Asymptotic Analysis and Spatial Coupling of Counter Braids

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Joint work with Eirik Rosnes Simula Research Lab/University of Bergen, Bergen, Norway

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Motivation •		Asymptotic analysis 000000		CHALMERS



• Per-flow measurements on high-speed links is a challenging problem.

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- Inter-arrival time between packets $\sim 40~{\rm ns}$ for a $10{\rm Gbps}$ link

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- Standard measurement architectures consist of large arrays of high-speed counters

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- Inter-arrival time between packets $\sim 40~{\rm ns}$ for a $10{\rm Gbps}$ link (Millions of flow sizes need to be measured over a few minutes!).
- Standard measurement architectures consist of large arrays of high-speed counters → Very costly!



• A counter architecture inspired by sparse graph codes.

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Motivation Counter braids Contribution Asymptotic analysis Numerical results Conclusion CHALMERS Counter braids Lu Montanari Prabhakar 2007]

- A counter architecture inspired by sparse graph codes.
- Cheap high-speed memory-efficient approximate counting.

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Motivation Counter braids Contribution Asymptotic analysis Numerical results Conclusion CHALMERS 0 0 0 0 0 0 0 CHALMERS Counter braids Lu Montanari Prabhakar 2007]

- A counter architecture inspired by sparse graph codes.
- Cheap high-speed memory-efficient approximate counting.
- Asymptotically optimal, i.e., the average number of bits needed to store the size of a flow tends to the information-theoretic limit (under ML decoding).

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• Can be described by a graph.

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- A layered structure.

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• Counter braids described by a graph → Low-complexity BP decoding!

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 Counter braids described by a graph → Low-complexity BP decoding! Start with the right-most layer and proceed layer-by-layer until the first layer is decoded.

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Belief propagation decoding

• Flow-to-counter message: $\mu_{f \to c}^{(\ell)} \in \mathbb{R}$.

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Belief propagation decoding

- Flow-to-counter message: $\mu_{f \to c}^{(\ell)} \in \mathbb{R}$.
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Belief propagation decoding

- Flow-to-counter message: $\mu_{f \to c}^{(\ell)} \in \mathbb{R}$.
- Counter-to-flow message: $\psi_{c \to f}^{(\ell)} \in \mathbb{R}$.
- Variable and check node updates:

$$\psi_{c \to f}^{(\ell)} = \max\left\{\phi(c) - \sum_{f' \in \delta(c) \setminus \{f\}} \mu_{f' \to c}^{(\ell-1)}, \ f_{\min}\right\}$$

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where f_{\min} is the minimum flow size (from the flow size distribution) of a flow node.

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where $f_{\rm min}$ is the minimum flow size (from the flow size distribution) of a flow node.

• Initialization:
$$\mu^{(0)}_{f \rightarrow c} = f_{\min}$$
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Example: Belief propagation decoding



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Example: Belief propagation decoding



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Spatially-coupled counter braids

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- Idea: Use spatial coupling to get to the MAP threshold [Rosnes 2015]

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Spatially-coupled counter braids

- CBs are asymptotically optimal under ML decoding.
- In practice...BP decoding entails suboptimality.
- Idea: Use spatial coupling to get to the MAP threshold [Rosnes 2015] \rightarrow Performance improvement, and saturation of the BP threshold to a given value.

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Our contribution

Two relevant questions

- 1. What is the MAP decoding threshold?
- 2. Threshold saturation to the MAP threshold?

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Our contribution

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- 1. What is the MAP decoding threshold? Need to derive it!
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Our contribution

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- 1. What is the MAP decoding threshold? Need to derive it!
- 2. Threshold saturation to the MAP threshold?

We know that...

For LDPC codes the area threshold is an upper bound on the MAP threshold.

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1. What is the MAP decoding threshold?

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1. What is the MAP decoding threshold?

Approach

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1. What is the MAP decoding threshold?

Approach

1. Derive the Maxwell decoder (message passing decoder with guesses).

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- 1. Derive the Maxwell decoder (message passing decoder with guesses).
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 - For LDPC codes the Maxwell decoder is a MAP decoder → Maxwell threshold = MAP threshold.
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- 2. Prove that the area threshold is an upper bound to the Maxwell decoding threshold.

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3. Conjecture that the area threshold is equal to the Maxwell decoding threshold, hence a lower bound on the MAP threshold.

	Contribution	Asymptotic analysis 000000		CHALMERS
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Second relevant question

2. Threshold saturation to the MAP threshold?

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Single-layer counter braids

• Regular-degree flow nodes,

$$L(z) = z^k, \qquad k \ge 2,$$

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Single-layer counter braids

• Regular-degree flow nodes,

$$L(z) = z^k, \qquad k \ge 2,$$

- The connection of flow nodes to counter nodes is performed randomly \rightarrow Asymptotically, the counter nodes distribution is Poisson,

$$R(z) = \sum_{i=0}^{\infty} R_i z^i = \sum_{i=0}^{\infty} \frac{\mathrm{e}^{-\gamma} (\gamma z)^i}{i!}$$

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- $\gamma = m_0 k/m_1$ is the average counter node degree.
- $\beta = m_1/m_0$ is the number of counters per flow

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- $\gamma = m_0 k/m_1$ is the average counter node degree.
- $\beta = m_1/m_0$ is the number of counters per flow $\rightarrow \beta d_1$ is the average number of bits needed to represent a flow.

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• $x^{(\ell)}(y^{(\ell)})$: error probability of a message from a flow (counter) node.

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- $x^{(\ell)}$ $(y^{(\ell)})$: error probability of a message from a flow (counter) node.
- DE updates:

$$x^{(\ell)} = \tilde{f}(y^{(\ell)}; \epsilon), \quad y^{(\ell)} = \tilde{g}(x^{(\ell-1)})$$

where

$$\begin{split} \tilde{f}(y;\epsilon) &= \begin{cases} y^{k-1}, & \text{if } \ell \text{ is odd} \\ \epsilon \cdot y^{k-1}, & \text{if } \ell \text{ is even} \end{cases} \\ \tilde{g}(x) &= 1 - \rho \left(1 - x\right) \end{split}$$

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• ϵ is the probability of observing a flow of size $> f_{\min}$.

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$$\beta_{\mathrm{BP}} = \beta_{\mathrm{BP}}(\epsilon) \triangleq \inf \left\{ \beta > 0 \mid x^{(\infty)}(\beta, \epsilon) = 0 \right\}.$$

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• Alternatively, fix β and find

$$\epsilon_{\mathrm{BP}} = \epsilon_{\mathrm{BP}}(\beta) \triangleq \sup \left\{ \epsilon \in \mathcal{E} \mid x^{(\infty)}(\beta, \epsilon) = 0 \right\}$$

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Equivalent graph representation of CBs



• Same (iteration-by-iteration) finite-length performance and asymptotic behavior.

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• DE on the equivalent bipartite graph:

$$x^{(2\ell)} = f(y^{(2\ell)}; \epsilon), \quad y^{(2\ell)} = g(x^{(2\ell-2)})$$

where

$$f(y;\epsilon) = \epsilon \cdot y^{k-1}$$

$$g(x) = 1 - \rho \left(1 - (1 - \rho (1 - x))^{k-1}\right)$$

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• The two DE recursions give the same BP decoding threshold!

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• Extended BP EXIT curve:

$$h^{\text{EBP}} = (\epsilon(x), (1 - \rho(1 - (1 - \rho(1 - x))^{k-1}))^k)$$

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• Extended BP EXIT curve:

$$h^{\text{EBP}} = (\epsilon(x), (1 - \rho(1 - (1 - \rho(1 - x))^{k-1}))^k)$$

Definition: Area threshold

Let $(\epsilon(x^*),h^{\rm EBP}(x^*))$ be a point on the EBP EXIT curve $h^{\rm EBP}$ of a single-layer CB such that

$$\int_{x^*}^1 h^{\text{EBP}}(x) \, \mathrm{d}\epsilon(x) = \int_0^1 h^{\text{EBP}}(x) \, \mathrm{d}\epsilon(x)$$

and there exist no $x' \in (x^*, 1]$ such that $\epsilon(x') = \epsilon(x^*)$. Then, the area threshold is defined as $\overline{\epsilon} = \epsilon(x^*)$.

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Area threshold and Maxwell threshold

Theorem

The area threshold is an upper bound on the Maxwell decoding threshold.

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Area threshold and Maxwell threshold

Theorem

The area threshold is an upper bound on the Maxwell decoding threshold.

Conjecture

The area threshold is equal to the Maxwell decoding threshold and thus a lower bound on the MAP threshold.

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Threshold saturation?



k = 3
k = 6
k = 8

dashed: uncoupled solid: coupled (N = 128, w = 5)

∃ →

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k = 3
k = 6
k = 8

dashed: uncoupled solid: coupled (N = 128, w = 5)

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k = 6 $\blacksquare \ k = 8$

dashed: uncoupled solid: coupled (N = 128, w = 5)

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k = 3k = 6 $\blacksquare \ k = 8$

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Motivation O	Counter braids 0000000	Contribution 00	Asymptotic analysis 000000	Numerical results 00	Conclusion •	CHALMERS
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Motivation O	Counter braids	Contribution OO	Asymptotic analysis	Numerical results	Conclusion	CHALMERS
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- SC-CBs exhibit improved iterative decoding thresholds compared to uncoupled CBs and their thresholds converge to a fixed value.
- The area threshold and potential thresholds coincide and we conjecture that they are equal to the Maxwell decoding threshold.

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Take-home message (II)

The tools we are used to in coding (message passing, density evolution) can be applied to many other areas!

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E. Rosnes and A. Graell i Amat, "Asymptotic analysis and spatial coupling of counter braids," submitted to *IEEE Trans. Inf. Theory.* Available on arXiv.