Theorem 2, which shows $\mathcal{N}^{(l)} = \mathcal{N}_0(\mathcal{I}_{\mathcal{N}^{(l)}})$. ∎

The results show a one-to-one correspondence between concave interference functions and UCCC sets. Every concave interference function $\mathcal{I}$ is uniquely associated with an UCCC set $\mathcal{N}_0(\mathcal{I})$. Conversely, every UCCC set $\mathcal{V}$ is uniquely associated with an interference function $\mathcal{I}_{\mathcal{V}}$. We have $\mathcal{I} = \mathcal{I}_{\mathcal{N}_0(\mathcal{I})}$ and $\mathcal{V} = \mathcal{N}_0(\mathcal{I}_{\mathcal{V}})$.

The UCCC set $\mathcal{N}_0(\mathcal{I})$ has an interesting interpretation in the context of network resource allocation. Suppose that w_k stands for some QoS measure, like bit error rate, or delay. For certain choices of system parameters the QoS region is convex (see, e.g., [17]). The weights p_l can be chosen such that individual

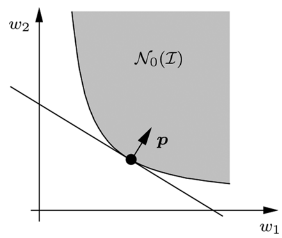

Fig. 2. The concave interference function $\mathcal{I}(\boldsymbol{p})$ can be interpreted as the minimum of a weighted sum-cost function optimized over the convex set $\mathcal{N}_0$. The "weighting vector" $\boldsymbol{p}$ controls the tradeoff between the utilities w_k.

user priorities are included. Then, $\mathcal{I}(\boldsymbol{p})$ is the minimum network cost obtained by optimizing over the boundary of the QoS region $\mathcal{N}_0(\mathcal{I})$, as illustrated in Fig. 2. This shows a connection between the axiomatic framework of interference functions and resource allocation problems. Further aspects will be discussed in Section IV.

III. Convex Interference Functions

In this section, we analyze the structure of *convex* interference functions.

Definition 5: A function $\mathcal{I} : \mathbb{R}^K_+ \mapsto \mathbb{R}_+$ is said to be a *convex interference function* if A1–A3 are fulfilled and in addition $\mathcal{I}$ is convex on $\mathbb{R}^K_+$.

Most results can be derived in a similar way to the concave functions studied in Section II. However, there are slight differences, which will be pointed out.

A. Representation of Convex Interference Functions

The conjugate function for the convex case is [38]

$$\overline{\mathcal{I}}^*(\boldsymbol{w}) = \sup_{\boldsymbol{p}>0}\left(\sum_{l\in\mathcal{K}} w_l p_l - \mathcal{I}(\boldsymbol{p})\right). \tag{30}$$

Exploiting the special properties A1–A3, we obtain the next result.

Lemma 5: The conjugate function (30) is either infinity or zero, i.e.,

$$\overline{\mathcal{I}}^*(\boldsymbol{w}) < +\infty \Leftrightarrow \overline{\mathcal{I}}^*(\boldsymbol{w}) = 0. \tag{31}$$

Proof: This is shown in a similar way to the proof of Lemma 1. ■

Furthermore, the monotonicity axiom A3 implies that we can focus on non-negative coefficients. This will become clear later, in the proof of Theorem 3. Therefore, the coefficient set of interest is

$$\mathcal{W}_0(\mathcal{I}) = \{\boldsymbol{w} \in \mathbb{R}^K_+ : \overline{\mathcal{I}}^*(\boldsymbol{w}) = 0\}. \tag{32}$$

Every $\boldsymbol{w} \in \mathcal{W}_0(\mathcal{I})$ is associated with a hyperplane which lower bounds the interference function.

Lemma 6: For any $\boldsymbol{w} \in \mathcal{W}_0(\mathcal{I})$

$$\sum_{l\in\mathcal{K}} w_l p_l \le \mathcal{I}(\boldsymbol{p}), \qquad \forall \boldsymbol{p} > 0. \tag{33}$$

Proof: For all $\boldsymbol{p} > 0$, we have

$$0 = \overline{\mathcal{I}}^*(\boldsymbol{w}) = \sup_{\hat{\boldsymbol{p}}>0}\left(\sum_{l\in\mathcal{K}} w_l \cdot \hat{p}_l - \mathcal{I}(\hat{\boldsymbol{p}})\right) \ge \sum_{l\in\mathcal{K}} w_l \cdot p_l - \mathcal{I}(\boldsymbol{p}).$$

Thus, (33) holds. ■

Based on this lemma, we will now show that every convex interference function can always be characterized as a maximum sum of weighted powers.

Theorem 3: Let $\mathcal{I}$ be an arbitrary convex interference function, then

$$\mathcal{I}(\boldsymbol{p}) = \max_{\boldsymbol{w}\in\mathcal{W}_0(\mathcal{I})} \sum_{k\in\mathcal{K}} w_k \cdot p_k, \text{ for all } \boldsymbol{p} > 0. \tag{34}$$

Proof: Consider an arbitrary fixed $\boldsymbol{p} > 0$. Since $\mathcal{I}(\boldsymbol{p})$ is convex, there exists a vector $\tilde{\boldsymbol{w}} \in \mathbb{R}^K$ such that [42, Theorem 1.2.1, p. 77]

$$\tilde{\boldsymbol{w}}^T\hat{\boldsymbol{p}} - \mathcal{I}(\hat{\boldsymbol{p}}) \le \tilde{\boldsymbol{w}}^T\boldsymbol{p} - \mathcal{I}(\boldsymbol{p}) \text{ for all } \hat{\boldsymbol{p}} > 0. \tag{35}$$

The vector $\tilde{\boldsymbol{w}}$ must be non-negative, otherwise (35) cannot be fulfilled for all $\hat{\boldsymbol{p}} > 0$. This can be shown by contradiction. Suppose that $\tilde{w}_r < 0$ for some index r, and we choose $\hat{\boldsymbol{p}}_\epsilon > 0$ such that $[\hat{\boldsymbol{p}}_\epsilon]_l = p_l$, $l \neq r$, and $[\hat{\boldsymbol{p}}_\epsilon]_r = p_r - \epsilon$, with $0 < \epsilon < p_r$. With A3 (monotonicity), we know that $\hat{\boldsymbol{p}}_\epsilon \le \boldsymbol{p}$ implies $\mathcal{I}(\hat{\boldsymbol{p}}_\epsilon) \le \mathcal{I}(\boldsymbol{p})$. Thus, (35) leads to $0 \ge \tilde{\boldsymbol{w}}^T(\hat{\boldsymbol{p}}_\epsilon - \boldsymbol{p}) = -\epsilon \cdot \tilde{w}_r$. This contradicts the assumption $\tilde{w}_r < 0$. Because of the non-negativity of $\mathcal{I}(\boldsymbol{p})$, we have

$$\tilde{\boldsymbol{w}}^T\boldsymbol{p} - \mathcal{I}(\boldsymbol{p}) < +\infty. \tag{36}$$

Inequality (35) holds for all $\hat{\boldsymbol{p}} > 0$. Taking the supremum and using (36), we have

$$\sup_{\hat{\boldsymbol{p}}>0}\left(\sum_{l\in\mathcal{K}} \tilde{w}_l \cdot \hat{p}_l - \mathcal{I}(\hat{\boldsymbol{p}})\right) \le \tilde{\boldsymbol{w}}^T\boldsymbol{p} - \mathcal{I}(\boldsymbol{p}) < +\infty. \tag{37}$$

Comparison with the conjugate (30) shows that $\overline{\mathcal{I}}^*(\tilde{\boldsymbol{w}}) < +\infty$ and therefore $\tilde{\boldsymbol{w}} \in \mathcal{W}_0(\mathcal{I})$. Lemma 6 implies

$$\tilde{\boldsymbol{w}}^T\boldsymbol{p} \le \mathcal{I}(\boldsymbol{p}), \qquad \forall \boldsymbol{p} > 0. \tag{38}$$

Inequality (35) holds for all $\hat{\boldsymbol{p}}$, so it holds as well for $\lambda\hat{\boldsymbol{p}}$, with an arbitrary $\lambda > 0$. With A2, we have

$$\tilde{\boldsymbol{w}}^T\boldsymbol{p} - \mathcal{I}(\boldsymbol{p}) \ge \lim_{\lambda\to 0}\left(\tilde{\boldsymbol{w}}^T\lambda\hat{\boldsymbol{p}} - \lambda\mathcal{I}(\hat{\boldsymbol{p}})\right) = 0. \tag{39}$$

By combining (38) and (39), it can be concluded that $\mathcal{I}(\boldsymbol{p}) = \tilde{\boldsymbol{w}}^T\boldsymbol{p}$. Thus, $\tilde{\boldsymbol{w}}$ is the maximizer of (34). ■

From the proof of Theorem 3, it becomes clear that the maximizer of (34) is always non-negative. Also, the set $\mathcal{W}_0(\mathcal{I})$ is nonempty.

B. Properties of the Set $\mathcal{W}_0(\mathcal{I})$

We now investigate how the properties of an arbitrary convex interference function $\mathcal{I}$ influence the properties of the resulting set $\mathcal{W}_0(\mathcal{I})$. We begin with some definitions.

Definition 6: A set $\mathcal{V} \subseteq \mathbb{R}_+^K$ is said to be downward-comprehensive if for all $\boldsymbol{w} \in \mathcal{V}$ and $\boldsymbol{w}' \in \mathbb{R}_+^K$

$$\boldsymbol{w}' \leq \boldsymbol{w} \Rightarrow \boldsymbol{w}' \in \mathcal{V}. \tag{40}$$

Definition 7: A set is said to be DCCC if it is downward-comprehensive closed convex.

Lemma 7: Let $\mathcal{I}$ be a convex interference function, then the set $\mathcal{W}_0(\mathcal{I})$, as defined by (32), is nonempty, bounded, and DCCC.

Proof: First, convexity is shown. Let $\hat{\boldsymbol{w}}, \check{\boldsymbol{w}} \in \mathcal{W}_0(\mathcal{I})$ and $\boldsymbol{w}(\lambda) = (1-\lambda)\hat{\boldsymbol{w}} + \lambda\check{\boldsymbol{w}}$. Similar to (23), we can show

$$\overline{\mathcal{I}}^*\Big(\boldsymbol{w}(\lambda)\Big) \leq (1-\lambda)\overline{\mathcal{I}}^*(\hat{\boldsymbol{w}}) + \lambda\overline{\mathcal{I}}^*(\check{\boldsymbol{w}}) < +\infty. \tag{41}$$

Thus, $\boldsymbol{w}(\lambda) \in \mathcal{W}_0(\mathcal{I})$.

Next, we show that the set is upper-bounded. Consider an arbitrary $\boldsymbol{w} \in \mathcal{W}_0(\mathcal{I})$. With (34), we have

$$\sum_{l\in\mathcal{K}} w_l \leq \max_{\boldsymbol{w}\in\mathcal{W}_0(\mathcal{I})} \sum_{k\in\mathcal{K}} w_k = \mathcal{I}(\mathbf{1}) \tag{42}$$

where $\mathbf{1}$ is the all-ones vector. Property A3 implies $\mathcal{I}(\mathbf{1}) < +\infty$, thus $\mathcal{W}_0(\mathcal{I})$ is bounded.

Now, we show that $\mathcal{W}_0(\mathcal{I})$ is closed. Let $\boldsymbol{w}^{(n)}$ be an arbitrary convergent Cauchy sequence in $\mathcal{W}_0(\mathcal{I})$, i.e., there exists a $\boldsymbol{w}^*$ such that $\lim_{n\to\infty} w_k^{(n)} = w_k^*$ for all components $k \in \mathcal{K}$. We need to show that the limit $\boldsymbol{w}^*$ is also contained in $\mathcal{W}_0(\mathcal{I})$.

Since $\boldsymbol{w}^{(n)} \in \mathbb{R}_+^K$, also $\boldsymbol{w}^* \in \mathbb{R}_+^K$. For an arbitrary fixed $\boldsymbol{p} > 0$, we have

$$\begin{aligned}\sum_{k\in\mathcal{K}} w_k^* p_k - \mathcal{I}(\boldsymbol{p}) &= \lim_{n\to\infty}\left(\sum_{k\in\mathcal{K}} w_k^{(n)} p_k - \mathcal{I}(\boldsymbol{p})\right)\\ &\leq \limsup_{n\to\infty}\left(\sup_{\tilde{\boldsymbol{p}}>0}\left(\sum_{k\in\mathcal{K}} w_k^{(n)}\tilde{p}_k - \mathcal{I}(\tilde{\boldsymbol{p}})\right)\right)\\ &= \limsup_{n\to\infty}\left(\overline{\mathcal{I}}^*(\boldsymbol{w}^{(n)})\right) = 0. \end{aligned}\tag{43}$$

The last step follows from $\boldsymbol{w}^{(n)} \in \mathcal{W}_0(\mathcal{I})$, which implies $\overline{\mathcal{I}}^*(\boldsymbol{w}^{(n)}) = 0$. Since inequality (43) holds for all $\boldsymbol{p} > 0$, we have

$$\overline{\mathcal{I}}^*(\boldsymbol{w}^*) = \sup_{\boldsymbol{p}>0}\left(\sum_{l\in\mathcal{K}} w_l^* p_l - \mathcal{I}(\boldsymbol{p})\right) \leq 0 < +\infty. \tag{44}$$

Thus, $\boldsymbol{w}^* \in \mathcal{W}_0(\mathcal{I})$, which proves that $\mathcal{W}_0(\mathcal{I})$ is closed. In order to show downward-comprehensiveness, consider an arbitrary $\hat{\boldsymbol{w}} \in \mathcal{W}_0(\mathcal{I})$. For any $\boldsymbol{w} \in \mathbb{R}_+^K$ with $\boldsymbol{w} \leq \hat{\boldsymbol{w}}$, we have

$$\sum_{l\in\mathcal{K}} p_l w_l - \mathcal{I}(\boldsymbol{p}) \leq \sum_{l\in\mathcal{K}} p_l \hat{w}_l - \mathcal{I}(\boldsymbol{p}) \leq \overline{\mathcal{I}}^*(\hat{\boldsymbol{w}}) < +\infty$$

for all $\boldsymbol{p} > 0$, thus $\boldsymbol{w} \in \mathcal{W}_0(\mathcal{I})$. ∎

The proof of Lemma 7 does not rely on convexity, except for showing nonemptiness and boundedness. Thus, $\mathcal{W}_0(\mathcal{I})$ is a DCCC set for any interference function fulfilling A1–A3. The result is illustrated in Fig. 3.

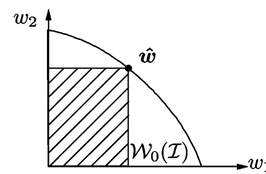

Fig. 3. Illustration of Lemma 7: the coefficient set $\mathcal{W}_0(\mathcal{I})$ is downward-comprehensive closed convex (DCCC). For any $\hat{\boldsymbol{w}} \in \mathcal{W}_0(\mathcal{I})$, all points $\boldsymbol{w} \leq \hat{\boldsymbol{w}}$ (shaded box) are also contained in $\mathcal{W}_0(\mathcal{I})$.

Next, consider the converse approach, i.e., the *synthesis* of a convex interference function from a bounded DCCC set. The function

$$\mathcal{I}_{\mathcal{V}}(\boldsymbol{p}) = \max_{\boldsymbol{w}\in\mathcal{V}} \sum_{l\in\mathcal{K}} p_l w_l \tag{45}$$

is a convex interference function which fulfills A1–A3.

Similar to the results of Section II, the operations *analysis* and *synthesis* are shown to be reversible:

Theorem 4: For any nonempty DCCC set $\mathcal{V} \subseteq \mathbb{R}_+^K$ we have

$$\mathcal{V} = \mathcal{W}_0(\mathcal{I}_{\mathcal{V}}). \tag{46}$$

Proof: Consider an arbitrary $\boldsymbol{v} \in \mathcal{V}$. Lemma 6 implies

$$\begin{aligned}\overline{\mathcal{I}}^*(\boldsymbol{v}) &= \sup_{\boldsymbol{p}>0}\left(\sum_{l\in\mathcal{K}} v_l p_l - \mathcal{I}_{\mathcal{V}}(\boldsymbol{p})\right)\\ &\leq \sup_{\boldsymbol{p}>0}\left(\sum_{l\in\mathcal{K}} v_l p_l - \sum_{l\in\mathcal{K}} v_l p_l\right) = 0.\end{aligned}\tag{47}$$

With Lemma 5, we have $\boldsymbol{v} \in \mathcal{W}_0(\mathcal{I}_{\mathcal{V}})$, and consequently $\mathcal{V} \subseteq \mathcal{W}_0(\mathcal{I}_{\mathcal{V}})$. Similar to the proof of Theorem 2, we can show by contradiction that this can only be fulfilled with equality. Suppose that $\mathcal{V} \neq \mathcal{W}_0(\mathcal{I}_{\mathcal{V}})$, then this implies the existence of a $\hat{\boldsymbol{w}} \in \mathcal{W}_0(\mathcal{I}_{\mathcal{V}})$ with $\hat{\boldsymbol{w}} \notin \mathcal{V}$ and $\hat{\boldsymbol{w}} > 0$. Applying the theorem of separating hyperplanes, we know that there is a $\hat{\boldsymbol{p}} > 0$ such that

$$\begin{aligned}\mathcal{I}_{\mathcal{V}}(\hat{\boldsymbol{p}}) &= \max_{\boldsymbol{w}\in\mathcal{V}} \sum_{l\in\mathcal{K}} w_l \hat{p}_l < \sum_{l\in\mathcal{K}} \hat{w}_l \hat{p}_l\\ &= \max_{\boldsymbol{w}\in\mathcal{W}_0(\mathcal{I}_{\mathcal{V}})} \sum_{l\in\mathcal{K}} w_l \hat{p}_l = \mathcal{I}_{\mathcal{V}}(\hat{\boldsymbol{p}})\end{aligned}\tag{48}$$

where the last equality follows from (3). This is a contradiction, thus $\mathcal{V} = \mathcal{W}_0(\mathcal{I}_{\mathcal{V}})$. ∎

The next corollary shows that there is a one-to-one correspondence between any convex interference function $\mathcal{I}$ and the respective DCCC set $\mathcal{W}_0(\mathcal{I})$.

Corollary 2: Let $\mathcal{W}_1$ and $\mathcal{W}_2$, be two arbitrary DCCC sets from $\mathbb{R}_+^K$. If $\mathcal{I}_{\mathcal{W}_1}(\boldsymbol{p}) = \mathcal{I}_{\mathcal{W}_2}(\boldsymbol{p})$ for all $\boldsymbol{p} > 0$, then $\mathcal{W}_1 = \mathcal{W}_2$.

Proof: The proof follows from Theorem 4. ∎

The results show that every convex interference function $\mathcal{I}$ can be interpreted as the maximum of the linear function $\sum_l p_l w_l$ over a bounded DCCC set $\mathcal{W}_0(\mathcal{I})$. Any convex interference function has an interpretation as a maximum of a weighted "sum utility function," as illustrated in Fig. 4.

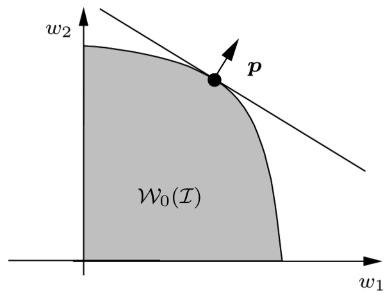

Fig. 4. Every convex interference function $\mathcal{I}(\boldsymbol{p})$ can be interpreted as the maximum of a weighted sum-utility function optimized over the convex set $\mathcal{W}_0(\mathcal{I})$. The "weighting vector" $\boldsymbol{p}$ controls the tradeoff between the utilities w_k.

C. *Log-Convex Interference Functions*

In this section, we study the class of *log-convex* interference functions. A function $f(\boldsymbol{s})$, with $\boldsymbol{s} \in \mathbb{R}^K$ is said to be log-convex if $\log f(\boldsymbol{s})$ is convex. An equivalent condition is [38]

$$f\left((1-\lambda)\hat{\boldsymbol{s}} + \lambda\check{\boldsymbol{s}}\right) \leq \left(f(\hat{\boldsymbol{s}})\right)^{1-\lambda} \cdot \left(f(\check{\boldsymbol{s}})\right)^{\lambda}, \quad \text{for all } \hat{\boldsymbol{s}}, \check{\boldsymbol{s}} \in \mathbb{R}^K.$$

Definition 8: A function $\mathcal{I} : \mathbb{R}_+^K \mapsto \mathbb{R}_+$ is said to be a *log-convex interference function* if $\mathcal{I}(\boldsymbol{p})$ fulfills A1–A3 and in addition $\mathcal{I}(\exp\{\boldsymbol{s}\})$ is log-convex on $\mathbb{R}^K$.

Here, we use a change of variable $\boldsymbol{p} = \mathrm{e}^{\boldsymbol{s}}$ (component-wise exponential). This technique was already used in [19]–[25] in order to exploit a "hidden convexity" of functions which are otherwise nonconvex.

Examples of log-convex interference functions are (7) and (8). Another well-known example is the case of linear interference functions (see, e.g., [26]–[30]). The log-convexity of linear interference functions can be exploited for the analysis of the SIR feasible set.

1) Example 1: Consider K linear interference functions $\mathcal{I}_k(\boldsymbol{p}) = [\boldsymbol{V}\boldsymbol{p}]_k$, $k \in \mathcal{K}$, where $\boldsymbol{V} \in \mathbb{R}_+^{K \times K}$ is an irreducible matrix containing the interference coupling coefficients. The SIR feasible region can be defined as

$$\mathcal{S} = \{\boldsymbol{\gamma} > 0 : \rho\left(\mathrm{diag}\{\boldsymbol{\gamma}\}\boldsymbol{V}\right) \leq 1\} \tag{49}$$

where $\rho(\boldsymbol{\gamma}) := \rho\big(\mathrm{diag}\{\boldsymbol{\gamma}\}\boldsymbol{V}\big)$ is the *spectral radius* of the weighted coupling matrix.

It was observed in [19], with extensions in [20], [22], and [23], that the SIR set $\mathcal{S}$ is convex on a logarithmic scale. This is because the spectral radius $\rho(\boldsymbol{\gamma})$ is log-convex after a change of variable $\boldsymbol{\gamma} = \mathrm{e}^{\boldsymbol{q}}$, where $\boldsymbol{q} \in \mathbb{R}^K$ is the logarithmic SIR.

Note, that $\rho(\boldsymbol{\gamma})$ also fulfills the axioms A1–A3. That is, the spectral radius is a special case of the more general framework of log-convex interference functions.

Some elementary properties of log-convex interference functions are as follows.

- The sum of log-convex interference functions is a log-convex interference function.
- Let $\mathcal{I}^{(1)}$ and $\mathcal{I}^{(2)}$ be log-convex interference functions, then
$$\mathcal{I}(\boldsymbol{p}) = \left(\mathcal{I}^{(1)}(\boldsymbol{p})\right)^{1-\alpha} \cdot \left(\mathcal{I}^{(2)}(\boldsymbol{p})\right)^{\alpha}, \qquad 0 \leq \alpha \leq 1$$
is also a log-convex interference function.
- Let $\mathcal{I}^{(n)}(\boldsymbol{p})$ be a sequence of log-convex interference functions, which converges to a limit $\lim_{n\to\infty} \mathcal{I}^{(n)}(\boldsymbol{p}) = \hat{\mathcal{I}}(\boldsymbol{p}) > 0$ for all $\boldsymbol{p} > 0$, then $\hat{\mathcal{I}}$ is also a log-convex interference function.

It was already shown in [17] that every convex interference function is log-convex after the change of variable $\boldsymbol{s} = \log \boldsymbol{p}$. The same result can be shown in a simpler and more direct way by exploiting the structure result of Theorem 3.

Theorem 5: Every convex interference function is a log-convex interference function in the sense of Definition 8.

Proof: Theorem 3 shows that every convex function $\mathcal{I}(\boldsymbol{p})$ can be expressed as $\max_{\boldsymbol{w} \in \mathcal{W}_0(\mathcal{I})} \sum_k w_k p_k$. The function $g(\mathrm{e}^{\boldsymbol{s}}) = \sum_k w_k \mathrm{e}^{s_k}$ is log-convex, i.e., $\log g(\mathrm{e}^{\boldsymbol{s}})$ is convex. Maximization preserves convexity, thus $\max_{\boldsymbol{w} \in \mathcal{W}_0(\mathcal{I})} \log g(\mathrm{e}^{\boldsymbol{s}})$ is convex as well. The result follows from interchanging log and max.

Theorem 5 shows that the class of log-convex interference functions contains convex interference functions as a special case.

Next, consider the function $\xi(\boldsymbol{p}) = \prod_{l \in \mathcal{K}} (p_l)^{w_l}$. The function ξ is a log-convex interference function if and only if the coefficients $\boldsymbol{w} = [w_1, \ldots, w_K]^T$ fulfill $\boldsymbol{w} \in \mathbb{R}_+^K$ and $\|\boldsymbol{w}\|_1 = 1$. Non-negativity is required for axiom A3 (monotonicity), and unity norm $\|\boldsymbol{w}\|_1 = 1$ is required for A2 (scale invariance). This becomes clear when writing

$$\xi(\alpha\boldsymbol{p}) = \prod_{l \in \mathcal{K}} (\alpha p_l)^{w_l} = \alpha^{\left(\sum_{l \in \mathcal{K}} w_l\right)} \cdot \prod_{l \in \mathcal{K}} (p_l)^{w_l} = \alpha \cdot \xi(\boldsymbol{p}).$$

The function $\xi(\boldsymbol{p})$ is an elementary log-convex interference function and can be regarded as a basic building block of every log-convex interference function. This will become clear from the following Theorem 6, which was proved in [43]. To this end, we introduce the function

$$f_{\mathcal{I}}(\boldsymbol{w}) = \inf_{\boldsymbol{p} > 0} \frac{\mathcal{I}(\boldsymbol{p})}{\prod_{l \in \mathcal{K}} (p_l)^{w_l}}, \qquad \boldsymbol{w} \in \mathbb{R}_+^K. \tag{50}$$

The function $f_{\mathcal{I}}(\boldsymbol{w})$ plays a similar role as the conjugate function which has been used in the context of convex/concave interference functions. Using (50), we define the coefficient set

$$\mathcal{L}(\mathcal{I}) = \left\{\boldsymbol{w} \in \mathbb{R}_+^K : f_{\mathcal{I}}(\boldsymbol{w}) > 0\right\}. \tag{51}$$

It can be shown that $\boldsymbol{w} \in \mathcal{L}(\mathcal{I})$ implies $\|\boldsymbol{w}\|_1 = 1$, so this property is not explicitly required in (51).

The following theorem shows that every log-convex interference function has an elementary product representation.

Theorem 6: Every log-convex interference function $\mathcal{I}(\boldsymbol{p})$, on $\mathbb{R}^K_{++}$, can be represented as

$$\mathcal{I}(\boldsymbol{p}) = \max_{\boldsymbol{w}\in\mathcal{L}(\mathcal{I})}\left(f_{\mathcal{I}}(\boldsymbol{w})\cdot\prod_{l\in\mathcal{K}}(p_l)^{w_l}\right). \tag{52}$$

The proof is given in [43]. Applications of Theorem 6 and the connection with convex interference functions with will be discussed in Section IV.

IV. Applications

We begin by deriving convex/concave approximations for general interference functions. By exploiting the equivalence between interference functions and certain feasible sets, these results can be used to derive convex approximations of otherwise nonconvex feasible sets. This is potentially useful, e.g., for resource allocation algorithms operating on the boundary of the set.

Later, in Section IV-F it will be shown how convexity/concavity of underlying interference function can be exploited directly, in order to solve the power minimization problem [14] with super-linear convergence rate.

A. Greatest Log-Convex Minorant

Minorants and Majorants are defined as follows:

Definition 9: An interference function $\underline{\mathcal{I}}(\boldsymbol{p})$ is said to be a *minorant* of $\mathcal{I}(\boldsymbol{p})$ if $\underline{\mathcal{I}}(\boldsymbol{p}) \le \mathcal{I}(\boldsymbol{p})$ for all $\boldsymbol{p}\in\mathcal{P}$, where $\mathcal{P}\subseteq\mathbb{R}^K_+$ is the domain of $\mathcal{I}$. An interference function $\overline{\mathcal{I}}(\boldsymbol{p})$ is said to be a *majorant* if $\overline{\mathcal{I}}(\boldsymbol{p}) \ge \mathcal{I}(\boldsymbol{p})$ for all $\boldsymbol{p}\in\mathcal{P}$.

Consider an arbitrary interference function $\mathcal{I}(\boldsymbol{p}) > 0$, defined on $\mathbb{R}^K_{++}$, which needs neither be (log-)convex nor concave. Using Theorem 6, we can construct a log-convex approximation of $\mathcal{I}(\boldsymbol{p})$

$$\underline{\mathcal{I}}^{(l)}(\boldsymbol{p}) = \sup_{\boldsymbol{w}\in\mathcal{L}(\mathcal{I})}\left(f_{\mathcal{I}}(\boldsymbol{w})\cdot\prod_{l\in\mathcal{K}}(p_l)^{w_l}\right). \tag{53}$$

It was shown in [43] that function $\underline{\mathcal{I}}^{(l)}(\boldsymbol{p})$ is a log-convex minorant of $\mathcal{I}(\boldsymbol{p})$, i.e.,

$$\mathcal{I}(\boldsymbol{p}) \ge \underline{\mathcal{I}}^{(l)}(\boldsymbol{p}) \text{ for all } \boldsymbol{p} > 0. \tag{54}$$

The next theorem shows that it is not possible to find a tighter log-convex minorant.

Theorem 7: Let $\mathcal{I}$ be an arbitrary interference function, then (53) is its greatest log-convex minorant. Precisely, let $\tilde{\mathcal{I}}$ be a log-convex interference function which fulfills

$$0 < \underline{\mathcal{I}}^{(l)}(\boldsymbol{p}) \le \tilde{\mathcal{I}}(\boldsymbol{p}) \le \mathcal{I}(\boldsymbol{p}), \qquad \forall \boldsymbol{p} > 0 \tag{55}$$

then $\underline{\mathcal{I}}^{(l)}(\boldsymbol{p}) = \tilde{\mathcal{I}}(\boldsymbol{p})$. Proof: The functions $f_{\underline{\mathcal{I}}^{(l)}}(\boldsymbol{w})$, $f_{\tilde{\mathcal{I}}}(\boldsymbol{w})$, and $f_{\mathcal{I}}(\boldsymbol{w})$ are defined as in (50). Because of (55) we have

$$f_{\underline{\mathcal{I}}^{(l)}}(\boldsymbol{w}) \le f_{\tilde{\mathcal{I}}}(\boldsymbol{w}) \le f_{\mathcal{I}}(\boldsymbol{w}), \text{ for all } \boldsymbol{w}\ge 0, \|\boldsymbol{w}\|_1 = 1.$$

This implies

$$\begin{aligned}\underline{\mathcal{I}}^{(l)}(\boldsymbol{p}) &= \sup_{\boldsymbol{w}\in\mathcal{L}(\underline{\mathcal{I}}^{(l)})}\left(f_{\underline{\mathcal{I}}^{(l)}}(\boldsymbol{w})\cdot\prod_{l\in\mathcal{K}}(p_l)^{w_l}\right)\\ &\le \sup_{\boldsymbol{w}\in\mathcal{L}(\tilde{\mathcal{I}})}\left(f_{\tilde{\mathcal{I}}}(\boldsymbol{w})\cdot\prod_{l\in\mathcal{K}}(p_l)^{w_l}\right)\\ &\le \sup_{\boldsymbol{w}\in\mathcal{L}(\mathcal{I})}\left(f_{\mathcal{I}}(\boldsymbol{w})\cdot\prod_{l\in\mathcal{K}}(p_l)^{w_l}\right) = \underline{\mathcal{I}}^{(l)}(\boldsymbol{p})\end{aligned} \tag{56}$$

from which we can conclude $\underline{\mathcal{I}}^{(l)}(\boldsymbol{p}) = \tilde{\mathcal{I}}(\boldsymbol{p})$. ■

B. Least Concave Majorant and Greatest Convex Minorant

Consider again the general interference function $\mathcal{I}$, defined on $\mathbb{R}^K_{++}$. We can use the results of Sections II and III to derive concave and convex approximations. To this end, we construct sets $\mathcal{N}_0(\mathcal{I})$ and $\mathcal{W}_0(\mathcal{I})$, as defined by (12) and (32), respectively. We know from Lemma 3 that for any $\boldsymbol{w}\in\mathcal{N}_0(\mathcal{I})$ we have $\mathcal{I}(\boldsymbol{p}) \le \sum_l w_l p_l$, thus

$$\mathcal{I}(\boldsymbol{p}) \le \min_{\boldsymbol{w}\in\mathcal{N}_0(\mathcal{I})}\sum_{l\in\mathcal{K}} w_l p_l \text{ for all } \boldsymbol{p} > 0. \tag{57}$$

This means that the function

$$\overline{\mathcal{I}}^{(v)}(\boldsymbol{p}) = \min_{\boldsymbol{w}\in\mathcal{N}_0(\mathcal{I})}\sum_{l\in\mathcal{K}} w_l p_l \tag{58}$$

is a concave majorant of $\mathcal{I}$.

In a similar way, it follows from Lemma 6 that

$$\underline{\mathcal{I}}^{(x)}(\boldsymbol{p}) = \max_{\boldsymbol{w}\in\mathcal{W}_0(\mathcal{I})}\sum_{l\in\mathcal{K}} w_l p_l \le \mathcal{I}(\boldsymbol{p}) \tag{59}$$

is a convex minorant.

Now, it will be shown that these approximations are best-possible.

Theorem 8: $\overline{\mathcal{I}}^{(v)}$ is the least concave majorant of $\mathcal{I}$, and $\underline{\mathcal{I}}^{(x)}$ is the greatest convex minorant of $\mathcal{I}$.

Proof: We prove the first statement by contradiction. The proof of the second statement follows in the same way.

Suppose that there exists a concave interference function $\hat{\mathcal{I}}$, such that

$$\mathcal{I}(\boldsymbol{p}) \le \hat{\mathcal{I}}(\boldsymbol{p}) \le \overline{\mathcal{I}}^{(v)}(\boldsymbol{p}), \qquad \forall \boldsymbol{p} > 0. \tag{60}$$

Both $\hat{\mathcal{I}}$ and $\overline{\mathcal{I}}^{(v)}$ are concave interference functions, so we know from Theorem 1 that they can be represented as (14).

If the conjugate of $\overline{\mathcal{I}}^{(v)}$ is finite for some $\boldsymbol{w}\ge 0$, i.e., $\underline{\mathcal{I}}^*(\boldsymbol{w}) > -\infty$, then it follows from inequality (60) that also the conjugates of $\hat{\mathcal{I}}$ and $\mathcal{I}$ are finite. Thus

$$\mathcal{N}_0(\overline{\mathcal{I}}^{(v)}) \subseteq \mathcal{N}_0(\hat{\mathcal{I}}) \subseteq \mathcal{N}_0(\mathcal{I}). \tag{61}$$

The set $\mathcal{N}_0(\mathcal{I})$ is UCCC, as shown Section II-B, so with Theorem 2 we have

$$\mathcal{N}_0(\mathcal{I}) = \mathcal{N}_0(\overline{\mathcal{I}}^{(v)}). \tag{62}$$

Combining (61) and (62), we have $\mathcal{N}_0(\overline{\mathcal{I}}^{(v)}) = \mathcal{N}_0(\hat{\mathcal{I}})$. Hence, $\overline{\mathcal{I}}^{(v)}(\boldsymbol{p}) = \hat{\mathcal{I}}(\boldsymbol{p})$ for all $\boldsymbol{p} > 0$. ■

In Section IV-C, the convex minorant will be compared with the log-convex minorant.

C. Comparison of Convex and Log-Convex Minorants

In the previous sections it was shown that every general interference function $\mathcal{I}(\boldsymbol{p})$ has a greatest convex minorant $\underline{\mathcal{I}}^{(x)}(\boldsymbol{p})$ and a greatest log-convex minorant $\underline{\mathcal{I}}^{(l)}(\boldsymbol{p})$. Now, an interesting question is which class of functions provides the tightest minorant.

From Theorem 5 we know that $\underline{\mathcal{I}}^{(x)}(\mathrm{e}^{\boldsymbol{s}})$ is also log-convex. Thus, the set of log-convex interference functions is more general as the set of convex interference functions. That is, every convex interference function is log-convex, but the converse is not true. This means that the greatest log-convex minorant is better or as good as the greatest convex minorant, i.e.,

$$\underline{\mathcal{I}}^{(x)}(\boldsymbol{p}) \le \underline{\mathcal{I}}^{(l)}(\boldsymbol{p}) \le \mathcal{I}(\boldsymbol{p}), \qquad \forall \boldsymbol{p} > 0. \tag{63}$$

If the log-convex minorant $\underline{\mathcal{I}}^{(l)}$ is trivial, i.e., $\underline{\mathcal{I}}^{(l)}(\boldsymbol{p}) = 0, \forall\, \boldsymbol{p} > 0$, then also the convex minorant $\underline{\mathcal{I}}^{(x)}$ will be trivial. Conversely, if $\underline{\mathcal{I}}^{(x)}$ is trivial, then this does not imply that $\underline{\mathcal{I}}^{(l)}$ is trivial as well. This is shown by the following example.

1) Example 2: Consider the log-convex interference function

$$\mathcal{I}(\boldsymbol{p}) = \prod_{l\in\mathcal{K}} (p_l)^{w_l}, \qquad \boldsymbol{w} \ge 0, \qquad \|\boldsymbol{w}\|_1 = 1 \tag{64}$$

with the convex minorant

$$\underline{\mathcal{I}}^{(x)}(\boldsymbol{p}) = \max_{\boldsymbol{v}\in\mathcal{W}_0(\mathcal{I})} \sum_{l\in\mathcal{K}} v_l p_l. \tag{65}$$

It was already shown that $\underline{\mathcal{I}}^{(x)}(\boldsymbol{p}) \le \mathcal{I}(\boldsymbol{p}), \forall\, \boldsymbol{p} > 0$. Suppose that there is a $\boldsymbol{v} \in \mathcal{W}_0(\mathcal{I})$ such that $v_r > 0$ for some index r. That is

$$\prod_{l\in\mathcal{K}} (p_l)^{w_l} \ge \sum_{l\in\mathcal{K}} v_l p_l \ge v_r p_r > 0, \text{ for all } \boldsymbol{p} > 0.$$

This would lead to the contradiction

$$0 = \lim_{p_r\to\infty} \frac{1}{p_r} \prod_{l\in\mathcal{K}} (p_l)^{w_l} \ge v_r > 0.$$

Hence, $\mathcal{W}_0(\mathcal{I}) = 0$. The only convex minorant of the log-convex interference function (64) is the trivial function $\underline{\mathcal{I}}^{(x)}(\boldsymbol{p}) = 0$.

D. Convex and Concave Approximations of SIR Feasible Sets

Now, we show how the results can be applied to the SIR feasible region of a multiuser system. Consider K users with interference functions $\mathcal{I}_k(\boldsymbol{p}) > 0$ for all $k \in \mathcal{K}$. Certain SIR targets $\boldsymbol{\gamma} = [\gamma_1, \dots, \gamma_K] > 0$ are said to be feasible if there exists a $\boldsymbol{p} > 0$ such that

$$\frac{p_k}{\mathcal{I}_k(\boldsymbol{p})} \ge \gamma_k - \epsilon, \text{ for all } \epsilon > 0 \text{ and } k \in \mathcal{K}.$$

That is, the SIR targets $\boldsymbol{\gamma}$ can be achieved, at least in an asymptotic sense. Whether or not this condition can be fulfilled depends on how the users are coupled by interference [17]. A point $\boldsymbol{\gamma}$ is feasible if and only if $C(\boldsymbol{\gamma}, \mathcal{I}) \le 1$, where

$$C(\boldsymbol{\gamma}, \mathcal{I}) = \inf_{\boldsymbol{p}>0} \left(\max_{k\in\mathcal{K}} \frac{\gamma_k \mathcal{I}_k(\boldsymbol{p})}{p_k} \right). \tag{66}$$

The feasible region F is the sublevel set

$$F = \{\boldsymbol{\gamma} > 0 : C(\boldsymbol{\gamma}, \mathcal{I}) \le 1\}. \tag{67}$$

This generalizes the region (49) to arbitrary interference functions characterized by A1–A3.

If $\mathcal{I}_1(\mathrm{e}^{\boldsymbol{s}}), \dots, \mathcal{I}_K(\mathrm{e}^{\boldsymbol{s}})$ are log-convex, then $C(\boldsymbol{\gamma}, \mathcal{I})$ is a log-convex interference function [17]. Thus, the sublevel set F is convex on a logarithmic scale. We will refer to such sets as "log-convex" in the following.

Now, consider general interference functions, with no further assumption on convexity or concavity. The corresponding region F need not be convex, which complicates the development of algorithms operating on the boundary of the region. But with the results from the previous sections, we can derive convex and concave approximations.

For each $\mathcal{I}_k$, we have a log-convex minorant $\underline{\mathcal{I}}_k^{(l)}(\boldsymbol{p})$, as defined by (53). This leads to a region $\underline{F}^{(l)}$, characterized by $C(\boldsymbol{\gamma}, \underline{\mathcal{I}}^{(l)})$. Because $\mathcal{I}_k(\boldsymbol{p}) \ge \underline{\mathcal{I}}_k^{(l)}(\boldsymbol{p})$, for all $\boldsymbol{p} > 0$, we have $F \subseteq \underline{F}^{(l)}$. That is, the feasible region F is contained in the log-convex region $\underline{F}^{(l)}$. According to Theorem 7, this is the smallest region associated with log-convex interference functions. Moreover, the SIR region $\underline{F}$ has a useful property. For every mapping $QoS = \phi(\mathrm{SIR})$, with a log-convex inverse function $\phi^{[-1]}$, the resulting QoS region is log-convex [17].

Instead of approximating the underlying interference functions $\mathcal{I}(\boldsymbol{p})$, it is also possible to approximate the function $C(\boldsymbol{\gamma}) := C(\boldsymbol{\gamma}, \mathcal{I})$ directly. It can be verified that $C(\boldsymbol{\gamma})$ fulfills the axioms A1–A3. Thus, the SIR feasible region F can also be regarded as a sublevel set of an interference function.

As shown in Section IV-B, we can construct the least concave majorant $\overline{C}$ and the greatest convex minorant $\underline{C}$. Consider the sublevel sets

$$\underline{F} = \{\boldsymbol{\gamma} > 0 : \underline{C}(\boldsymbol{\gamma}) \le 1\}, \tag{68}$$

$$\overline{F} = \{\boldsymbol{\gamma} > 0 : \overline{C}(\boldsymbol{\gamma}) \le 1\}. \tag{69}$$

Because $\underline{C}(\boldsymbol{\gamma}) \le C(\boldsymbol{\gamma}) \le \overline{C}(\boldsymbol{\gamma})$ for all $\boldsymbol{\gamma} > 0$, the resulting level sets fulfill

$$\underline{F} \supseteq F \supseteq \overline{F}. \tag{70}$$

Sublevel sets of convex interference functions are downward-comprehensive convex. Because $\underline{C}$ is the *greatest* convex minorant, the set $\underline{F}$ is the smallest closed downward-comprehensive convex subset of $\mathbb{R}_+^K$ containing F (the "convex comprehensive hull").

The other sublevel $\overline{F}$ is generally not convex, but it has a convex complementary set $\overline{F}^c = \{\boldsymbol{\gamma} > 0 : \overline{C}(\boldsymbol{\gamma}) > 1\}$. The

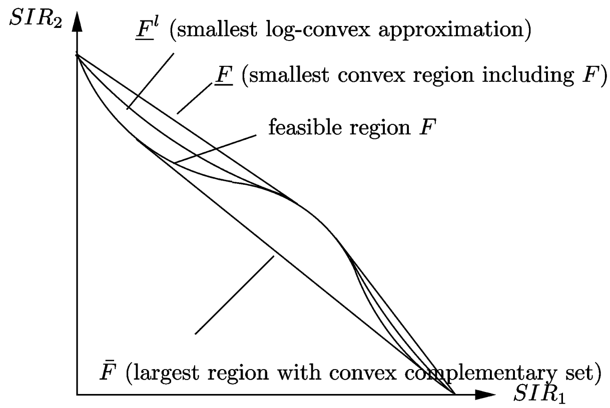

Fig. 5. Illustration: An arbitrary SIR feasible region F can be approximated by convex regions.

complementary set $\overline{F}^c$ is a superlevel set of a concave interference functions, so it is upward-comprehensive convex. The set $\overline{F}$ is downward-comprehensive. Thus, $\overline{F}$ is the largest closed downward-comprehensive subset of F such that the complementary set $\overline{F}^c$ is convex.

These regions provide best-possible convex approximations of the original region. Of course, there can exist other bounds, which are nonconvex, but tighter. For example, it is possible to construct a log-convex minorant $\underline{C}^l(\gamma)$, which fulfills $\underline{C}(\gamma) \leq \underline{C}(\gamma)^l \leq C(\gamma)$. The resulting sublevel set

$$\underline{F}^l = \{\boldsymbol{\gamma} > 0 : \underline{C}(\gamma)^l(\boldsymbol{\gamma}) \leq 1\}$$

fulfills $F \subseteq \underline{F}^l \subseteq \underline{F}$. This is illustrated in Fig. 5. Note that these bounds need not be good. It can happen that only a trivial bound exists, as shown in Section IV-C.

1) Example 3: Consider the SIR supportable region $\mathcal{S}$ resulting from linear interference functions $\mathcal{I}_k(\boldsymbol{p}) = [\boldsymbol{V}\boldsymbol{p}]_k$, as defined by (49). For $K = 2$, we have a coupling matrix $\boldsymbol{V} = \begin{bmatrix} 0 & V_{12} \\ V_{21} & 0 \end{bmatrix}$. The closure of the nonsupportable region is the set $\{\boldsymbol{\gamma} : \rho(\boldsymbol{\Gamma}\boldsymbol{V}) \geq 1\}$, where $\boldsymbol{\Gamma} = \mathrm{diag}\{\boldsymbol{\gamma}\}$. It can be verified that the function $\rho(\boldsymbol{\gamma}) = \rho\big(\mathrm{diag}\{\boldsymbol{\gamma}\}\boldsymbol{V}\big)$ fulfills the axioms A1–A3, thus $\rho(\boldsymbol{\gamma})$ is an interference function. The spectral radius is

$$\rho(\boldsymbol{\gamma}) = \sqrt{\gamma_1\gamma_2 V_{12}V_{21}} \tag{71}$$

thus $\rho(\boldsymbol{\gamma}) \geq 1$ if and only if $\gamma_2 \geq (V_{12}V_{21}\gamma_1)^{-1}$, which shows that the nonsupportable region is convex. Perhaps interestingly, this set can be shown to be convex for $K < 4$ users [44]. However, this property does not extend to larger numbers $K \geq 4$, as shown in [45].

With the proposed theory, this problem can be understood in a more general context. This result shows that certain properties of the Perron root [44], [45] can be generalized to the min–max optimum $C(\boldsymbol{\gamma})$ for arbitrary convex/concave interference functions. The function $C(\boldsymbol{\gamma})$ is an indicator for feasibility of SIR targets $\boldsymbol{\gamma}$, and the level set (67) is the SIR region, i.e., the set of jointly feasible SIR.

E. Convex Comprehensive Level Sets

In the previous section, we have discussed the SIR region, which is a comprehensive sublevel set of the interference function $C(\boldsymbol{\gamma}, \mathcal{I})$. This can be generalized to other level sets. It was shown in [18] that any closed downward-comprehensive subset of $\mathbb{R}^K_{++}$ can be expressed as a sublevel set of an interference function. Also, any closed upward-comprehensive subset of $\mathbb{R}^K_{++}$ can be expressed as a superlevel set of an interference function. Here, "closed" means *relatively closed on* $\mathbb{R}^K_{++}$.

In this section, we derive necessary and sufficient conditions for convexity. Consider an interference function $\mathcal{I}$ with the sublevel set

$$\underline{\mathcal{R}} = \{\boldsymbol{p} > 0 : \mathcal{I}(\boldsymbol{p}) \leq 1\} \tag{72}$$

and the superlevel set

$$\overline{\mathcal{R}} = \{\boldsymbol{p} > 0 : \mathcal{I}(\boldsymbol{p}) \geq 1\}. \tag{73}$$

Note, that the meaning of the vector $\boldsymbol{p}$ depends on the context. In the first part of the paper, $\boldsymbol{p}$ was introduced as a "power vector." However, $\boldsymbol{p}$ can stand for any other parameter, like the SIR vector $\boldsymbol{\gamma}$ used in the previous section.

Theorem 9: The set $\overline{\mathcal{R}}$ is nonempty UCCC and $\overline{\mathcal{R}} \neq \mathbb{R}^K_{++}$ if and only if the interference function $\mathcal{I}$ is concave and there exists a $\boldsymbol{p} > 0$ such that $\mathcal{I}(\boldsymbol{p}) > 0$.

Proof: Assume that the interference function $\mathcal{I}$ is concave. It was shown in [18] that the resulting superlevel set (73) is upward-comprehensive (this follows from axiom A3), closed (relatively on $\mathbb{R}^K_{++}$), and $\overline{\mathcal{R}} \neq \mathbb{R}^K_{++}$. The set $\overline{\mathcal{R}}$ is also convex since every superlevel set of a concave function is convex (see, e.g., [38, p. 75]). Conversely, assume that the superlevel set $\overline{\mathcal{R}}$ is a UCCC set. It was shown in [18] that $\overline{\mathcal{R}} \neq \mathbb{R}^K_{++}$ implies the existence of a $\boldsymbol{p} > 0$ such that $\mathcal{I}(\boldsymbol{p}) > 0$. It remains to show that the interference function $\mathcal{I}(\boldsymbol{p})$ is concave. Consider arbitrary boundary points $\hat{\boldsymbol{p}}, \check{\boldsymbol{p}} \in \mathbb{R}^K_{++}$, such that $\mathcal{I}(\hat{\boldsymbol{p}}) = \mathcal{I}(\check{\boldsymbol{p}}) = 1$. Defining $\boldsymbol{p}(\lambda) = (1-\lambda)\hat{\boldsymbol{p}} + \lambda\check{\boldsymbol{p}}$, we have $\mathcal{I}\big(\boldsymbol{p}(\lambda)\big) \geq 1$ for all $\lambda \in (0,1)$. For arbitrary $\alpha, \beta > 0$, we define

$$1 - \lambda = \frac{\alpha}{\alpha+\beta} \text{ and } \lambda = \frac{\beta}{\alpha+\beta},$$

which ensures the desired property $\lambda \in (0,1)$. With property A2, we have

$$1 \leq \mathcal{I}\left(\frac{\alpha}{\alpha+\beta}\cdot\hat{\boldsymbol{p}} + \frac{\beta}{\alpha+\beta}\cdot\check{\boldsymbol{p}}\right) = \frac{1}{\alpha+\beta}\cdot\mathcal{I}\left(\alpha\cdot\hat{\boldsymbol{p}} + \beta\cdot\check{\boldsymbol{p}}\right). \tag{74}$$

Using $\mathcal{I}(\hat{\boldsymbol{p}}) = \mathcal{I}(\check{\boldsymbol{p}}) = 1$ and (74), we have

$$\alpha\mathcal{I}(\hat{\boldsymbol{p}}) + \beta\mathcal{I}(\check{\boldsymbol{p}}) = \alpha + \beta \leq \mathcal{I}\left(\alpha\cdot\hat{\boldsymbol{p}} + \beta\cdot\check{\boldsymbol{p}}\right). \tag{75}$$

Next, consider arbitrary points $\hat{\boldsymbol{p}}', \check{\boldsymbol{p}}' \in \mathbb{R}^K_{++}$, from which we can construct boundary points $\hat{\boldsymbol{p}} = \hat{\boldsymbol{p}}'/\mathcal{I}(\hat{\boldsymbol{p}}')$ and $\check{\boldsymbol{p}} = \check{\boldsymbol{p}}'/\mathcal{I}(\check{\boldsymbol{p}}')$. It can be observed from A2 that $\mathcal{I}(\hat{\boldsymbol{p}}) = 1$ and $\mathcal{I}(\check{\boldsymbol{p}}) = 1$ holds.

Defining $\hat{\alpha} = \alpha/\mathcal{I}(\hat{\boldsymbol{p}}')$ and $\check{\beta} = \beta/\mathcal{I}(\check{\boldsymbol{p}}')$, and using (75), we have

$$\hat{\alpha}\mathcal{I}(\hat{\boldsymbol{p}}') + \check{\beta}\mathcal{I}(\check{\boldsymbol{p}}') \leq \mathcal{I}\left(\hat{\alpha}\cdot\hat{\boldsymbol{p}}' + \check{\beta}\cdot\check{\boldsymbol{p}}'\right). \quad (76)$$

Inequality (76) holds for arbitrary $\hat{\alpha}, \check{\beta} > 0$ and $\hat{\boldsymbol{p}}', \check{\boldsymbol{p}}' \in \mathbb{R}^K_{++}$, thus implying concavity of $\mathcal{I}$. ■

A similar result can be shown for the set $\underline{\mathcal{R}}$. The proof is similar to the proof of Theorem 9, but the directions of the inequalities are reversed.

Theorem 10: The set $\underline{\mathcal{R}}$ is nonempty DCCC and $\underline{\mathcal{R}} \neq \mathbb{R}^K_{++}$ if and only if the interference function $\mathcal{I}$ is convex and there exists a $\boldsymbol{p} > 0$ such that $\mathcal{I}(\boldsymbol{p}) > 0$.

Applying the result to the nonsupportable SIR region introduced in Example 3, it follows from Theorem 9 that the spectral radius $\rho(\boldsymbol{\gamma}) = \rho\big(\mathrm{diag}\{\boldsymbol{\gamma}\}\boldsymbol{V}\big)$ needs to be concave in order for the nonsupportable SIR region to be convex. It was shown [44] that $\rho(\mathrm{e}^{\boldsymbol{s}}\boldsymbol{V})$ is log-convex when using the substitution $\boldsymbol{\gamma} = \exp \boldsymbol{s}$. This does not imply that $\rho(\boldsymbol{\gamma})$ is concave.

Theorem 5 shows that every convex interference function $\mathcal{I}(\boldsymbol{p})$ is log-convex when we substitute $\boldsymbol{p} = \mathrm{e}^{\boldsymbol{s}}$. However, this does not mean that a concave function cannot be log-convex. For example, the function $\rho(\boldsymbol{\gamma})$, as defined by (71), is a concave interference function, even though $\rho(\mathrm{e}^{\boldsymbol{s}}\boldsymbol{V}) = \mathrm{e}^{s_1/2}\mathrm{e}^{s_2/2}\sqrt{V_{12}V_{21}}$ is log-convex.

The following example shows a case where an interference function $\mathcal{I}(\boldsymbol{p})$ is log-convex, but not concave. This discussion shows that log-convex interference functions need neither be convex nor concave. Both cases are possible, however.

1) Example 4: Consider two log-convex interference functions $\mathcal{I}_1$ and $\mathcal{I}_2$, where $\mathcal{I}_1(\boldsymbol{p})$ only depends on $p_1, \ldots, p_r$ and $\mathcal{I}_2(\boldsymbol{p})$ only depends on $p_{r+1}, \ldots, p_K$. We define

$$\mathcal{I}(\boldsymbol{p}) = \max\left(\mathcal{I}_1(\boldsymbol{p}), \mathcal{I}_2(\boldsymbol{p})\right). \quad (77)$$

The maximum of log-convex interference functions is a log-convex interference function. However, (77) is not concave. In order to show this, let $\boldsymbol{p}^{(1)} = \left[p_1^{(1)}, \ldots, p_r^{(1)}, 0, \ldots, 0\right]^T$ and $\boldsymbol{p}^{(2)} = \left[0, \ldots, 0, p_{r+1}^{(1)}, \ldots, p_K^{(1)}\right]^T$ be two arbitrary vectors such that $\mathcal{I}_1(\boldsymbol{p}^{(1)}) = 1$ and $\mathcal{I}_2(\boldsymbol{p}^{(2)}) = 1$. Defining $\boldsymbol{p}(\lambda) = (1-\lambda)\boldsymbol{p}^{(1)} + \lambda\boldsymbol{p}^{(2)}$, $\lambda \in (0,1)$, we have

$$\mathcal{I}_1\left(\boldsymbol{p}(\lambda)\right) = (1-\lambda)\mathcal{I}_1\left(\boldsymbol{p}^{(1)}\right) = 1-\lambda$$
$$\mathcal{I}_2\left(\boldsymbol{p}(\lambda)\right) = \lambda\mathcal{I}_2\left(\boldsymbol{p}^{(2)}\right) = \lambda.$$

Thus

$$\mathcal{I}\left(\boldsymbol{p}(\lambda)\right) = \max\left((1-\lambda), \lambda\right) < \mathcal{I}(\boldsymbol{p}^{(1)}) = \mathcal{I}(\boldsymbol{p}^{(2)}) = 1.$$

The super-level set $\{\boldsymbol{p} : \mathcal{I}(\boldsymbol{p}) \geq 1\}$ is not convex and $\mathcal{I}$ is not concave. This example shows that log-convex interference functions need not be concave.

The results can be further generalized by assuming a bijective mapping between a QoS vector $\boldsymbol{q}$ and the associated SIR values $\gamma(\boldsymbol{q}) = [\gamma_1(q_1), \ldots, \gamma_K(q_K)]^T$. For a linear interference model with a coupling matrix $\boldsymbol{V}$, the QoS region is defined as

$$F_q = \left\{\boldsymbol{q} : \rho\left(\mathrm{diag}\{\gamma(\boldsymbol{q})\}\boldsymbol{V}\right) \leq 1\right\}. \quad (78)$$

Under which condition is the QoS region F_q a convex set? This question is probably difficult and only partial answers exist. It was shown in [44] that if the function $\gamma(\boldsymbol{q})$ is log-convex, then $\rho\big(\mathrm{diag}\{\gamma(\boldsymbol{q})\}\boldsymbol{V}\big)$ is convex for all irreducible $K \times K$ matrices $\boldsymbol{V}$. In this case, convexity of $\rho\big(\mathrm{diag}\{\gamma(\boldsymbol{q})\}\boldsymbol{V}\big)$ implies convexity of the QoS feasible region F_q. However, the converse is not true. That is, convexity of F_q does not imply convexity of $\rho\big(\mathrm{diag}\{\gamma(\boldsymbol{q})\}\boldsymbol{V}\big)$. Note, that $\rho\big(\mathrm{diag}\{\gamma(\boldsymbol{q})\}\boldsymbol{V}\big)$ is generally not an interference function with respect to $\boldsymbol{q}$ (except, e.g., for the trivial case $\boldsymbol{\gamma} = \boldsymbol{q}$), so Theorem 10 cannot be applied.

F. Super-Linearly Convergent Algorithm

We can exploit that every concave or convex interference function can be expressed as (14) or (34), respectively. Assume that the first $K' = K - 1$ powers are caused by users with arbitrary concave interference functions $\mathcal{I}_1, \ldots, \mathcal{I}_{K'}$. The last power component $p_K = \sigma^2$ is constant noise. All functions $\mathcal{I}_1, \ldots, \mathcal{I}_{K'}(\boldsymbol{p})$ are strictly monotonic with respect to p_K. We are interested in the global power minimum

$$\min_{\boldsymbol{p}>0: p_K=\sigma^2} \sum_{k=1}^{K'} p_k \text{ s.t. } p_k \geq \gamma_k \mathcal{I}_k(\boldsymbol{p}), \quad k = 1, 2, \ldots, K' \quad (79)$$

where γ_k is a target SIR. Collecting all targets in a diagonal matrix $\boldsymbol{\Gamma} = \mathrm{diag}\{\gamma_1, \ldots, \gamma_{K'}\}$, the global optimum of (79) is found by the following iteration. Superscript (n) stands for the iteration step, and an arbitrary "feasible" initialization $\boldsymbol{p}^{(0)}$ is assumed

$$\boldsymbol{p}^{(n+1)} = \begin{bmatrix} \sigma^2\left(\boldsymbol{\Gamma}^{-1} - \boldsymbol{A}^{(n)}\right)^{-1} \boldsymbol{b}^{(n)} \\ \sigma^2 \end{bmatrix} \quad (80)$$

$$\boldsymbol{w}_k^{(n)} = \arg\min_{\boldsymbol{w}_k \in \mathcal{N}_0(\mathcal{I}_k)} \boldsymbol{w}_k^T \boldsymbol{p}^{(n)}, \quad k = 1, 2, \ldots K' \quad (81)$$

where $\boldsymbol{A}^{(n)}$ is the first $K' \times K'$ block of the $K' \times K$ matrix $\left[\boldsymbol{w}_1^{(n)}, \ldots, \boldsymbol{w}_{K'}^{(n)}\right]^T$. The vector $\boldsymbol{b}^{(n)}$ is the last column of this matrix. The following result is an immediate consequence of the results in [34].

Theorem 11: The sequence $\boldsymbol{p}^{(n)}$ obtained by the iteration (80) converges monotonically (component-wise) to the global optimum (79), with super-linear convergence.

For convex functions, the algorithm is similar, except that "min" is replaced by "max" and the coefficient set $\mathcal{W}_0(\mathcal{I}_k)$ is used instead (see [13] for details).

If the interference function $\mathcal{I}(\boldsymbol{p})$ is neither convex nor concave, then the power minimization problem (79) can still be solved by the fixed-point iteration proposed in [14] (see also [17], [34]. In this case, the convergence is only linear [16], [34]. Theorem 11 shows that a better convergence rate can be achieved by exploiting convexity.

V. Conclusion

In this paper, we analyze the basic building blocks of concave, convex, and log-convex interference functions. Every such interference function can be expressed as an optimum over elementary functions, with coefficients that adapt to the current power allocation. This shows that previously proposed axiomatic interference models are equivalent to matrix-based models, which have evolved from practical problem formulations, like interference mitigation or robust designs. This shows the existence of a unifying framework connecting different lines of research.

There are some potentially useful applications. Fundamental results which were previously derived for matrix-based interference functions can now be generalized. For example, the problem of minimizing the total power subject to QoS constraints can be solved by an iterative algorithm with super-linear convergence. This has applications in the area of robust signal processing (convex functions) and receiver optimization (concave functions).

Another main result of the paper is to show a one-to-one correspondence between convex/concave interference functions and certain convex "feasible sets." As an example, we have studied the feasible SIR region, which can be expressed as a sublevel set of an interference function. For arbitrary comprehensive feasible sets, best possible convex/concave approximations have been derived by exploiting the structure of interference functions. This shows that the theory of interference functions is useful beyond the area of power control, from which it originated.

This paper has focused on deriving a theoretical basis. More details on algorithmic aspects can be found in [13], [33], and [34].

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