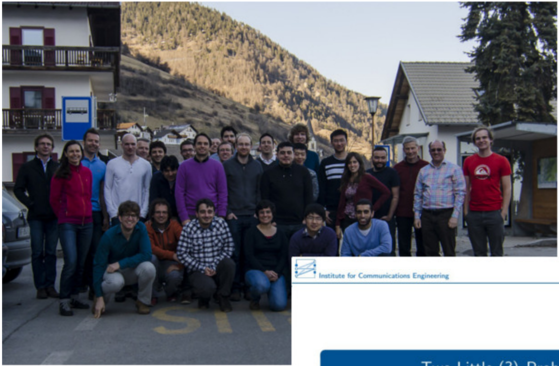
The Fractality of Polar Codes

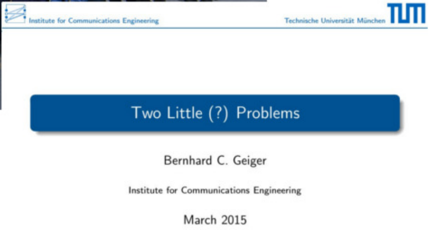
Bernhard C. Geiger

Institute for Communications Engineering

July 26th, 2016

A look back in time. . . (JCCC 2015)





Institute for Communications Engineering   Technische Universität München

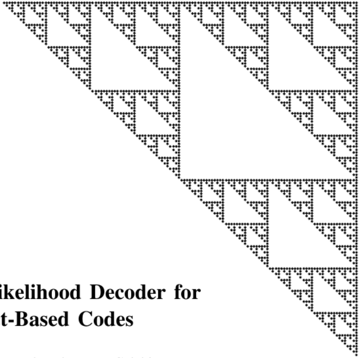
Two Little (?) Problems

Bernhard C. Geiger

Institute for Communications Engineering

March 2015

The Sierpinski Triangle



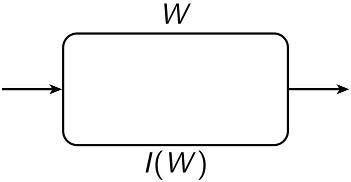
Fifty-first Annual Allerton Conference
Allerton House, UIUC, Illinois, USA
October 2 - 3, 2013

Folded Tree Maximum-Likelihood Decoder for Kronecker Product-Based Codes

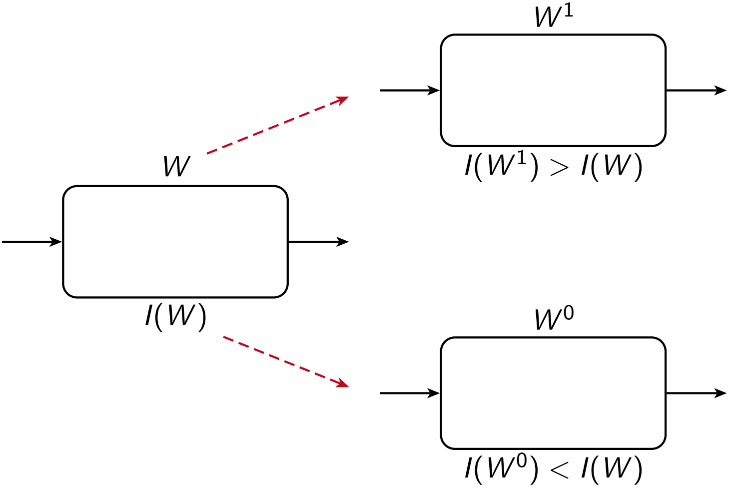
Sinan Kahraman, Emanuele Viterbo, and Mehmet E. Çelebi

Fig. 2. The matrix $\mathbf{G} = \mathbf{F}^{\otimes 7}$ has the fractal form of a Sierpinski triangle.

Polar Codes



Polar Codes



Polar Codes

Continue polarizing:

$$\begin{aligned}(W, W) &\rightarrow (W^1, W^0)\\(W, W, W, W) &\rightarrow (W^1, W^1, W^0, W^0) \rightarrow (W^{11}, W^{10}, W^{01}, W^{00})\\&\vdots\end{aligned}$$

Polar Codes

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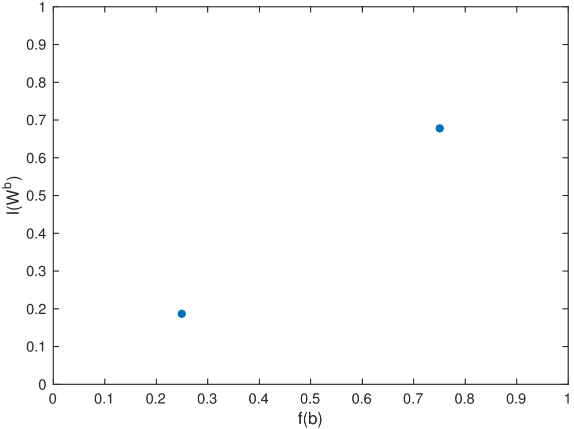
Proposition (Polarization)

For almost every $b \in \{0, 1\}^\infty$, either $I(W^b) = 1$ or $I(W^b) = 0$.

Some intuition... (BEC, $I(W) = 0.433$)



2^1

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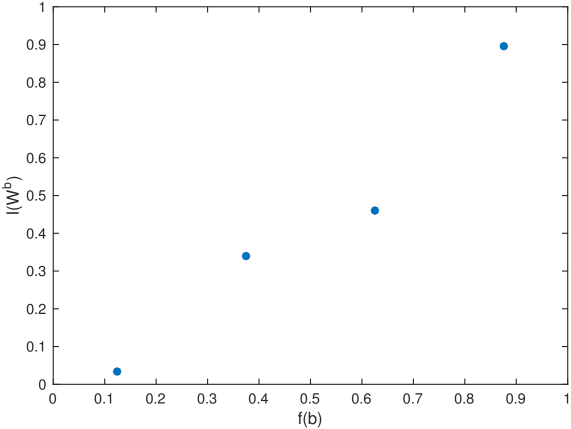

2^2

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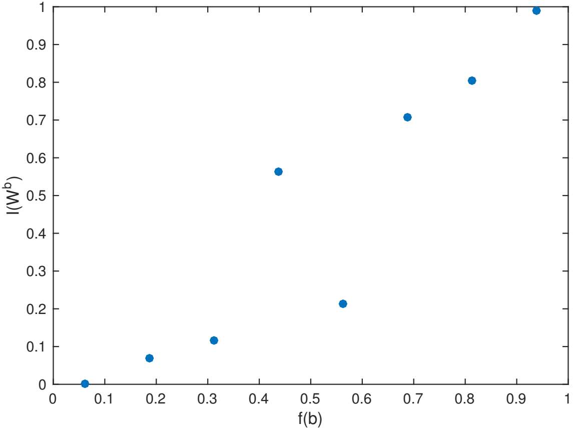
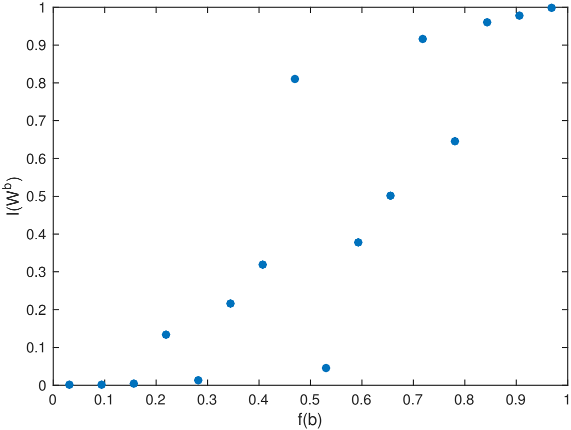


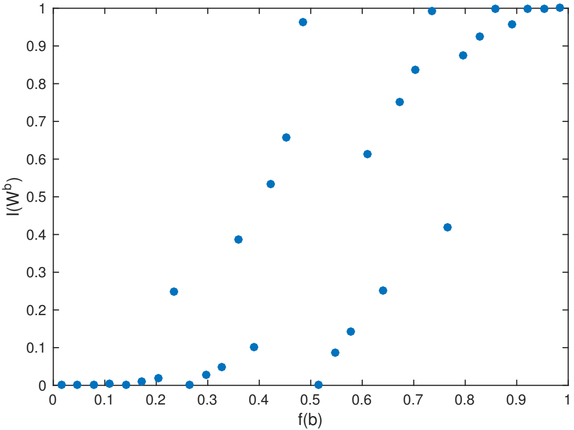
2^3

Some intuition. . . (BEC, $I(W) = 0.433$)



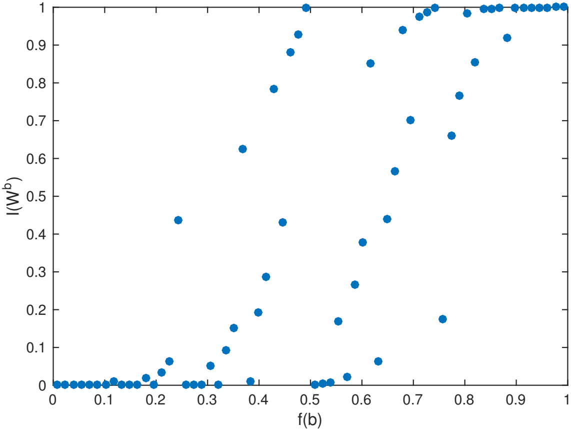
2^4

Some intuition. . . (BEC, $I(W) = 0.433$)



2^5

Some intuition... (BEC, $I(W) = 0.433$)



2^6

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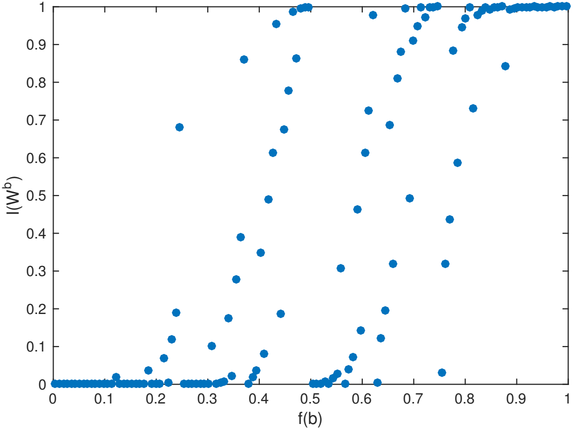

2^7

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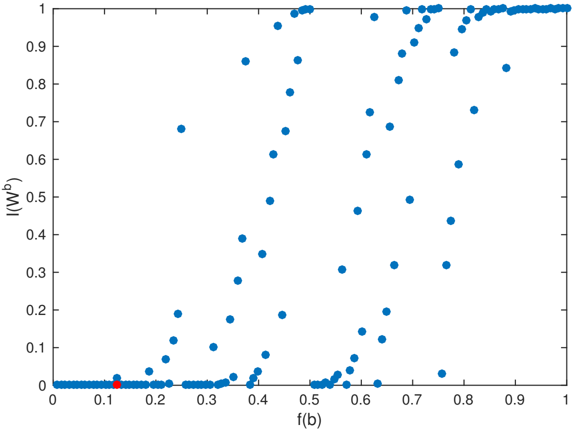

2^8

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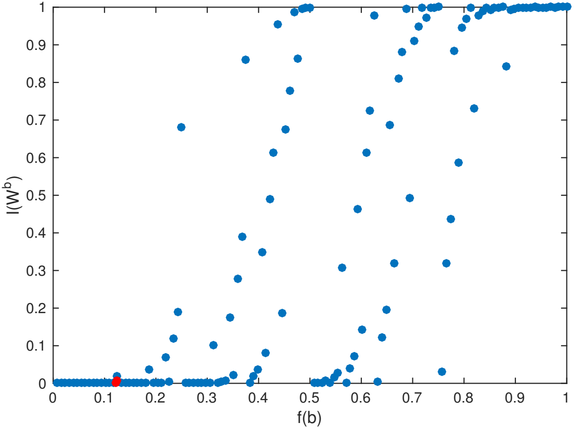
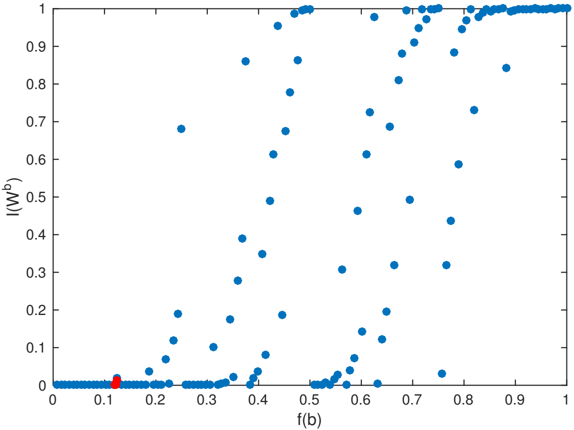


2^{16}

Some intuition... (BEC, $I(W) = 0.433$)



2^{17}

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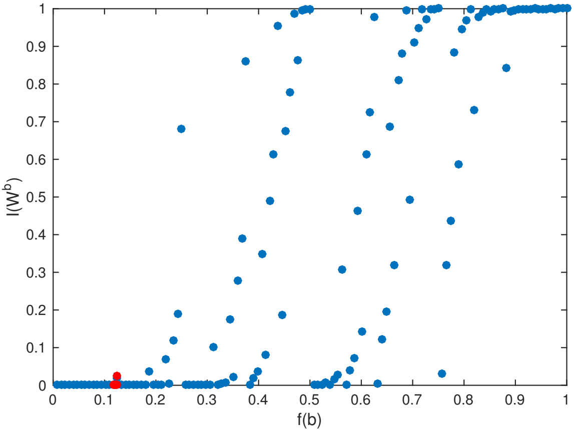

2^{18}

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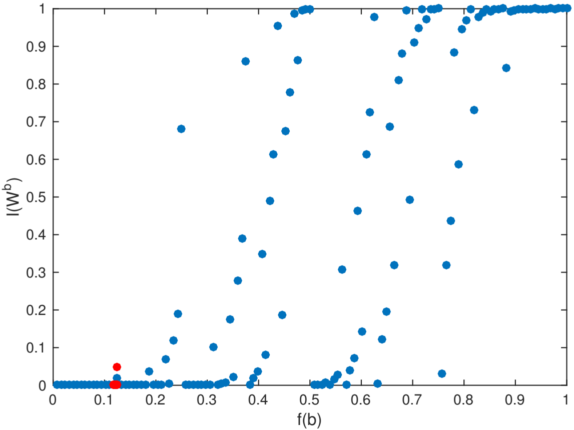

2^{19}

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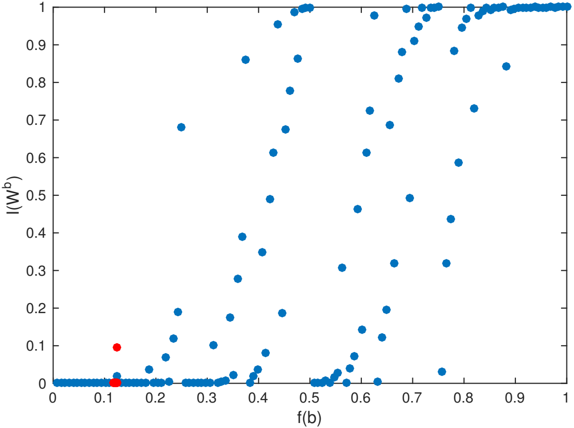
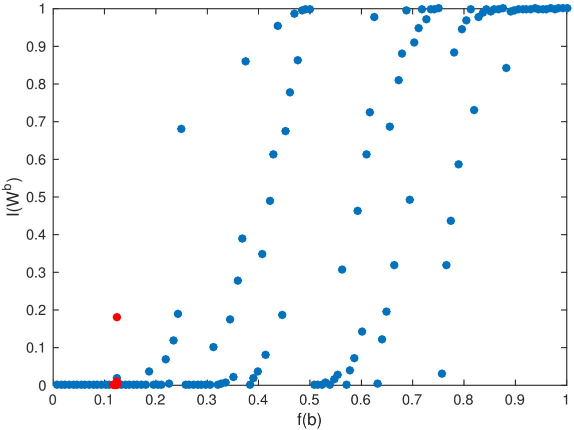


2^{20}

Some intuition. . . (BEC, $I(W) = 0.433$)



2^{21}

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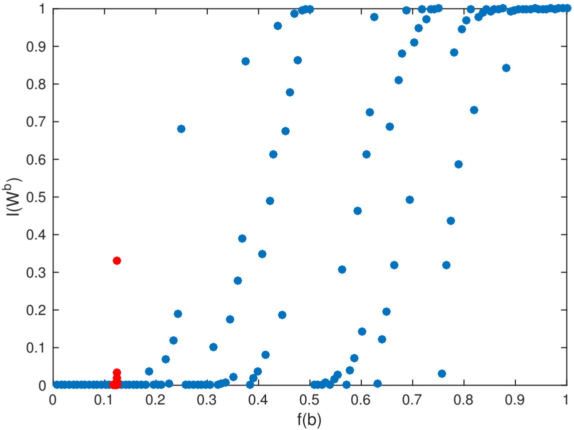

2^{22}

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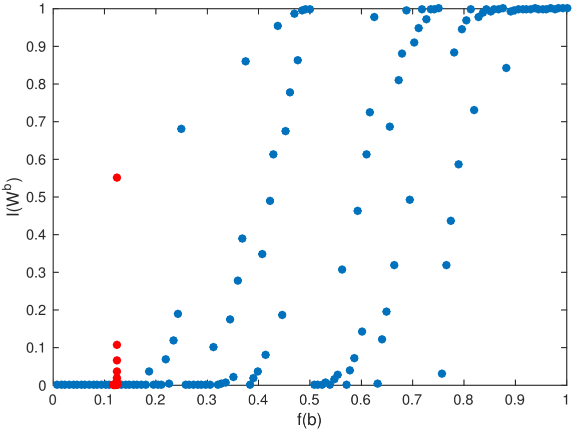

2^{23}

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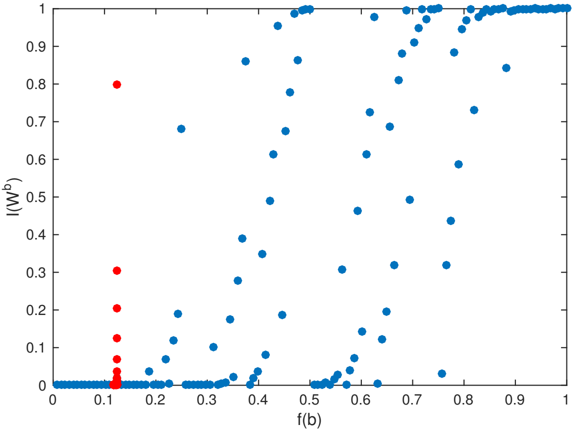

2^{24}

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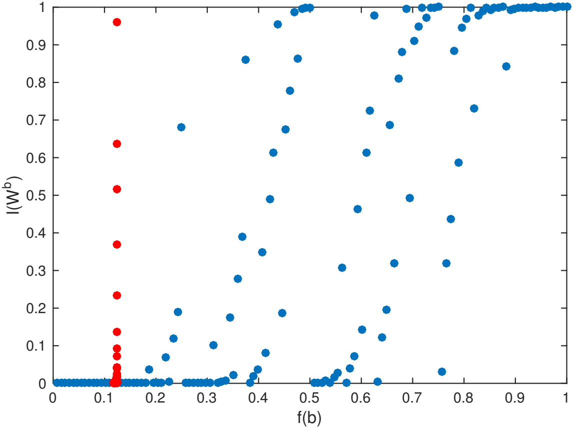
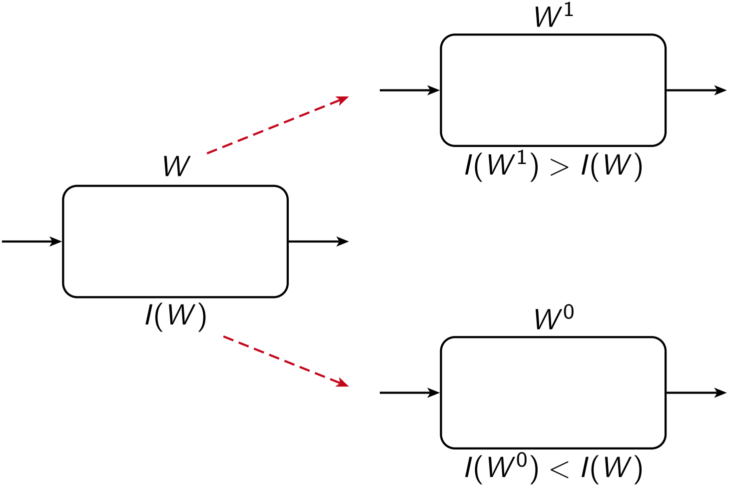


2^{25}

Polar Codes: Up- & Degrading



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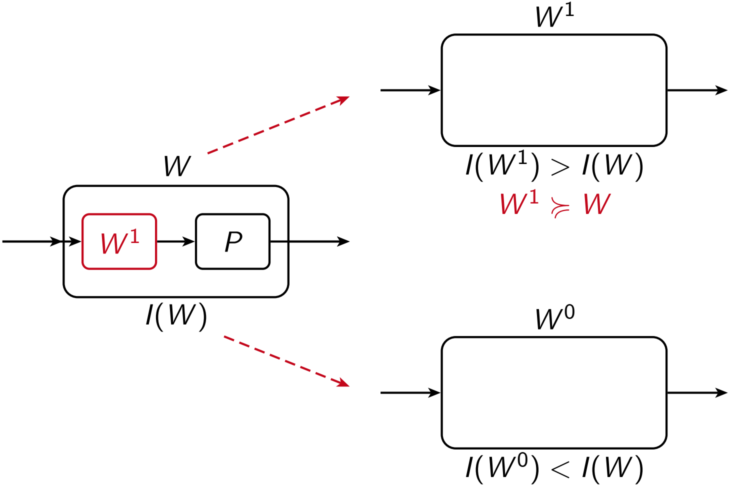
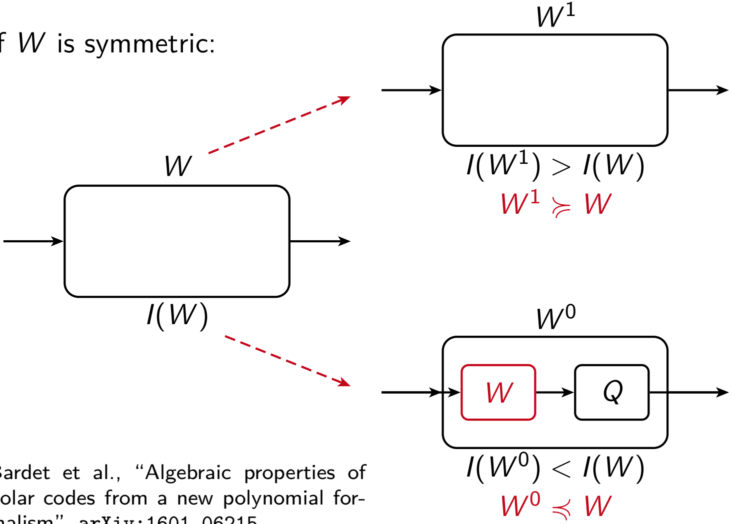


Polar Codes: Up- & Degrading

If W is symmetric:



Bardet et al., "Algebraic properties of polar codes from a new polynomial formalism", arXiv:1601.06215.

Polar Codes: Up- & Degrading

For symmetric channels,

$$\begin{aligned} W^0 \preccurlyeq \quad & W \quad \preccurlyeq W^1 \\ I(W^0) < \quad & I(W) \quad < I(W^1) \end{aligned}$$

Polar Codes: Up- & Degrading

For symmetric channels,

$$\begin{array}{ccccc} W^0 & \preccurlyeq & W & \preccurlyeq & W^1 \\ I(W^0) & < & I(W) & < & I(W^1) \end{array}$$

For every binary sequence $b^n \in \{0,1\}^n$,

$$\begin{array}{ccccc} W^{0b^n} & \preccurlyeq & W^{b^n} & \preccurlyeq & W^{1b^n} \\ I(W^{0b^n}) & \leq & I(W^{b^n}) & \leq & I(W^{1b^n}) \end{array}$$

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For every b^n and every $a \in \{0,1\}^\infty$,

$$\begin{aligned} W^{b^n000\cdots} \preccurlyeq \quad & W^{b^n a} \quad \preccurlyeq W^{b^n111\cdots} \\ I(W^{b^n000\cdots}) \leq \quad & I(W^{b^n a}) \quad \leq I(W^{b^n111\cdots}) \end{aligned}$$

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For every b^n and every $a \in \{0,1\}^\infty$,

$$\begin{array}{ccc} W^{b^n 000\cdots} \preccurlyeq & W^{b^n a} & \preccurlyeq W^{b^n 111\cdots} \\ 0 = I(W^{b^n 000\cdots}) \leq & I(W^{b^n a}) & \leq I(W^{b^n 111\cdots}) = 1 \end{array}$$

Simplifying Notation

Map binary sequences $b \in \{0,1\}^\infty$ to real numbers $x \in [0,1]$:

$$x = f(b) = \sum_{i=1}^{\infty} \frac{b_i}{2^{-i}}$$

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$\Rightarrow$ f is bijective on $\{0,1\}^\infty \setminus f^{-1}(\mathbb{D})$

The Good Channels and the Bad Channels

Definition

Let $\mathcal{G}$ denote the set of good channels, i.e.,

$$I(W^b) = 1 \Rightarrow f(b) \in \mathcal{G};$$

let $\mathcal{B}$ denote the set of bad channels, i.e.,

$$I(W^b) = 0 \Rightarrow f(b) \in \mathcal{B}.$$

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Proposition

$\lambda(\mathcal{G}) = I(W)$ and $\lambda(\mathcal{B}) = 1 - I(W)$.

The Good (and the Bad) Channels are Dense

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$\mathcal{G}$ and $\mathcal{B}$ are dense in $[0, 1]$.

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Idea (symmetric W):

- $x = 0.5$: $f^{-1}(0.5) = \{(1000\cdots),(0111\cdots)\}$
- For every a, $W^{1000\cdots} \preccurlyeq W^{1a} \preccurlyeq W^{1111\cdots}$

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Proposition

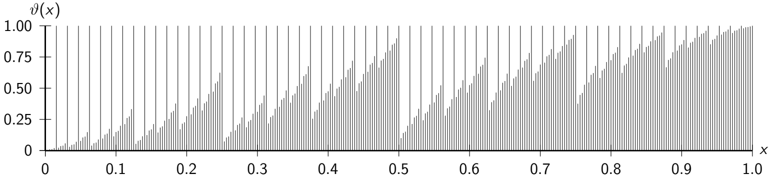
$\mathcal{G} \setminus \mathbb{D}$ is dense in $[0, 1]$. If W is a BEC, also $\mathcal{B} \setminus \mathbb{D}$ is dense in $[0, 1]$.

The Good Channels are Self-Similar

Proposition

Let W be symmetric. Then, $\mathcal{G}$ is quasi self-similar in the sense that it is a subset of its "right halves", and a superset of its "left halves". That is true on all scales.

The Polar Fractal



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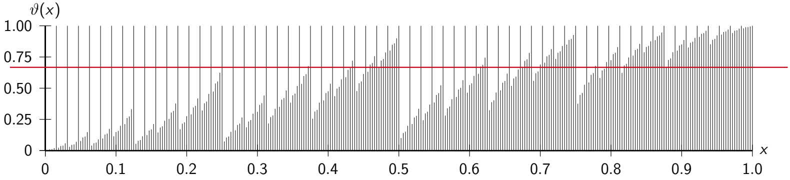

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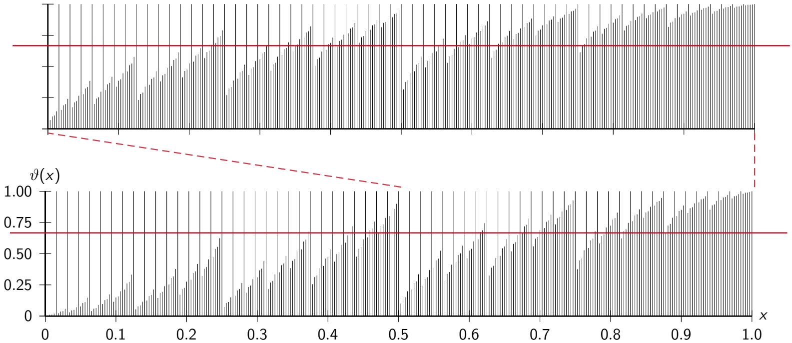

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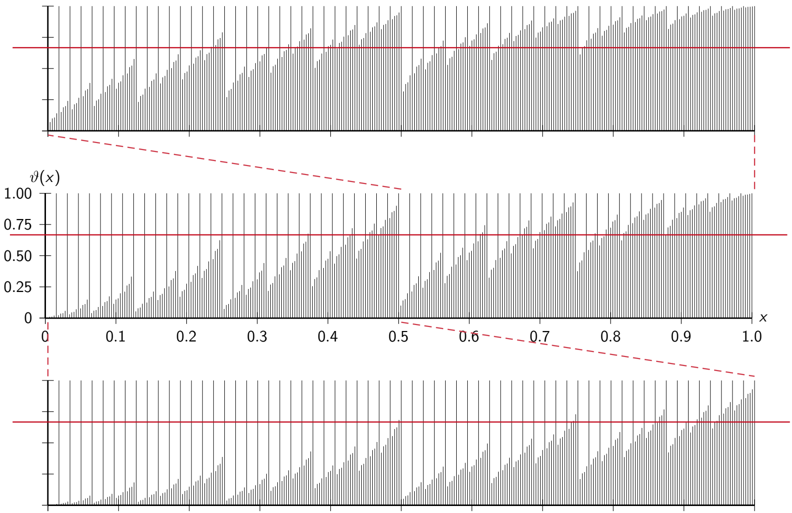

Conclusion and Open Questions

`arxiv:1506.05231 [cs.IT]`

- Polar codes are fractal: Self-similarity
- Reed-Muller codes are fractal, too: Self-similarity and Hausdorff dimension
- Extension to non-binary polar codes?
- Are there practical implications for polar code construction?

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Thanks for your attention!