Coding over Interference Channels: An Information-Estimation View

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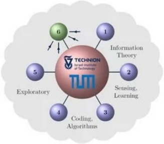


Based on joint work with R. Bustin, General Motors and A. Dytso, QUALCOMM

Outline

* I-MMSE scalar and vector relations.
* Good and 'bad' codes.
* Information versus MMSE disturbance.
* Degrees-of-Freedom: an MMSE dimension perspective.
* Outlook.

I-MMSE

- The I-MMSE relation [Guo-Shamai-Verdú, IT'05].

$$Y = \sqrt{\mathsf{snr}}\, X + N$$

X – Input signal, Y – Output signal, N – Gaussian noise $\sim \mathcal{N}(0,1)$,
snr – Signal-to-Noise Ratio.

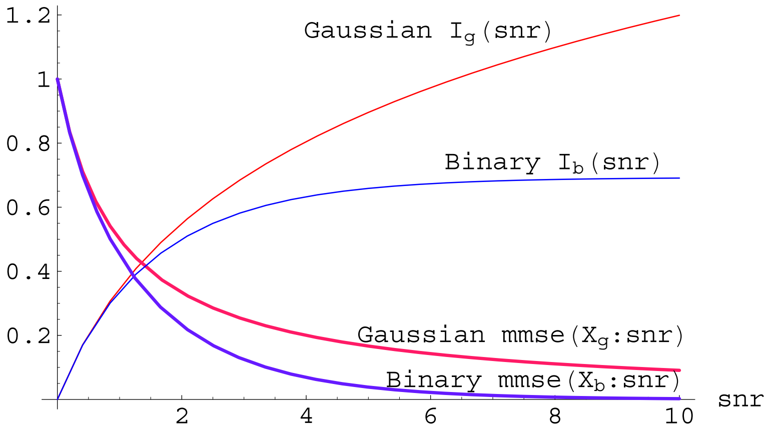
$$\boxed{\frac{d}{d\mathsf{snr}} I(X;Y) = \frac{1}{2}\mathsf{mmse}(X:\mathsf{snr})}$$

$$\mathsf{mmse}(X|Y) = \mathsf{mmse}(X:\mathsf{snr}) = \mathsf{mmse}(\mathsf{snr}) = E\Big(X - E(X|Y)\Big)^2.$$

I-MMSE - examples

- I-MMSE: Gaussian Example: $X \sim \mathcal{N}(0,1)$.
  - $\text{mmse}(X : \text{snr}) = E\left(X - \frac{\sqrt{\text{snr}}}{1+\text{snr}} Y\right)^2 = \frac{1}{1+\text{snr}}$,
  - $I(X;Y) = I_g(\text{snr}) = \frac{1}{2}\log(1+\text{snr})$.
- I-MMSE: Binary Example: $X = \pm 1$, symmetric.
  - $\text{mmse}(X : \text{snr}) = 1 - \int_{-\infty}^{\infty} \frac{e^{-y^2/2}}{\sqrt{2\pi}} \tanh(\text{snr} - \sqrt{\text{snr}}\, y)\, dy$
  - $I(X;Y) = I_b(\text{snr}) = \text{snr} - \int_{-\infty}^{\infty} \frac{e^{-y^2/2}}{\sqrt{2\pi}} \log\cosh(\text{snr} - \sqrt{\text{snr}}\, y)\, dy$

$$\frac{d}{d\mathsf{snr}} I(X;Y) = \frac{1}{2}\mathsf{mmse}(X:\mathsf{snr})$$



Vector Channel

Theorem (Guo-Shamai-Verdú, IT'05)

Let $\boldsymbol{Y} = \sqrt{\mathsf{snr}}\,\boldsymbol{HX} + \boldsymbol{N}$.

If $\mathsf{E}\|\boldsymbol{X}\|^2 < \infty$, $\boldsymbol{X}, \boldsymbol{Y}, \boldsymbol{N} \sim \mathcal{N}(0, I)$ *vectors*

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{dsnr}} I(\boldsymbol{X}; \sqrt{\mathsf{snr}}\,\boldsymbol{HX} + \boldsymbol{N}) &= \frac{1}{2}\mathrm{mmse}(\boldsymbol{HX} | \sqrt{\mathsf{snr}}\,\boldsymbol{HX} + \boldsymbol{N}) \\ = \frac{1}{2}\mathrm{mmse}(\mathsf{snr}) &= \frac{1}{2}\mathsf{E}\,\|\boldsymbol{H}\,\boldsymbol{X} - \boldsymbol{H}\,\mathsf{E}\{\boldsymbol{X}|\boldsymbol{Y}\}\|^2 \\ &= \frac{1}{2}\mathrm{Tr}(\boldsymbol{H}\mathbf{E}_{\boldsymbol{X}}(\mathsf{snr})\boldsymbol{H}^{\mathsf{T}})\end{aligned}$$

where $\mathbf{E}_{\boldsymbol{X}}(\mathsf{snr})$ *is the MMSE matrix of estimating* $\boldsymbol{X}$ *from* $\boldsymbol{Y}$.

Special Case: MIMO channel with Gaussian Inputs

$$\boldsymbol{Y} = \sqrt{\mathsf{snr}} \cdot \boldsymbol{H}\,\boldsymbol{X} + \boldsymbol{N}$$

$\boldsymbol{X}$ – iid standard Gaussian

$$I(\boldsymbol{X};\boldsymbol{Y}) = \frac{1}{2}\mathrm{logdet}\left(\boldsymbol{I} + \mathsf{snr}\boldsymbol{H}^{\mathsf{T}}\boldsymbol{H}\right)$$

Error covariance matrix:

$$\mathsf{E}\left\{\left(\boldsymbol{X} - \widehat{\boldsymbol{X}}\right)\left(\boldsymbol{X} - \widehat{\boldsymbol{X}}\right)^{\mathsf{T}}\right\} = \left(\boldsymbol{I} + \mathsf{snr}\boldsymbol{H}^{\mathsf{T}}\boldsymbol{H}\right)^{-1}$$

$$\begin{aligned} \mathsf{mmse}(\mathsf{snr}) &= \mathsf{E}\left\{\left\|\boldsymbol{H}\boldsymbol{X} - \boldsymbol{H}\widehat{\boldsymbol{X}}\right\|^2\right\}, \\ &= \mathrm{Tr}\left\{\left(\boldsymbol{I} + \mathsf{snr}\boldsymbol{H}^{\mathsf{T}}\boldsymbol{H}\right)^{-1}\boldsymbol{H}^{\mathsf{T}}\boldsymbol{H}\right\} \end{aligned}$$

Some Abbreviations

γ - designates signal-to-noise ratio.
n - length of the input/output vectors.

$$I_n(\gamma) = \frac{1}{n} I\left(\boldsymbol{X}; \boldsymbol{Y}(\gamma)\right)$$

$$I(\gamma) = \lim_{n\to\infty} \frac{1}{n} I\left(\boldsymbol{X}; \boldsymbol{Y}(\gamma)\right)$$

$$\mathrm{MMSE}^{\mathrm{c}_n}(\gamma) = \frac{1}{n} \mathrm{Tr}(\mathbf{E}_{\boldsymbol{X}}(\gamma))$$

$$\mathrm{MMSE}^{\mathrm{c}}(\gamma) = \lim_{n\to\infty} \mathrm{MMSE}^{\mathrm{c}_n}(\gamma)$$

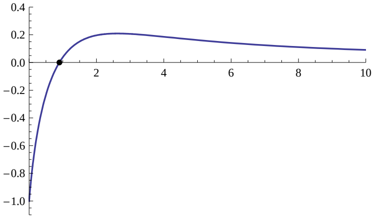
The Single Crossing Point Property

Theorem (Guo-Wu-Shamai-Verdú, IT'11)

$$f(\gamma) \triangleq (1+\gamma)^{-1} - \mathsf{mmse}(X:\gamma)$$

If X is not standard Gaussian, f has at most one zero ($\mathrm{Var}(X) > 1$).
If $f(\mathsf{snr}_0) = 0$, then

1. $f(0) \leq 0$;
2. $f(\gamma)$ *is strictly increasing on* $\gamma \in [0, \mathsf{snr}_0]$;
3. $f(\gamma) > 0$ *for every* $\gamma \in (\mathsf{snr}_0, \infty)$; *and*
4. $\lim_{\gamma\to\infty} f(\gamma) = 0$.



The Single Crossing Point Property - Extension

- Several MIMO extensions are given in [Bustin, Payaró, Palomar and Shamai, IT'13]. We give here only the simplest extension.

Theorem (Bustin, Payaró, Palomar and Shamai, IT'13)

The function

$$q(\boldsymbol{X}, \sigma^2, \gamma) = \frac{\sigma^2}{1 + \gamma\sigma^2} - \frac{1}{n}\mathrm{Tr}(\mathbf{E}_{\boldsymbol{X}}(\gamma))$$

has no nonnegative-to-negative zero crossings and, at most, a single negative-to-nonnegative zero crossing in the range $\gamma \in [0, \infty)$. *Moreover, assume* $\mathrm{snr}_0 \in [0, \infty)$ *is a negative-to-nonnegative crossing point. Then,*

- $q(\boldsymbol{X}, \sigma^2, 0) \leq 0$.
- $q(\boldsymbol{X}, \sigma^2, \gamma)$ *is a strictly increasing function in the range* $\gamma \in [0, \mathrm{snr}_0)$.
- $q(\boldsymbol{X}, \sigma^2, \gamma) \geq 0$ *for all* $\gamma \in [\mathrm{snr}_0, \infty)$.
- $\lim_{\gamma \to \infty} q(\boldsymbol{X}, \sigma^2, \gamma) = 0$.

Optimal Point-to-Point Codes

Looking at code-sequences over the scalar Gaussian channel.

Theorem (Peleg, Sanderovich and Shamai, ETT'07)

For every capacity achieving code-sequence, C_n*, over the Gaussian channel, the mutual information, when* $n \to \infty$*, is as follows:*

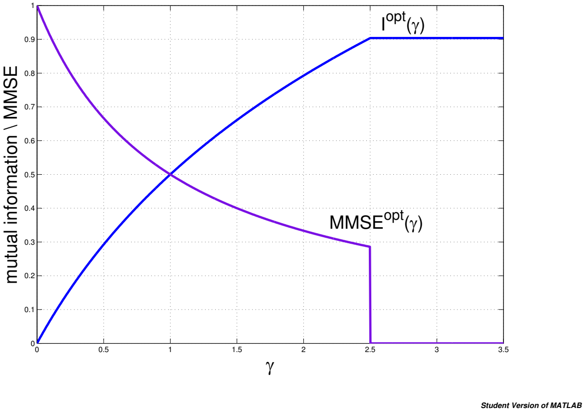
$$I(\gamma) = \lim_{n\to\infty} \frac{1}{n} I(\boldsymbol{X}; \sqrt{\gamma}\boldsymbol{X} + \boldsymbol{N}) = \begin{cases} \frac{1}{2}\log(1+\gamma), & \gamma \leq \mathsf{snr} \\ \frac{1}{2}\log(1+\mathsf{snr}), & \text{o/w} \end{cases}$$

and the MMSE is:

$$\mathrm{MMSE}^{\mathrm{c}}(\gamma) = \begin{cases} \frac{1}{1+\gamma}, & \gamma \leq \mathsf{snr} \\ 0, & \text{o/w} \end{cases}$$

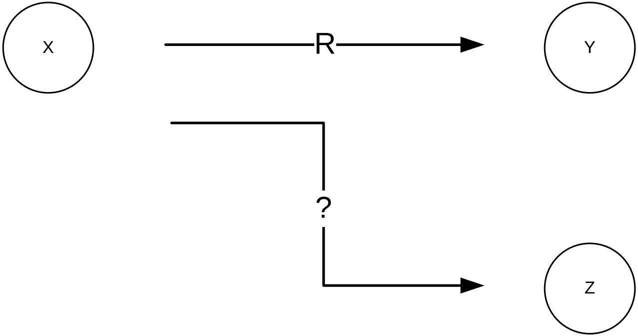
Optimal Point-to-Point Codes - Cont.

⇒ The mutual information (and MMSE) of optimal point-to-point codes follow the behavior of an i.i.d. Gaussian input up to snr.



⇒ Area under curve $= R \Rightarrow$ Good code must exhibit a threshold (phase transition).

What is the effect on an unintended receiver?



Assumption: the unintended receiver, Z, has smaller snr, that is, $\text{snr}_z < \text{snr}_y$.

How should we measure the effect (disturbance)?

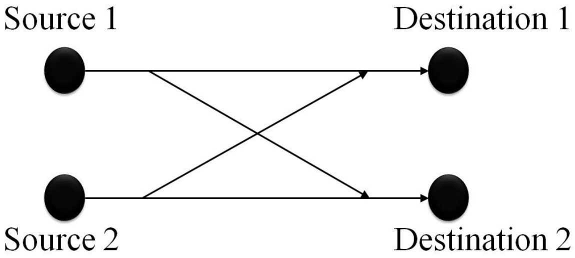
For optimal point-to-point codes both the mutual information and MMSE are completely known.

But what about non-optimal code (that do not attain capacity)?

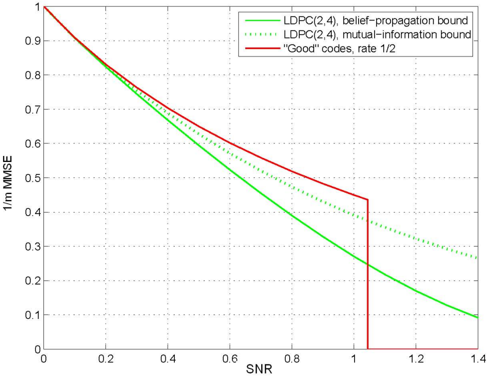
'Bad' Codes & The Interference Channel

- 'Bad' Codes: Trade rate versus MMSE (interference) in lower snr ($R >$ capacity) [Bennatan-Shamai-Calderbank, IT-S'11]
- Interference channels:
  - Interfering codeword not decodable
  - Interference Networks with Point-to-Point Codes [Baccelli-El Gamal-Tse, ISIT'11]

  ⇒ Soft partial interference cancellation



'Bad' Codes – Multiterminal Systems: MMSE Bounds



* I-MMSE Application: deterministic finite length codes
- MI-Bound: I-MMSE invoking properties in [Wiechman-Sason, IT'07]
- BP-Bound: based on tools in [Richardson-Urbanke, IT'01]

Problem Definition - Single MMSE Constraint

Assuming $\mathsf{snr}_1 < \mathsf{snr}_2$ and $\mathsf{MMSE}^c(\mathsf{snr}_1)$ is limited to some value.

What is the maximum possible rate at snr_2?

In other words:

$$\begin{aligned} \max \quad & I(\mathsf{snr}_2) \\ \text{s.t.} \quad & \mathsf{MMSE}^c(\mathsf{snr}_1) \leq \frac{\beta}{1+\beta\mathsf{snr}_1} \end{aligned}$$

for some $\beta \in [0, 1]$.

Superposition codes

The mutual information, $I(\gamma)$, and $\mathsf{MMSE}^c(\gamma)$ of an optimal Gaussian superposition code are known exactly, for all γ.

Theorem (Merhav, Guo and Shamai, IT'10)

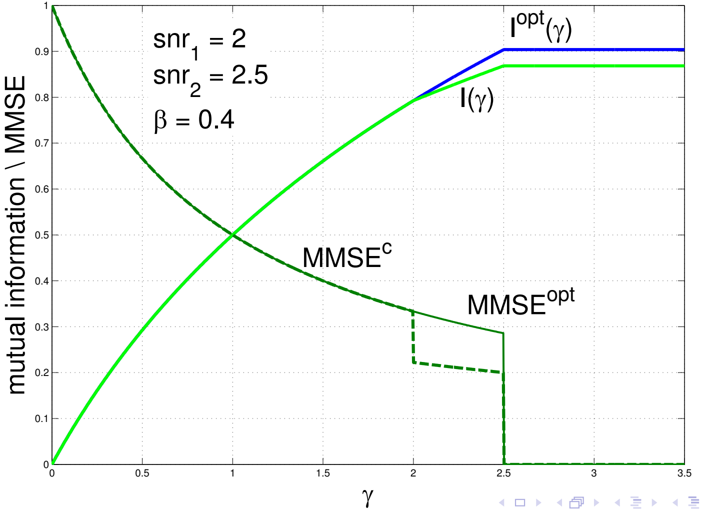
A superposition codebook designed for $(\mathsf{snr}_1, \mathsf{snr}_2)$ *with the rate-splitting coefficient* $\beta < 1$ *has the following mutual information:*

$$I(\gamma) = \begin{cases} \frac{1}{2}\log\left(1+\gamma\right), & \text{if} \quad 0 \le \gamma < \mathsf{snr}_1 \\ \frac{1}{2}\log\left(\frac{1+\mathsf{snr}_1}{1+\beta\mathsf{snr}_1}\right) + \frac{1}{2}\log\left(1+\beta\gamma\right), & \text{if} \quad \mathsf{snr}_1 \le \gamma \le \mathsf{snr}_2 \\ \frac{1}{2}\log\left(\frac{1+\mathsf{snr}_1}{1+\beta\mathsf{snr}_1}\right) + \frac{1}{2}\log\left(1+\beta\mathsf{snr}_2\right), & \text{if} \quad \mathsf{snr}_2 < \gamma \end{cases}$$

and the following $\mathsf{MMSE}^c(\gamma)$*:*

$$\mathsf{MMSE}^c(\gamma) = \begin{cases} \frac{1}{1+\gamma}, & 0 \le \gamma < \mathsf{snr}_1 \\ \frac{\beta}{1+\beta\gamma}, & \mathsf{snr}_1 \le \gamma \le \mathsf{snr}_2 \\ 0, & \mathsf{snr}_2 < \gamma \end{cases}$$

Superposition codes - Example



Solution - Single MMSE Constraint

Theorem (Bustin and Shamai, IT'13)

Assuming $\mathsf{snr}_1 < \mathsf{snr}_2$ *the solution of the following optimization problem,*

$$\begin{aligned} \max \quad & I(\mathsf{snr}_2) \\ \text{s.t.} \quad & \mathsf{MMSE}^c(\mathsf{snr}_1) \leq \frac{\beta}{1+\beta \mathsf{snr}_1} \end{aligned}$$

for some $\beta \in [0,1]$, *is the following*

$$I(\mathsf{snr}_2) = \frac{1}{2}\log\left(1+\beta \mathsf{snr}_2\right) + \frac{1}{2}\log\left(\frac{1+\mathsf{snr}_1}{1+\beta \mathsf{snr}_1}\right)$$

and is attainable when using the optimal Gaussian superposition codebook designed for $(\mathsf{snr}_1, \mathsf{snr}_2)$ *with a rate-splitting coefficient* β.

The Effect at Other snrs

Theorem (Bustin and Shamai, IT'13)

From the set of reliable codes of rate

$$\mathsf{R}_c = \frac{1}{2}\log\left(1+\beta\mathsf{snr}_2\right) + \frac{1}{2}\log\left(\frac{1+\mathsf{snr}_1}{1+\beta\mathsf{snr}_1}\right)$$

complying with the MMSE constraint at snr_1*:*

$$\mathsf{MMSE}^c(\mathsf{snr}_1) \leq \frac{\beta}{1+\beta\mathsf{snr}_1}$$

the superposition codebook provides the minimum MMSE for all snrs.

Extension to K MMSE Constraints

Theorem (Bustin and Shamai, IT'13)

Assuming $\mathsf{snr}_0 < \mathsf{snr}_1 < \ldots < \mathsf{snr}_K$ *the solution of,*

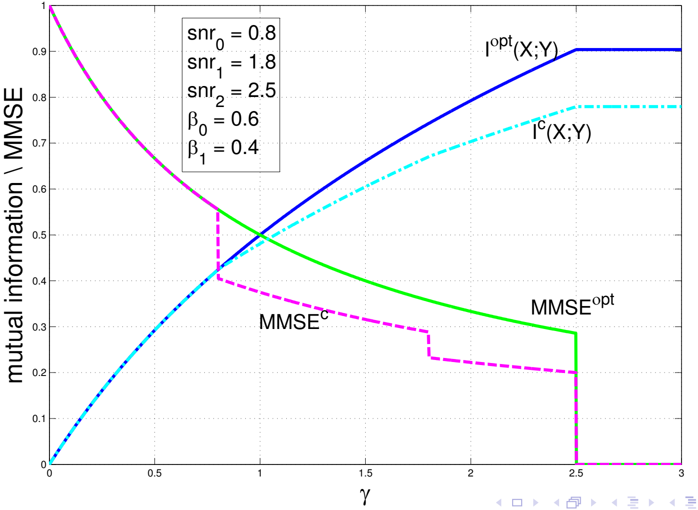
$$\begin{aligned} \max \quad & I(\mathsf{snr}_K) \\ \text{s.t.} \quad & \mathrm{MMSE}^c(\mathsf{snr}_i) \leq \frac{\beta_i}{1+\beta_i \mathsf{snr}_i}, \quad \forall i \in \{0,1,\ldots,K-1\} \end{aligned}$$

for some positive $\beta_i, i \in \{0,1,\ldots,K-1\}$ *such that* $\sum_{i=0}^{K-1} \beta_i \leq 1$ *and* $\beta_{K-1} < \beta_{K-2} < \ldots < \beta_1 < \beta_0$ *is the following*

$$I(\mathsf{snr}_K) = \frac{1}{2}\log\left(\frac{1+\mathsf{snr}_0}{1+\beta_0\mathsf{snr}_0}\prod_{j=1}^{K-1}\frac{1+\beta_{j-1}\mathsf{snr}_j}{1+\beta_j\mathsf{snr}_j}\right) + \frac{1}{2}\log\left(1+\beta_{K-1}\mathsf{snr}_K\right)$$

and is attainable when using the optimal K-layers Gaussian superposition codebook designed for $(\mathsf{snr}_0, \mathsf{snr}_1, \ldots, \mathsf{snr}_K)$ *with power-splitting coefficients* $(\beta_0, \ldots, \beta_{K-1})$*. Additional constraints of the following form:* $\mathrm{MMSE}^c(\mathsf{snr}_\ell) \leq \frac{\beta_\ell}{1+\beta_\ell \mathsf{snr}_\ell}$ *for* $\mathsf{snr}_{i-1} \leq \mathsf{snr}_\ell \leq \mathsf{snr}_i$*, when* $\beta_\ell \geq \beta_{i-1}$*, do not affect the above result.*

K-layer Superposition Code, $K = 3$



Coding Interpretation

* Using MMSE as a disturbance measure, and the resultant optimality of superposition coding, provides an intuitive engineering support for Han-Kobayashi coding strategies over the Gaussian Interference Channel.
* Can an I-MMSE methodology be used to prove the capacity region corner points (Proved Costa's conjecture): Statement: [Sason, Allerton'13], Attempt: [Bustin, Poor, Shamai, '15], Different Proof: [Wasserstein distance based, Polyanskiy, Wo, ISIT'16].
* While [Han-Kobayashi, IT'81] is known to be good over the two users Gaussian interference channel [Etkin-Tse-Wang, IT'08] (capacity within 1 bit), and can not be improved by ML decoders (with random coding) [Bandemer-El Gamal-Kim, arXiv'12], this is not the case for many users Gaussian interference channel.

  For example: Gaussian superposition coding yields DoF=1, immaterial of the number of users for a fully connected Gaussian interference channel.
* Can an I-MMSE perspective be useful, for many users Gaussian interference channels ?

Basic DoF Concepts: Information/MMSE Dimension

* Information dimension [Renyi, Acta-Math. Hung'59], [Wu-Verdú, IT'10]
  X-real valued random variable:

$$d(X) = \lim_{m\to\infty} \frac{H(\langle X\rangle_m)}{\log m}\,, \quad \langle X\rangle_m \triangleq \frac{\lfloor mX\rfloor}{m}$$

- $d(X) < \infty \Longleftarrow \mathbb{E}\log(1+|X|) < \infty$
- vector generalizations

* MMSE dimension [Wu-Verdú, IT'11]

$$Y = \sqrt{\mathsf{snr}}X + N\,,$$
$$\mathcal{D}(X) = \lim_{\mathsf{snr}\to\infty} \mathsf{snr}\,\mathsf{mmse}(X:\mathsf{snr})\,, \quad \mathsf{mmse} = \frac{\mathcal{D}(X)}{\mathsf{snr}} + o\left(\frac{1}{\mathsf{snr}}\right)$$

- $d(X) = \mathcal{D}(X)$ for $X \sim$ discrete, continuous, mixture
- $\lim_{\mathsf{snr}\to\infty} \frac{I(X,\mathsf{snr})}{\frac{1}{2}\log\mathsf{snr}} = d(X)\,,\ d(X) < \infty, \qquad I(X,\mathsf{snr}) \triangleq I(X;Y)$

Interference Channel

$$Y_i = \sum_{j=1}^{K} \sqrt{\text{snr}}\, h_{ij} X_j + N_i\,, \quad H = \{h_{ij}\}\,, \quad (i,j) = 1, \ldots K$$

$\{X_i\}$ independent users' signals, $E(X_i^2) \leq 1$, $N_i \in \mathcal{N}(0,1)$,

$\mathbb{C}(H, \text{snr})$ - capacity region

$$C(H, \text{snr}) \triangleq \left\{ \sum_{i=1}^{K} R_i,\; R^K \in \mathbb{C}(H, \text{snr}) \right\} \text{ sum rate capacity}$$

$$\text{DoF}(H) = \lim_{\text{snr}\to\infty} \frac{C(H, \text{snr})}{\frac{1}{2}\log \text{snr}} = \text{sum DoF}$$

$\mathcal{D}(H) = \text{DoF - region}$

Central results: [Host-Madsen-Nosratinia, ISIT'05], [Cadambe-Jafar, IT'08], [Etkin-Ordentlich, IT'09], [Motahari-Gharan-Maddah Ali-Khandani, IT'11], Tutorial-[Jafar, FnT'11]

$\Rightarrow$ Unlike previously believed DoF(H) "almost surely for all H" is $K/2$ and not 1!

- central tool: number theoretic – 'Diophantine Approximation Theory'

An Information/MMSE Dimension Approach

[Wu-Shamai-Verdú, ISIT'11]:
Non Single Letter Capacity Region [Ahlswede, ISIT'71] $\Rightarrow$

$$C(\mathbf{H}, \mathsf{snr}) = \lim_{n\to\infty} \frac{1}{n} \sup_{\mathbb{P}(x_1^n,\dots x_K^n)} \sum_{i=1}^{K} I(X_i^n; Y_i^n)$$

$$\mathbb{C}(\mathbf{H}, \mathsf{snr}) = \lim_{n\to\infty} \sup_{\mathbb{P}(x_1^n,\dots x_K^n)} \left\{ R_i \le I(X_i^n; Y_i^n), \ \ i = 1, 2 \dots K \right\}$$

$$\Rightarrow I(X_i^n; Y_i^n) = I(X_1^n, X_2^n \dots X_K^n; Y_i^n) - I(X_1^n, \dots, X_K^n; Y_i^n | X_i^n)$$

$\Rightarrow$single-letterized expressions!

$$\mathrm{dof}\,(X^{(K)} : \mathbf{H}) \triangleq \sum_{i=1}^{K} \left\{ d\left(\sum_{j=1}^{K} h_{ij} X_j \right) - d\left(\sum_{j\neq i}^{K} h_{ij} X_j \right) \right\}$$

$$\mathrm{DoF}\,(\mathbf{H}) = \sup_{\mathbb{P}(X^{(K)})\ (\text{independent } X_1 X_2 \dots X_K)} \mathrm{dof}\,(X^{(K)} : \mathbf{H})$$

Information/MMSE Dimension-Results

- DoF $(\mathbf{H})$ is invariant under row or column scaling
- Removing cross-links does not decrease DoF
- Suboptimality of discrete-continuous mixtures!

$\Rightarrow$ to get DoF > 1, necessary to emply singular components $\sim$ MMSE dimension oscillates periodically in snr (dB) around information dimension

- If off-diagonal entries $\mathbf{H}$ are rational and diagonal entries irrational $\Rightarrow$ DoF $(\mathbf{H}) = K/2$ (no need for irrational algebraic numbers [Etkin-Ordentlich, IT'09])
- Example: $\mathbf{H} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 2 & 0 \\ 1 & 1 & 1 \end{pmatrix}$, $\text{DoF}(\mathbf{H}) \geq 1 + \log_6 \frac{1+\sqrt{5}}{2} > \frac{2+\log_2 3}{6}$ [Etkin-Ordentlich, IT'09].
- DoF Region: $\mathbb{D}(\mathbf{H}) \subset co\{e_1, \ldots e_K, \frac{1}{2}I\}$, fully connected $\mathbf{H}$

Outlook: Finite n MMSE Constraint

* Assume $0 < \mathsf{snr}_1 < \mathsf{snr}_2$.

$$\begin{aligned} \max \quad & I_n(\mathsf{snr}_2) \\ \text{s.t.} \quad & \mathrm{MMSE}^{c_n}(\mathsf{snr}_1) \leq \frac{\beta}{1+\beta\mathsf{snr}_1} \end{aligned}$$

where the maximization is over $\mathbb{P}(\boldsymbol{X})$, $\boldsymbol{X}$ - length n random vector complying with the power constraint.

* Conjecture for $n = 1$: For $\beta < 1$ discrete optimizing measure $\mathbb{P}(x)$.

Mutual information disturbance: single constraint

Bandemer and El Gamal, 2011, measure the disturbance at the unintended receiver using the mutual information at Z. That is, assuming this mutual information is at most R_d what is the maximum possible rate to the intended receiver, Y.

Theorem (Bandemer and El Gamal, ISIT'11)

Assuming $\mathsf{snr}_1 < \mathsf{snr}_2$ *the solution of the following optimization problem,*

$$\begin{aligned} \max \quad & I_n(\mathsf{snr}_2) \\ \text{s.t.} \quad & I_n(\mathsf{snr}_1) \leq \frac{1}{2}\log\left(1+\alpha^\star \mathsf{snr}_1\right) \end{aligned}$$

for some $\alpha^\star \in [0,1]$, *is the following*

$$I_n(\mathsf{snr}_2) = \frac{1}{2}\log\left(1+\alpha^\star \mathsf{snr}_2\right).$$

Equality is attained, for any n, by choosing X Gaussian with i.i.d. components of variance $\alpha^\star$. *For* $n \to \infty$ *equality is also attained by a Gaussian codebook designed for* snr_2 *with limited power of* $\alpha^\star$.

Simple I-MMSE proof

* Since, $0 \leq I_n(\mathsf{snr}_1) \leq \frac{1}{2}\log(1+\mathsf{snr}_1)$ there exists an $\alpha^\star \in [0,1]$ such that

$$I_n(\mathsf{snr}_1) = \frac{1}{n}\log(1+\alpha^\star \mathsf{snr}_1).$$

$\Longrightarrow$ $\mathrm{MMSE}^{c_n}(\gamma)$ and $\mathrm{mmse}_G(\gamma)$ of $X_G \sim \mathcal{N}(0,\alpha^\star)$ cross in $[0,\mathsf{snr}_1]$.

* Using the I-MMSE

$$\begin{aligned} I_n(\mathsf{snr}_2) &= \frac{1}{2}\log(1+\alpha^* \mathsf{snr}_1) + \frac{1}{2}\int_{\mathsf{snr}_1}^{\mathsf{snr}_2} \mathrm{MMSE}^{c_n}(\gamma)d\gamma \\ &\leq \frac{1}{2}\log(1+\alpha^* \mathsf{snr}_2) \end{aligned}$$

due to the "single crossing point" property which ensures

$$\mathrm{MMSE}^{c_n}(\gamma) \leq \mathrm{mmse}_G(\gamma), \quad \forall\gamma \in [\mathsf{snr}_1,\infty)$$

K Mutual Information Constraints

Theorem

Assuming $\mathsf{snr}_1 < \mathsf{snr}_2 < \ldots < \mathsf{snr}_K$ *the solution of*

$$\begin{aligned} &\max \quad I_n(\mathsf{snr}_K) \\ &\text{s.t.} \quad \forall i \in \{1, \ldots, K-1\}, \quad I_n(\mathsf{snr}_i) \le \frac{1}{2}\log\left(1 + \alpha_i \mathsf{snr}_i\right) \end{aligned}$$

for some $\alpha_i \in [0, 1]$, *is the following*

$$I_n(\mathsf{snr}_K) = \frac{1}{2}\log\left(1 + \alpha_\ell \mathsf{snr}_K\right)$$

where α_ℓ, $\ell \in \{1, \ldots, K-1\}$, *is defined such that*

$$\forall i \in \{1, \ldots, K-1\} \quad \frac{1}{2}\log\left(1 + \alpha_\ell \mathsf{snr}_i\right) \le \frac{1}{2}\log\left(1 + \alpha_i \mathsf{snr}_i\right)$$

The maximum rate is attained, for any n, *by choosing* X *Gaussian with i.i.d. components of variance* α_ℓ. *For* $n \to \infty$ *equality is also attained by a Gaussian codebook designed for* snr_K *with limited power of* α_ℓ.

Gaussian MIMO Channels

* Measuring disturbance by MMSE provides some "engineering" insights into the relative efficiency of rate splitting (Han-Kobayashi) for simple scalar Gaussian interference channels. This is not the case, when disturbance is measured by mutual information.
* Bandemer and El Gamal have extended their solution to the Gaussian MIMO channel under a mutual information constraint <arXiv:1103.0996v2>, and this solution implies rate-splitting. Challenge: extend the equivalent MMSE constraint to the Gaussian MIMO setting. Can I-MMSE considerations reflect the best achievable performance in all snr regimes? DoF via information (MMSE) dimensions for the vector channel [Stotz-Bölcskei, 2012].
* Challenge: Optimal rate-MMSE disturbance constraints, tradeoffs for single codes (not allowing rate splitting) and practical implications (belief propagation soft decoding for LDPC codes).

Thank You!

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Munich Workshop on Shannon Coding Techniques (MSCT April 2024)

‘‘Coding over Interference Channels: An Information-Estimation View”

The information-estimation relation is used to gain insight into useful coding schemes operating over the Gaussian interference channel.
After reviewing basic I-MMSE relations and their implications on point-to-point coding over the Gaussian channel, we focus on the Gaussian interference channel. Here the inflicted interference is measured by the associated minimum mean square error (MMSE). Structure of codes achieving reliable communication at some specific signal-to-noise ratio (SNR) and constrained by the permitted MMSE at a lower SNR values, modeling the interference, are discussed. It is shown that layered superposition codes attain optimal performance, providing thus some engineering insight to the relative efficiency of the Han-Kobayashi coding strategy. The Degrees-of-Freedom (DoF) behavior of the multi-user Gaussian interference channel is captured by considering the MMSE-Dimension concept, providing a general expression for the DoF. A short outlook concludes the presentation, addressing related research challenges, and also recent results, where interference is measured by the corresponding mutual information.

Joint work with R. Bustin, General Motors and A. Dytso, QUALCOMM.

A. Dytso, R. Bustin, D. Tuninetti, N. Devroye, H.V. Poor and S. Shamai (Shitz), “On Communications through a Gaussian Noise Channel with an MMSE Disturbance Constraint,” IEEE Trans. Information Theory, vol. 64, no. 1, pp. 513–530, January 2017.
A. Dytso, R. Bustin, H.V. Poor and Shlomo S. (Shitz), “A View of Information-Estimation Relations in Gaussian Networks,” Entropy, August 2017.