LDPC Codes for Quantitative Group Testing with a Non-Binary Alphabet

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Joint work with Mgeni Makambi Mashauri[†] and Alexandre Graell i Amat[‡]

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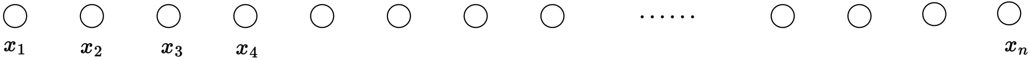
Munich Workshop on Shannon Coding Techniques

TUM, April 5, 2024

Background: Group Testing

- We have a large population of items



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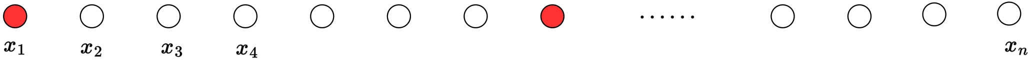
- We have a large population of items
- Very few of them are "defective" (probability of being defective, γ is very small)



- **Goal:** Identify $\boldsymbol{x}$: defective $(x_i = 1)$, non-defective $(x_i = 0)$

[Dorfman1943] Robert Dorfman, “The Detection of Defective Members of Large Populations,” *The Annals of Mathematical Statistics,*, vol. 14, no. 4, pp. 436–440, 1943.

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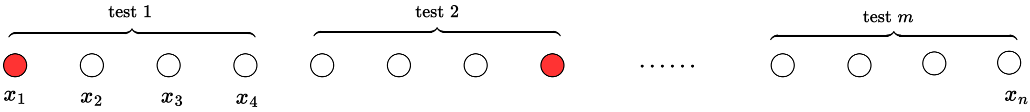

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- To reduce the number of tests: **test the items in groups** (pooling) [Dorfman1943]
- Rate, $\Omega = \frac{m}{n}$ (smaller is better)

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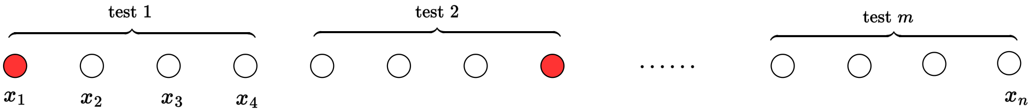

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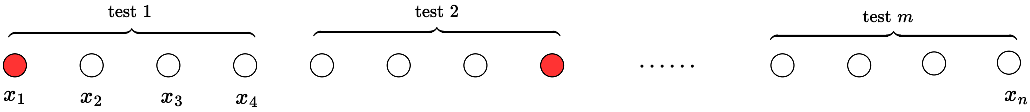

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- Rate, $\Omega = \frac{m}{n}$ (smaller is better)
- Adaptive vs non-adaptive test design
- We consider the asymptotic regime: $n \to \infty$

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Background: Graphical Representation

- For non-adaptive group testing the pooling can be represented by a test matrix $\boldsymbol{A}$

$$\boldsymbol{A} = \begin{array}{c} \begin{pmatrix} 1 & 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 & 1 \end{pmatrix} \\ \begin{matrix} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 \end{matrix} \end{array}$$

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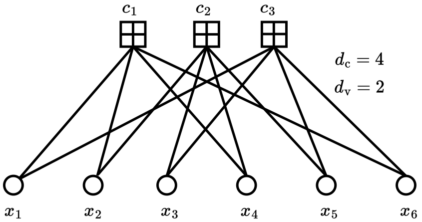
- The matrix can be represented by a bipartite graph G



- We consider the scenario where the graph is sparse

Non-quantitative vs Quantitative

- **Non-quantitative:** test result, $s_i = 1$ if at least one item is defective otherwise $s_i = 0$ (logical OR)

$$\begin{array}{ccc} \begin{array}{cccccc} \bullet & \bullet & \circ & \circ & \circ & \bullet \end{array} & & \text{Non-Quantitative} \\ \begin{pmatrix} 0 & 0 & 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 \end{pmatrix} & \begin{array}{c} s_1 \\ s_2 \\ s_3 \\ s_4 \end{array} & \begin{array}{c} 0 \\ 1 \\ 1 \\ 1 \end{array} \end{array}$$

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0 & 0 & 1 & 1 & 1 & 0 & s_1 & 0 & 0 \\
1 & 1 & 0 & 1 & 0 & 0 & s_2 & 1 & 2 \\
0 & 1 & 1 & 0 & 1 & 0 & s_3 & 1 & 1 \\
1 & 1 & 0 & 0 & 0 & 1 & s_4 & 1 & 3
\end{array}$$

- For **quantitative** group testing, a test result shows the number of defective items

$$s_i = \sum_{j=1}^{n} x_j a_{ij} \quad \rightarrow \quad \boldsymbol{s} = Ax$$

Quantitative Group Testing with Sparse Graphs: Prior work

- The test results show the number of defectives
- Best known scheme with sparse graph uses GLDPC [KAR2019]

$$\boldsymbol{U} = \begin{pmatrix} \overbrace{\begin{matrix}1 & 1 & 1 & 1 & 1 & 1 & 1\end{matrix}}^{d_{\mathsf{c}}} \\ \begin{matrix}0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1\end{matrix} \end{pmatrix} \quad t = 1, m_{\mathsf{u}} = 3$$

- A t-error-correcting BCH code is used as a component code
- An additional row of ones to identify # of defective items

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- $d_{\mathsf{c}} = 2^{m_{\mathsf{u}}} - 1 \rightarrow m_{\mathsf{u}} = \log_2(d_{\mathsf{c}} + 1)$
- Tests per subcode $= t \log_2(d_{\mathsf{c}} + 1) + 1$

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- Tests per subcode $= t \log_2(d_{\mathsf{c}} + 1) + 1$
- Rate, $\Omega = \frac{m}{n} = \frac{d_{\mathsf{v}}}{d_{\mathsf{c}}} \left(t \lceil \log_2(d_{\mathsf{c}} + 1) \rceil + 1 \right)$

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Prior Work

- **Density Evolution**

  For each iteration ℓ

  $q^{(\ell)}$: probability a test sends resolved to item
  $p^{(\ell)}$: probability a defective item is unresolved

  Test to item:

  $$q^{(\ell)} = \sum_{i=0}^{t-1} \binom{d_{\mathsf{c}}-1}{i} \left(p^{(\ell-1)}\right)^{i} \left(1-p^{(\ell-1)}\right)^{d_{\mathsf{c}}-1-i}$$

  Item to test:

  $$p^{(\ell)} = \gamma(1-q^{(\ell-1)})^{d_{\mathsf{v}}-1}$$

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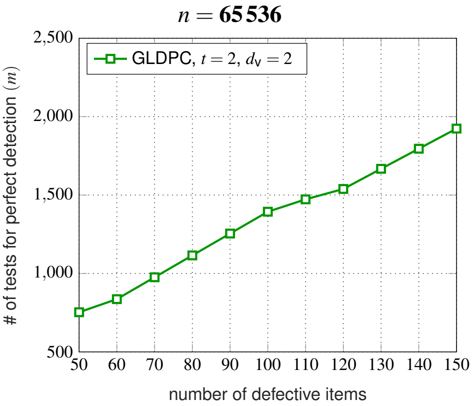

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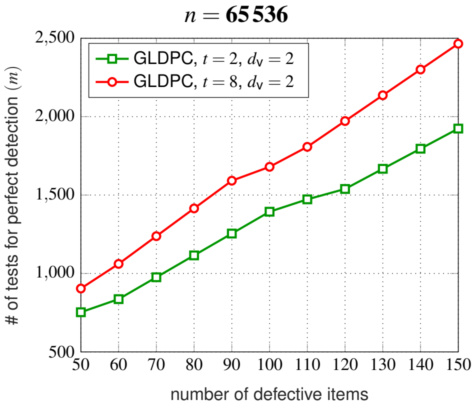

- Small number of tests for a large population size
- Increasing t improves error correction
- Penalized by increasing number of tests
  $m = n\frac{d_{\mathsf{v}}}{d_{\mathsf{c}}}\left(t\lceil\log_2(d_{\mathsf{c}}+1)\rceil+1\right)$

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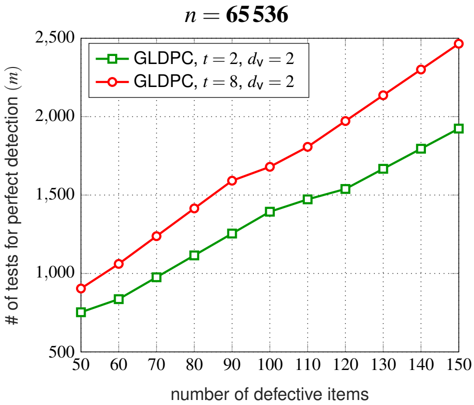

- What about using $t=0$?

Proposed scheme: Group Testing with LDPC

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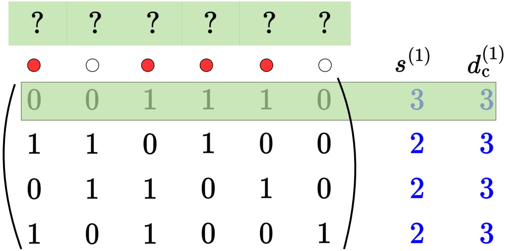

$$\begin{array}{cccccccc}
? & ? & ? & ? & ? & ? & & \\
\bullet & \circ & \bullet & \bullet & \bullet & \circ & \boldsymbol{s}^{(1)} & \boldsymbol{d}_{\mathrm{c}}^{(1)} \\
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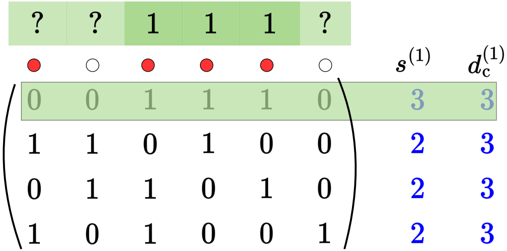

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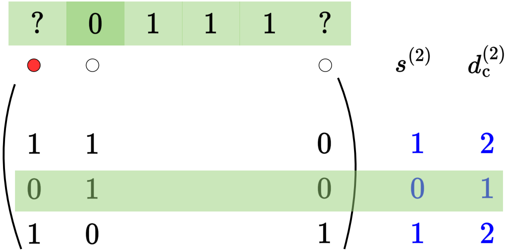

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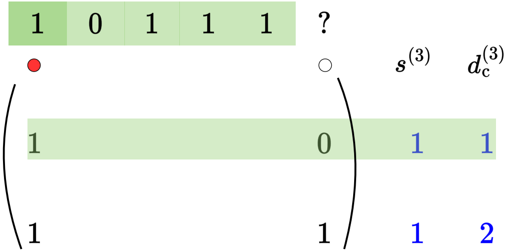

$$\begin{array}{cccccccc} 1 & 0 & 1 & 1 & 1 & ? & & \\ \bullet & & & & & \circ & s^{(3)} & d_\mathrm{c}^{(3)} \\ 1 & & & & & 0 & 1 & 1 \\ 1 & & & & & 1 & 1 & 2 \end{array}$$

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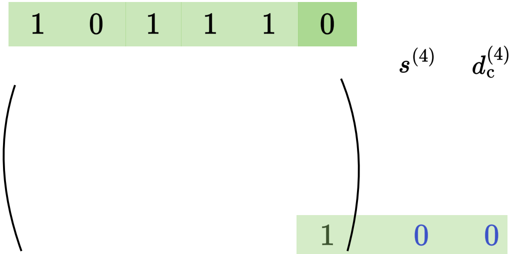

Density Evolution

- $p_1^{(\ell)}$: probability that a message from a defective is *unresolved*
- $q_0^{(\ell)}$: **probability that a message to a non-defective is *resolved***
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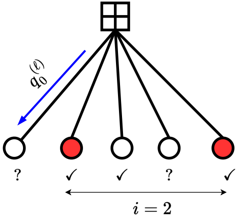

From test to item

$$q_0^{(\ell)} = \sum_{i=0}^{d_\mathsf{c}-1} \binom{d_\mathsf{c}-1}{i} \gamma^i (1-\gamma)^{d_\mathsf{c}-1-i} \left(1-p_1^{(\ell-1)}\right)^i$$

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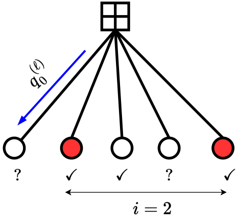

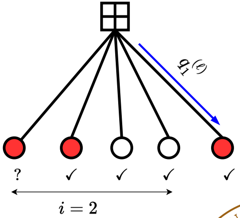

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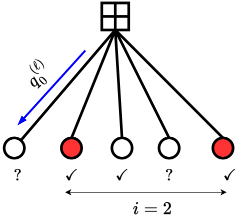

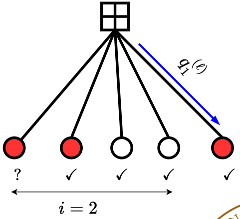

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From item to test

$$p_0^{(\ell)} = \left(1 - q_0^{(\ell-1)}\right)^{d_v-1}$$

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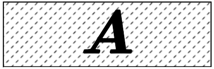
Performance Comparison

- We consider two scenarios
  - **Fixing the proportion of defective items γ and changing the rate $\Omega = \frac{m}{n}$**



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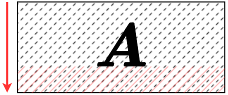

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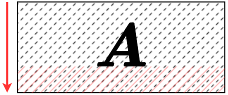

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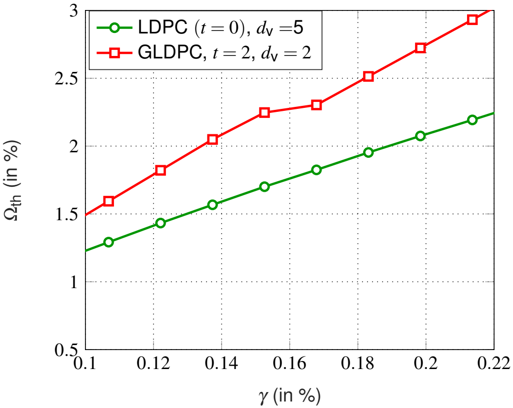
Minimum rate required for a fixed γ



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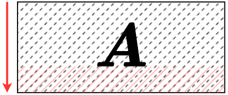
- We consider two scenarios
  - **Fixing the proportion of defective items γ and changing the rate $\Omega = \frac{m}{n}$**



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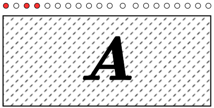
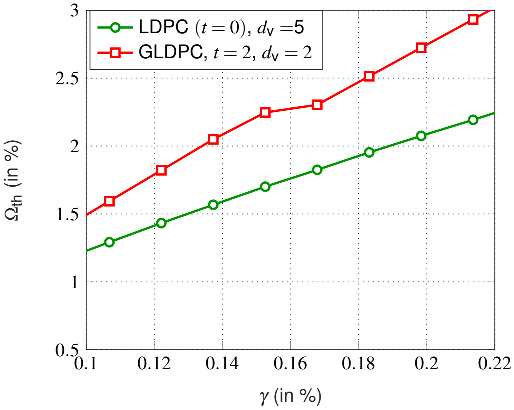


Minimum rate required for a fixed γ



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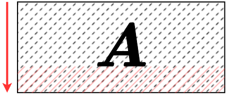

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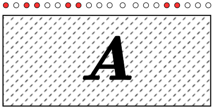
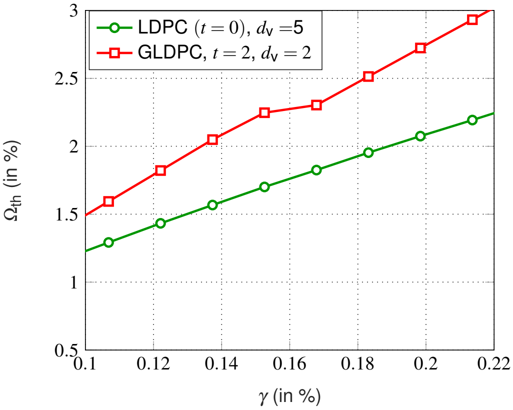


  - A new perspective considering $\boldsymbol{A}$ (code) as fixed

Minimum rate required for a fixed γ



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Performance Comparison: Fixed Rate, $\Omega = 5\%$

Table: GLDPC Based

t	d_v	γ_th
1	2	0.2487
	3	0.3708
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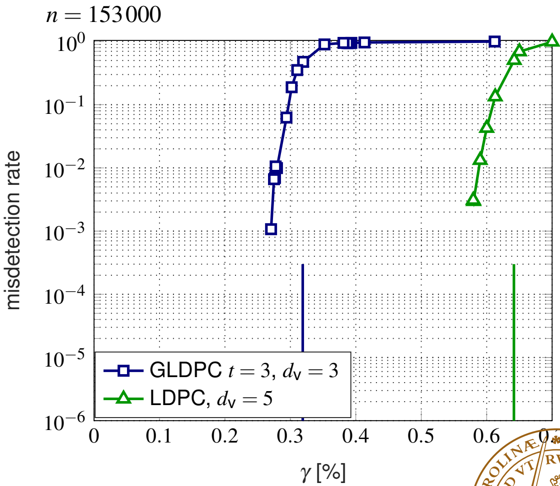

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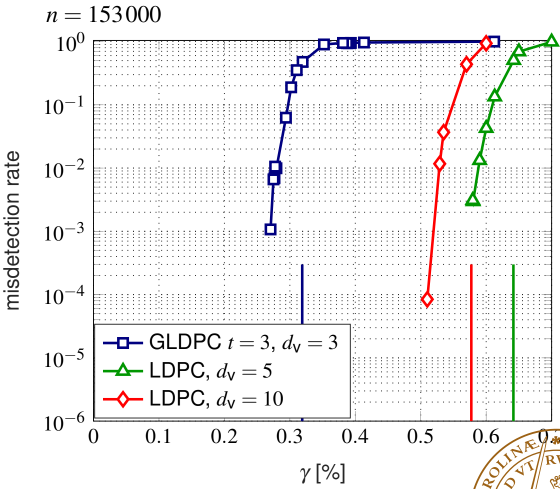

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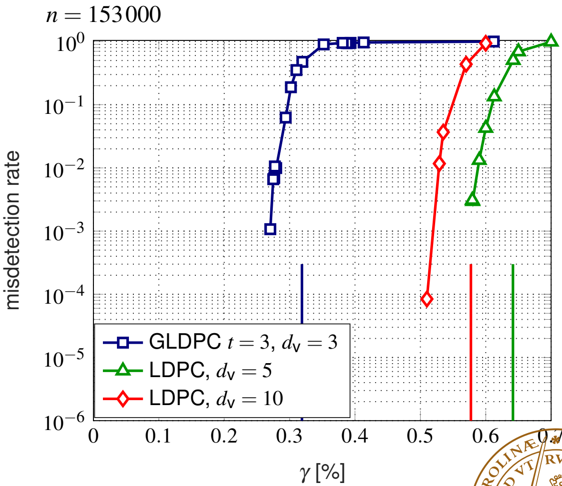

- Can we get consistently better with increasing d_{v}? What about spatial coupling?

Improved Decoding with a Non-Binary Alphabet

- **Idea:** for a subset of tests, items occur only in bundles of size q

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Example: $d_\mathsf{v} = 3$, $d_\mathsf{c} = 4$

$$\boldsymbol{A} = \begin{array}{cccccccc} \multicolumn{2}{c}{z_1} & \multicolumn{2}{c}{z_2} & \multicolumn{2}{c}{z_3} & \multicolumn{2}{c}{z_4} \\ \end{array}$$

$$\boldsymbol{A} = \left[\begin{array}{cccc} \boxed{1 \quad 1} & 0 \quad 0 & 0 \quad 0 & \boxed{1 \quad 1} \\ 0 \quad 0 & \boxed{1 \quad 1} & \boxed{1 \quad 1} & 0 \quad 0 \\ \boxed{1 \quad 1} & 0 \quad 0 & \boxed{1 \quad 1} & 0 \quad 0 \\ 0 \quad 0 & \boxed{1 \quad 1} & 0 \quad 0 & \boxed{1 \quad 1} \\ \hline 1 \quad 0 & 1 \quad 0 & 1 \quad 0 & 1 \quad 0 \\ 0 \quad 1 & 0 \quad 1 & 0 \quad 1 & 0 \quad 1 \end{array}\right] \qquad q = 2$$

$$x_1 \; x_2 \; x_3 \; x_4 \; x_5 \; x_6 \; x_7 \; x_8$$

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Factor graph representation:

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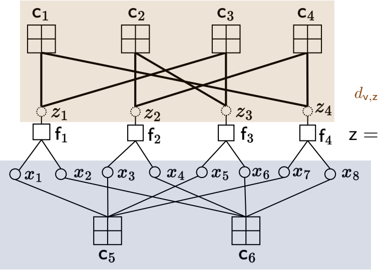

$d_{\mathsf{v,z}} = 2$

$\mathsf{z} = \mathsf{f}(\boldsymbol{x}) = \sum_{i \in \mathcal{T}(\mathsf{f})} x_i$

$d_{\mathsf{v,x}} = 1$

$d_{\mathsf{v}} = d_{\mathsf{v,z}} + d_{\mathsf{v,x}}$

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Local Decoding with q-ary Variables

Extension of the erasure decoder to $q > 1$:

- messages: $\mu \in \{0, q, ?\}$
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- messages $\mu = [L, U]$ consist of lower bound L and upper bound U on $z \in \{0, \ldots, q\}$
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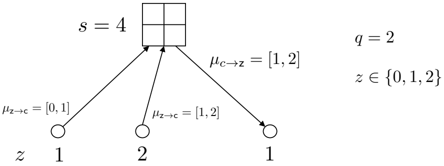

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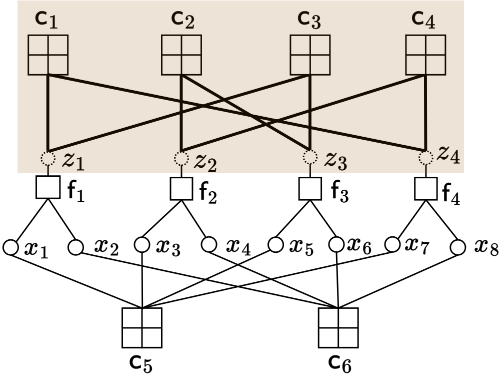
Message Passing between Bundles and Tests

$$\mathcal{CN}_{\mathsf{z}} = \{\mathsf{c}_1, \mathsf{c}_2, \mathsf{c}_3, \mathsf{c}_4\}$$



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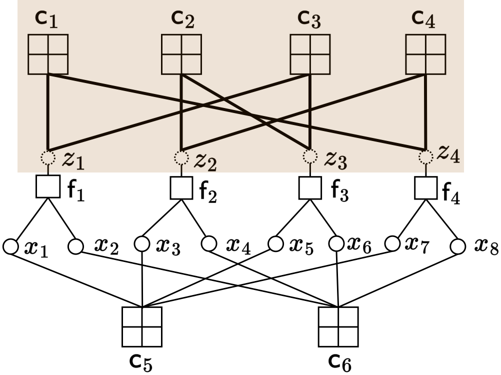

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Test to bundle:

$$\mathsf{L}^{(\ell)}_{\mathsf{c}\to\mathsf{z}} = \max\left\{ s(\mathsf{c}) - \sum_{\mathsf{z}'\in\mathcal{T}(\mathsf{c})\setminus\mathsf{z}} \mathsf{U}^{(\ell-1)}_{\mathsf{z}'\to\mathsf{c}}, 0 \right\}$$

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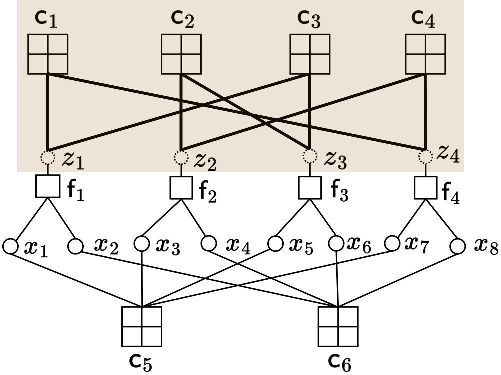

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Message Passing between Items and Tests

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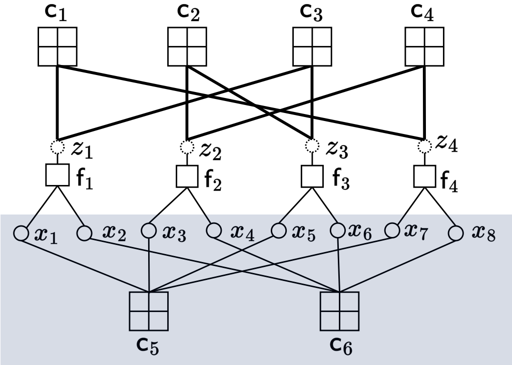

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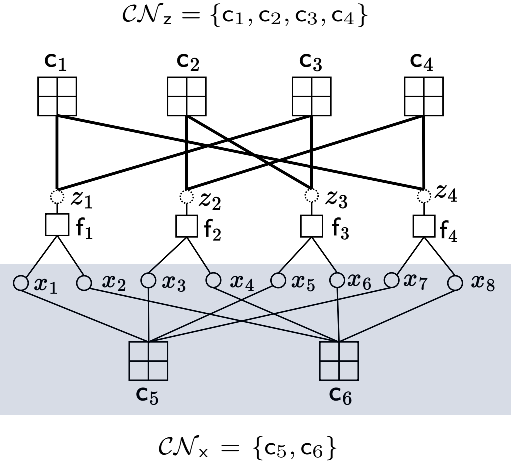

Test to item:

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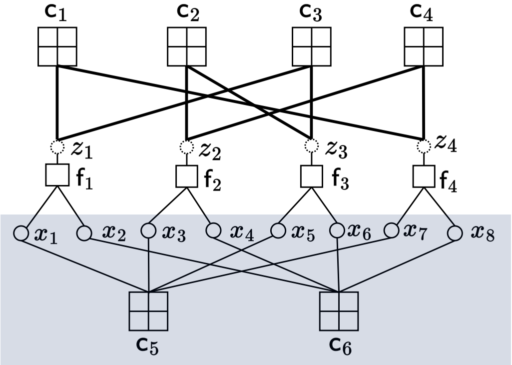

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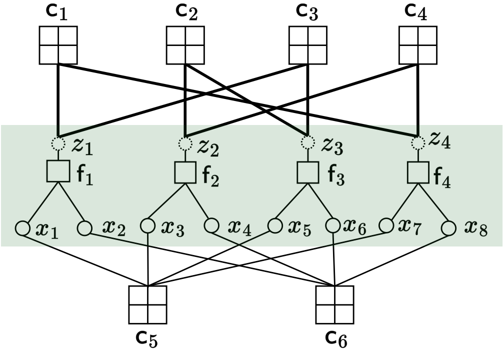

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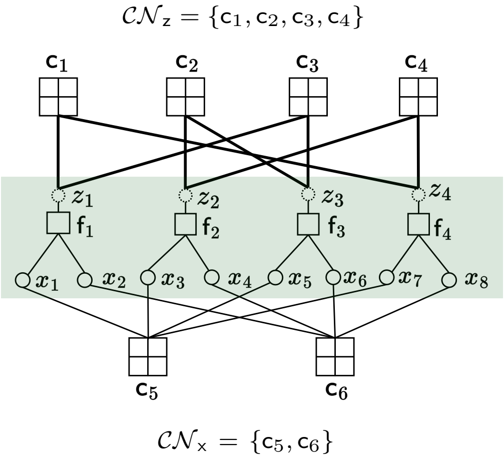

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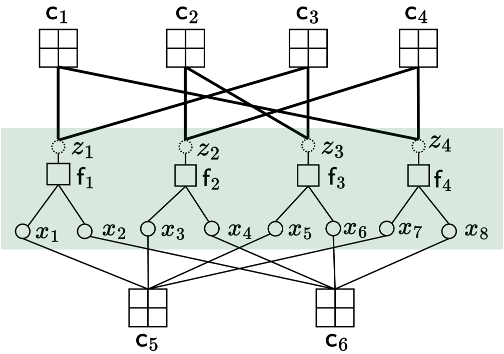

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Bundle to item:

$$\mathsf{L}^{(\ell)}_{\mathsf{z}\to\mathsf{f}} = \max_{\mathsf{c}\in\mathcal{T}(\mathsf{z})} \mathsf{L}^{(\ell)}_{\mathsf{c}\to\mathsf{z}} \text{ and } \mathsf{U}^{(\ell)}_{\mathsf{z}\to\mathsf{f}} = \min_{\mathsf{c}\in\mathcal{T}(\mathsf{z})} \mathsf{U}^{(\ell)}_{\mathsf{c}\to\mathsf{z}}\,.$$

$$\mathsf{L}^{(\ell)}_{\mathsf{f}\to\mathsf{x}} = \max\left\{\mathsf{L}^{(\ell-1)}_{\mathsf{z}\to\mathsf{f}} - \sum_{\mathsf{x}'\in\mathcal{N}(\mathsf{f})\setminus\mathsf{x}} \mathsf{U}^{(\ell-1)}_{\mathsf{x}'\to\mathsf{f}}, 0\right\}$$

$$\mathsf{U}^{(\ell)}_{\mathsf{f}\to\mathsf{x}} = \min\left\{\mathsf{U}^{(\ell-1)}_{\mathsf{z}\to\mathsf{f}} - \sum_{\mathsf{x}'\in\mathcal{N}(\mathsf{f})\setminus\mathsf{x}} \mathsf{L}^{(\ell-1)}_{\mathsf{x}'\to\mathsf{f}}, 1\right\},$$

Item to bundle:

$$\mathsf{L}^{(\ell)}_{\mathsf{x}\to\mathsf{f}} = \max_{\mathsf{c}\in\mathcal{T}_s(\mathsf{x})} \mathsf{L}^{(\ell-1)}_{\mathsf{c}\to\mathsf{x}} \quad \text{and} \quad \mathsf{U}^{(\ell)}_{\mathsf{x}\to\mathsf{f}} = \min_{\mathsf{c}\in\mathcal{T}_s(\mathsf{x})} \mathsf{U}^{(\ell-1)}_{\mathsf{c}\to\mathsf{x}}\,.$$

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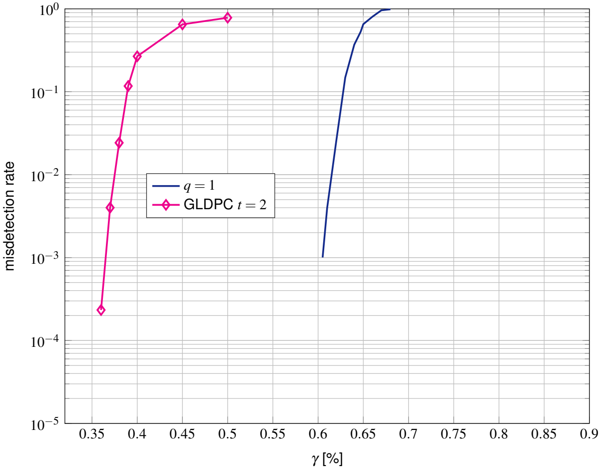
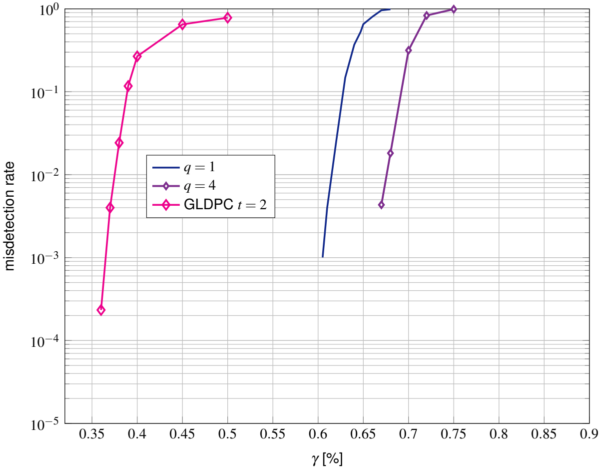
Performance Evaluation



Simulation results:

$n = 210\,000$ items
Fixed rate $\Omega = 5\,\%$, i.e., $m = 10\,500$ tests

Performance Evaluation



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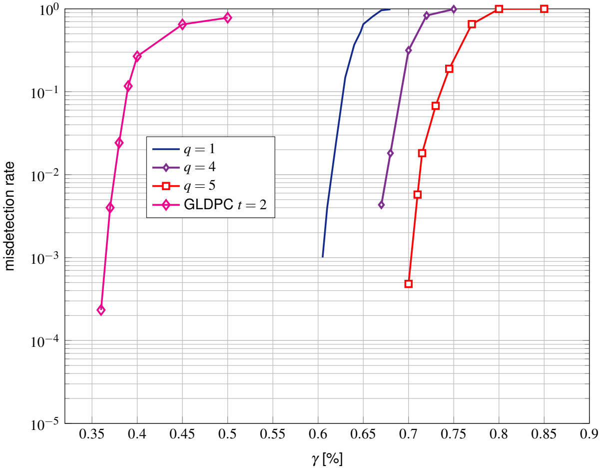

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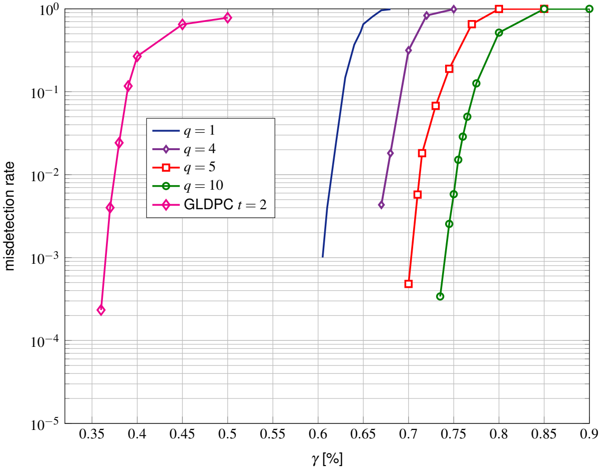

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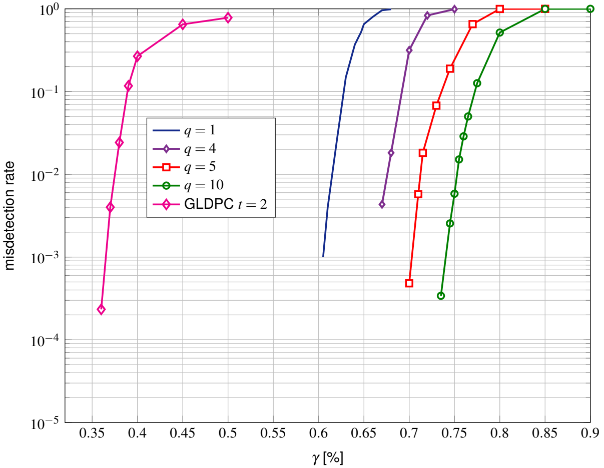

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Density evolution thresholds: γ_{th}

q	$d_{\mathrm{v,x}}$	$d_{\mathrm{v}} = 4$	$d_{\mathrm{v}} = 5$	$d_{\mathrm{v}} = 6$	$d_{\mathrm{v}} = 7$	$d_{\mathrm{v}} = 8$
1		0.598	0.641	0.646	0.635	0.618
4	2	0.590	0.660	0.694	0.706	0.702
5	2	0.592	0.672	0.725	0.746	0.744
10	3	0.549	0.636	0.693	0.774	0.694

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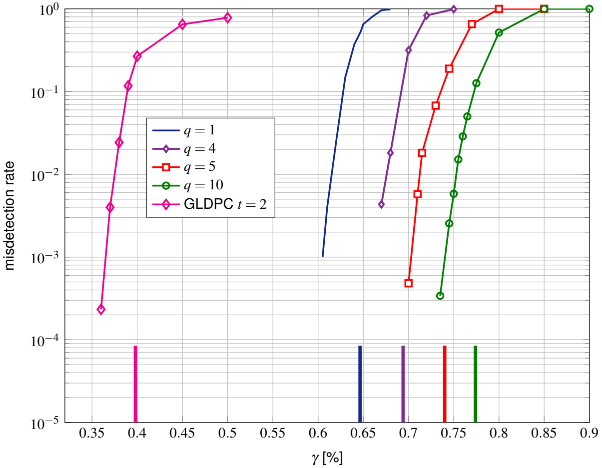

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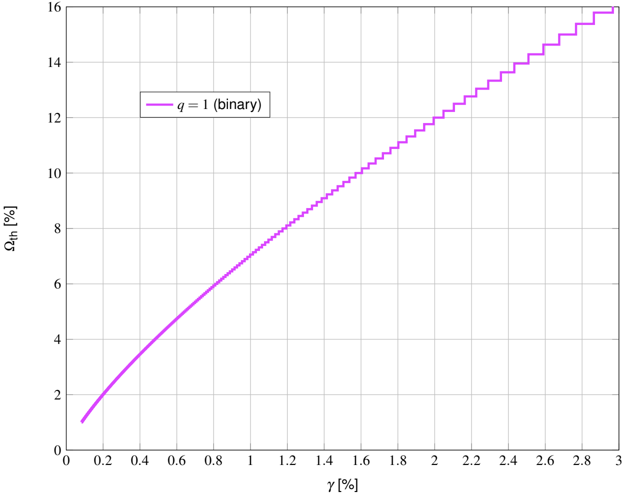
Minimum Rate Ω_{th} for a Fixed γ



$$\Omega^*_{\text{th}} = \frac{d_{\text{v}}}{d_{\text{c}}} = \frac{m}{n}$$

(smaller is better)

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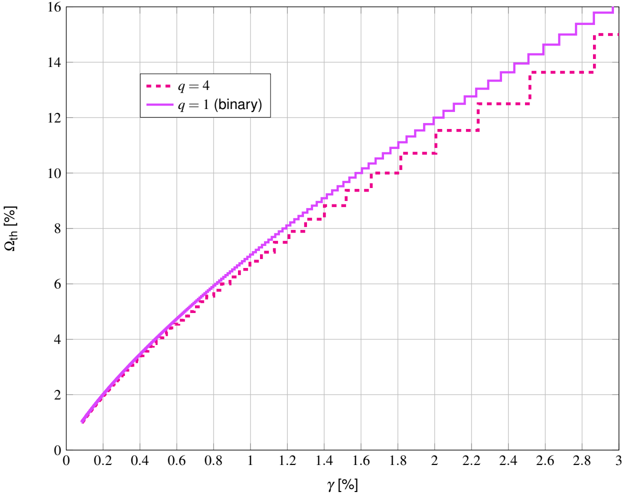

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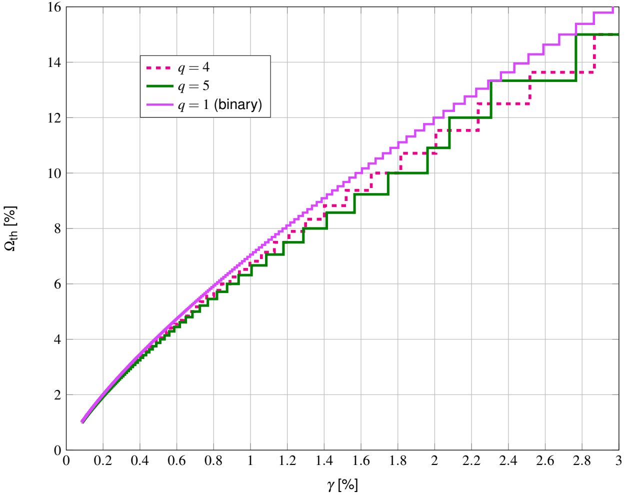

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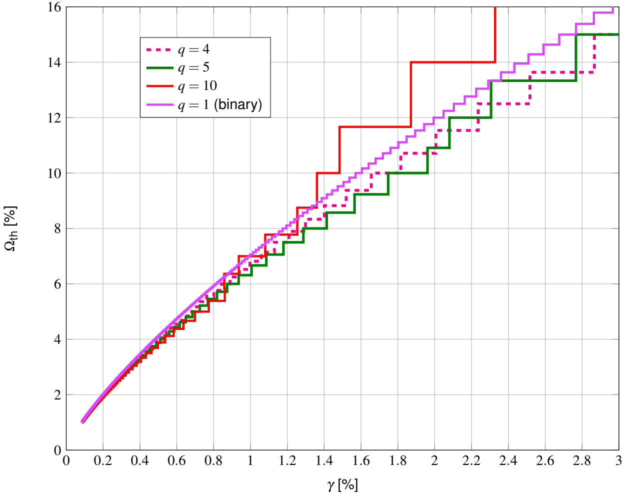
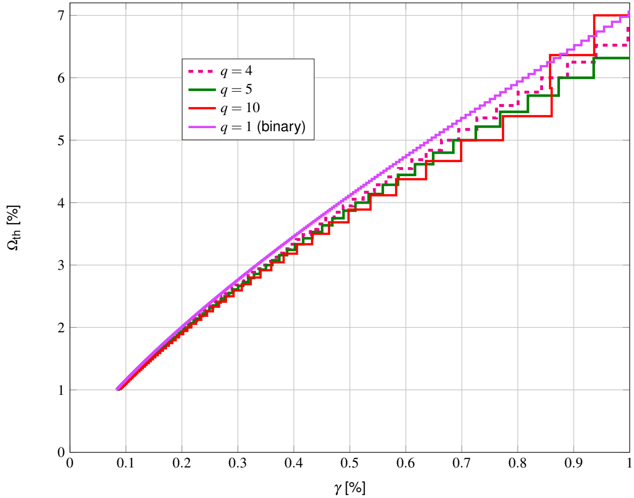


$$\Omega_{\mathrm{th}}^{*} = \frac{d_{\mathrm{v}}}{d_{\mathrm{c}}} = \frac{m}{n}$$

(smaller is better)

Minimum Rate Ω_{th} for a Fixed γ

Consider a smaller range:



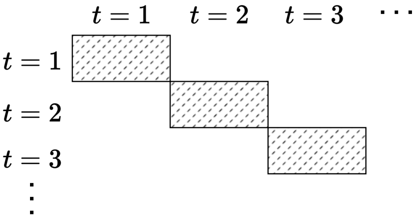
$$\Omega_{\text{th}}^* = \frac{d_{\text{v}}}{d_{\text{c}}} = \frac{m}{n}$$

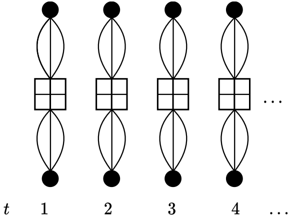
(smaller is better)

Observe:
best q depends on range of γ

Group Testing with Spatial Coupling

- **Classical approach**: test each block of items separately





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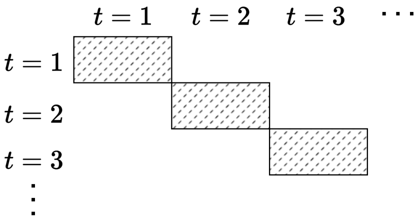
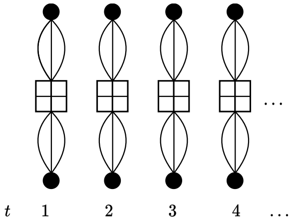
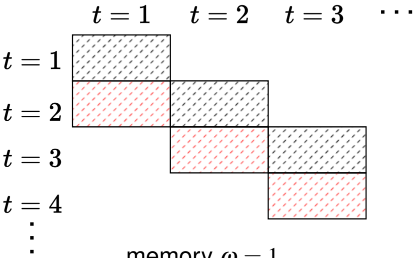
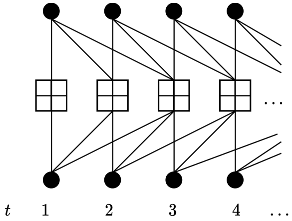




- **Spatial Coupling**: Interconnect blocks (motivated by results in coding theory)



memory $\omega = 1$



Spatial coupling: Performance for Fixed Rate $(q = 1)$

Table: γ_{th} for $\Omega = 5\%$ with GLDPC Code-Based

t	d_{v}	$\omega = 0$	$\omega = 1$	$\omega = 5$	$\omega = 10$
1	3	0.3708	0.4166	0.4166	0.4166
	4	0.3510	0.4395	0.4425	0.4425
3	3	0.3189	0.4257	0.4379	0.4395
	4	0.2441	0.3662	0.4028	0.4028
5	3	0.2686	0.3784	0.4089	0.4089
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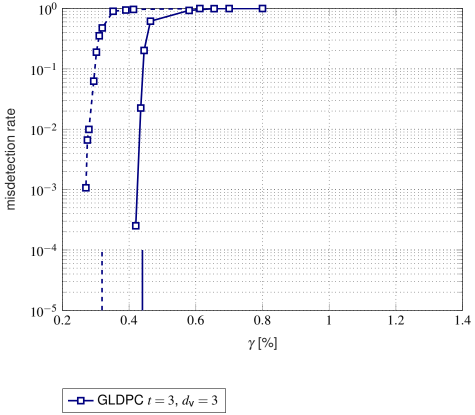

- $n = 153\,000$, $L = 200$, $\omega = 5$
- solid(coupled) dashed(uncoupled)

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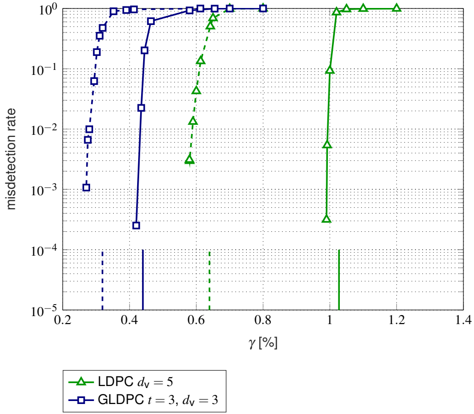
10	0.5773	0.9188	1.2814	1.2816



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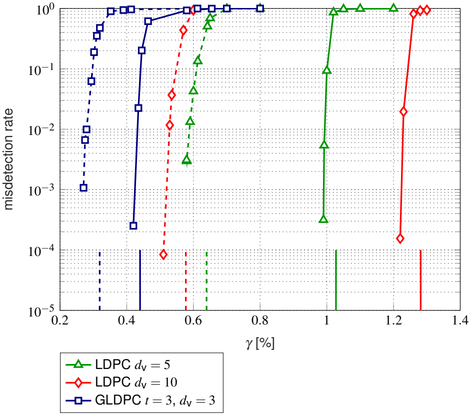

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Proving threshold saturation: minimum Ω for a fixed γ $(q=1)$

- **Vector admissible system:** [YED2012] a recursion $(\mathbf{f},\boldsymbol{g})$ with

$$\boldsymbol{x}^{(\ell)} = \mathbf{f}\left(\boldsymbol{g}(\boldsymbol{x}^{(\ell-1)});\varepsilon\right) \;, \quad \boldsymbol{x}^{(0)} = \mathbf{1} \;, \; \varepsilon \in [0,1]$$

where $\mathbf{f}(\boldsymbol{x}) = \;[f_1(\boldsymbol{x}),\cdots,f_d(\boldsymbol{x})]$ and $\boldsymbol{g}(\boldsymbol{x}) = \;[g_1(\boldsymbol{x}),\cdots,g_d(\boldsymbol{x})]$ are twice continuously differentiable and strictly increasing in all arguments.

[YED2012] A. Yedla, Y.-Y. Jian, P. S. Nguyen, and H. D. Pfister, “A simple proof of threshold saturation for coupled vector recursions,” in*Proc. IEEE Inf. Theory Workshop (ITW)*, 2012.

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- Setting $\varepsilon = 1 - \frac{1}{d_{\mathsf{c}}}$ we get from density evolution equations:

$$\mathbf{f}(y_0,y_1;\varepsilon) = \left[1-(1-y_1)^{\frac{\varepsilon}{1-\varepsilon}}, \quad 1-(1-y_0)^{\frac{\varepsilon}{1-\varepsilon}}\right]$$

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- Threshold saturation occurs
- The potential function is then given as

$$\mathrm{U}(\boldsymbol{x};\varepsilon)=\int_0^1\Big(\big(z(\lambda)-\mathbf{f}\left(\boldsymbol{g}(z(\lambda));\varepsilon\right)\big)\boldsymbol{D}\boldsymbol{g}'(z(\lambda))\Big)\cdot z'(\lambda)\mathrm{d}\lambda$$

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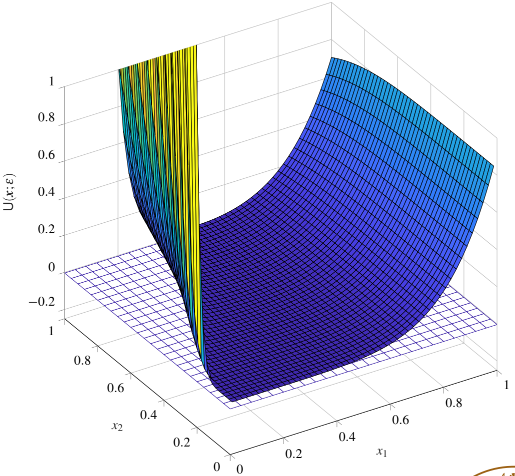
Potential function $(q = 1)$

$$\mathsf{U}(\boldsymbol{x};\varepsilon) = (1-p)x_1^{d_\mathsf{v}-1}\left((1-\varepsilon)\frac{1-\left(1-px_2^{d_\mathsf{v}-1}\right)^{\frac{1}{1-\varepsilon}}}{px_2^{d_\mathsf{v}-1}} + \frac{(d_\mathsf{v}-1)}{d_\mathsf{v}}x_1 - 1\right)$$
$$+ px_2^{d_\mathsf{v}-1}\left((1-\varepsilon)\frac{1-\left(1-(1-p)x_2^{d_\mathsf{v}-1}\right)^{\frac{1}{1-\varepsilon}}}{(1-p)x_1^{d_\mathsf{v}-1}} + \frac{(d_\mathsf{v}-1)}{d_\mathsf{v}}x_2 - 1\right)$$

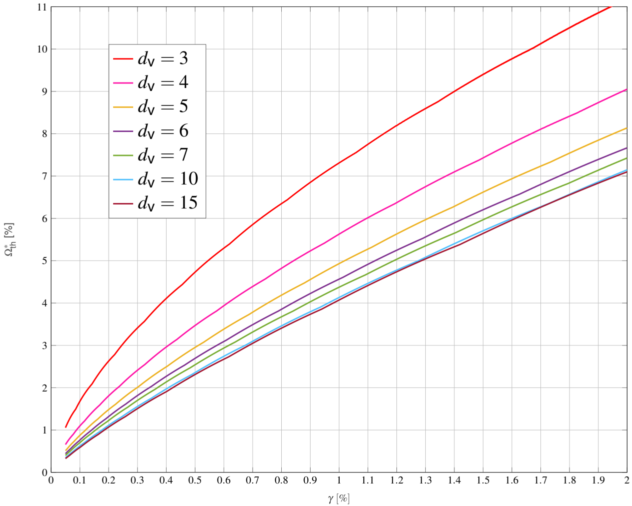


Potential threshold:

$$\varepsilon^* = \sup\{\varepsilon \in [0,1] \mid \min_{\boldsymbol{x}} \mathsf{U}(\boldsymbol{x};\varepsilon) \geq 0\}\,.$$

$d_\mathsf{v} = 6$, $\gamma = 1\%$ with $\varepsilon^* = 0.9924$. $\mathsf{U}(\boldsymbol{x};\varepsilon)$ is above the $z = 0$ plane since $\varepsilon = 0.9667 < \varepsilon^*$.

Potential thresholds $(q = 1)$



$$\Omega^*_{\mathsf{th}} = \frac{d_\mathsf{v}}{d_\mathsf{c}} = d_\mathsf{v}(1 - \varepsilon^*)\,.$$

$$\varepsilon^* = \sup\{\varepsilon \in [0,1] \,|\, \min_{\boldsymbol{x}} \mathsf{U}(\boldsymbol{x};\varepsilon) \geq 0\}\,.$$

The minimum rate Ω^*_{th} for a fixed γ computed from the potential threshold ε^*.

Conclusions and Outlook

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- Using a simple LDPC code significantly outperforms a GLDPC construction
- We can measure the performance by two different approaches
  - Fixing the proportion γ and determining minimum rate Ω
  - Fixing the rate, Ω and determining the maximum γ

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Outlook

- Spatial coupling with q-bundles
- Looking at soft message passing