Qualcomm

Energy-Based Methods for Probabilistic Amplitude Shaping

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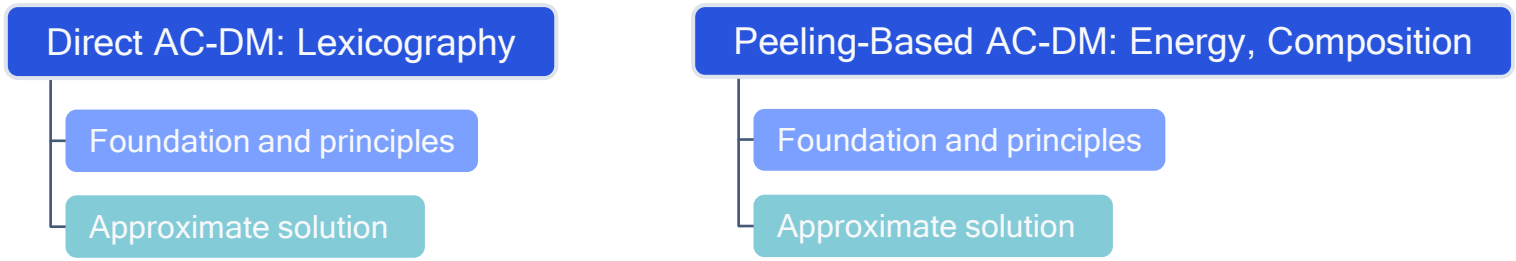
Outline

- Introduction to probabilistic amplitude shaping (PAS)
- Energy-based arithmetic coding (AC) for PAS
  - Direct AC-DM: based on lexicographical order of sequences: simple implementation
  - Peeling-based AC-DM: extension of constant-composition distribution matching (CCDM) to handle multiple compositions as well as more general extensions to include energy and other metrics

Two classes of energy-based AC methods

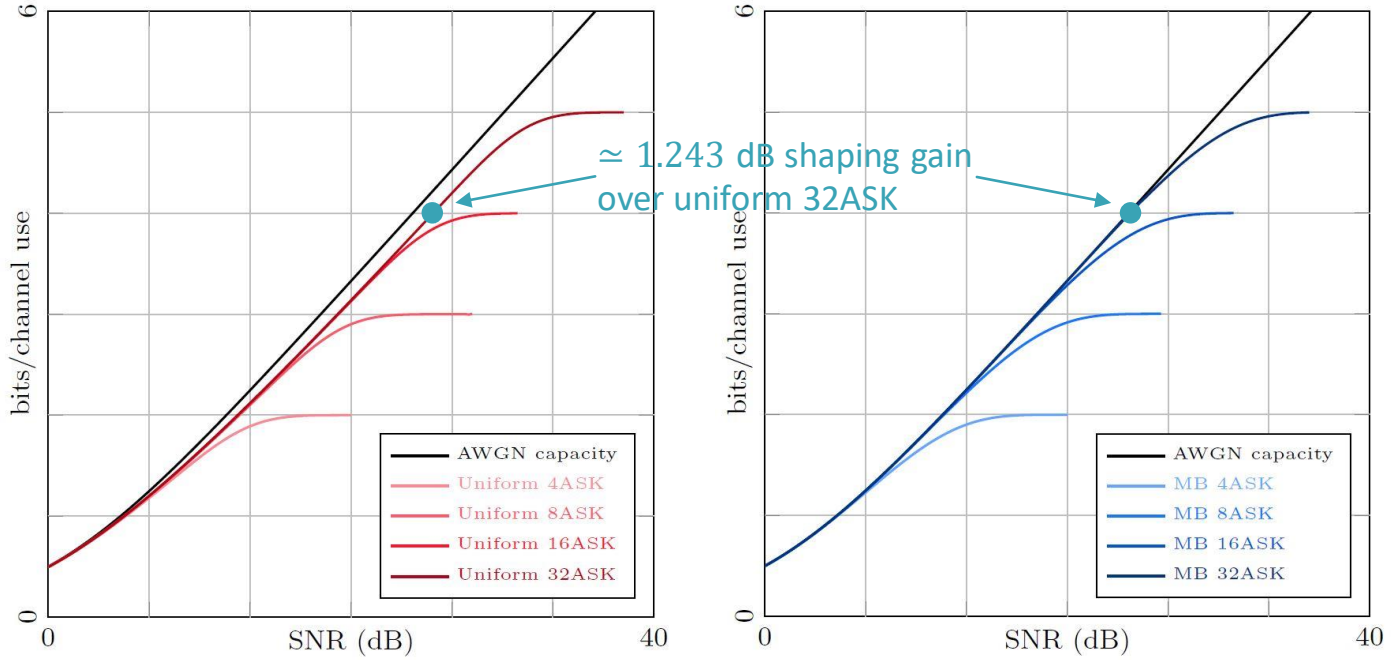


- Approximations
- Performance evaluation
- Conclusion

Motivation: Shaping Gain

- Coded modulation
  - In cellular wireless systems, high-order modulation is combined with binary forward-error-correction (FEC) to achieve high spectral efficiency for mobile data transmission
  - The coded modulation scheme usually endows a uniform distribution over the per-dimension constellations
  - Aim: preserve constellation architecture, change distribution.

- Shaping gain over AWGN channel
  - The optimal MB-distributed input exhibits a shaping gain over the uniformly distributed input over an ASK constellation
  - The shaping gain increases with increasing rates and constellation sizes

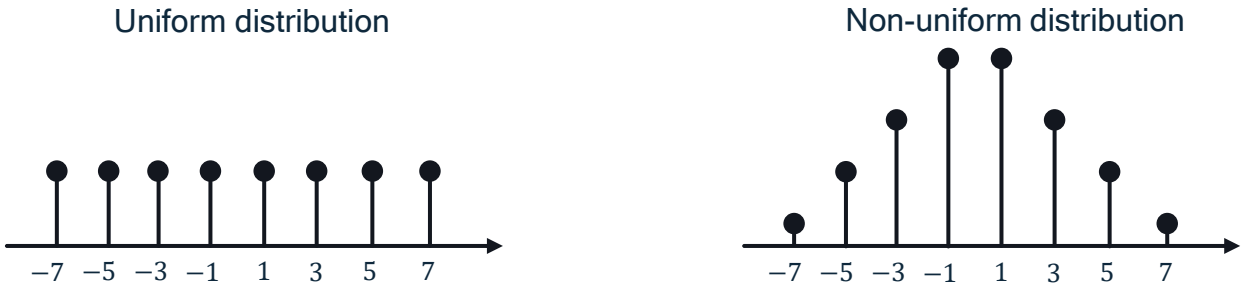


G. Forney, Jr., R. Gallager, G. Lang, F. Longstaff, and S. Qureshi, "Efficient modulation for band-limited channels," *IEEE Trans. J. Sel. Areas Commun.*, vol. 2, pp. 632-647, Sept. 1984.

Probabilistic Shaping and Maxwell-Boltzmann Distribution

The road to channel capacity

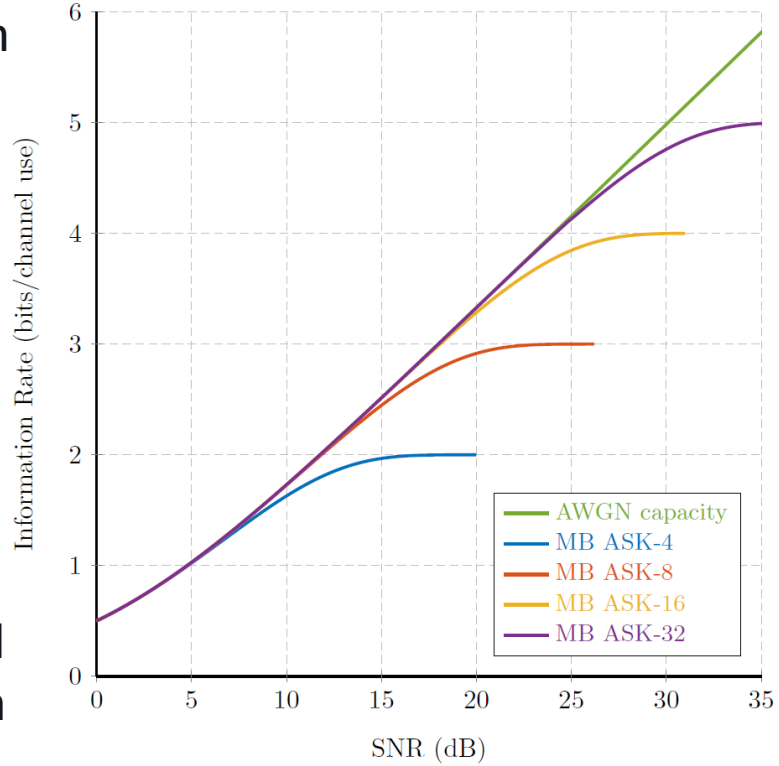
- Probabilistic shaping
  - Generally, it is a technique to generate a target non-uniform distribution on equidistant constellation points to reduce or close the shaping gap



Example: amplitude shift keying constellation (ASK-8)

- Maxwell-Boltzmann (MB) distribution
  - Over the AWGN channel, the mutual information obtained by optimized Maxwell-Boltzmann input distribution exhibits negligible difference from the capacity-achieving input distribution over ASK constellations

$$P^{\mathrm{MB}}(x) = \frac{1}{Z_\nu} \mathrm{e}^{-\nu x^2} \qquad x \in \{\pm 1, \pm 3, \ldots, \pm(2^M - 1)\}$$



Maximize mutual information subject to average energy constraint over AWGN

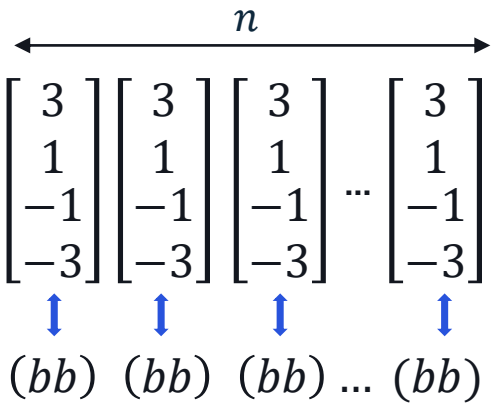
$\max_{\nu,\Delta}\{\mathbb{I}(X_\nu;\, \Delta X_\nu + z)\}$ subject to $\mathbb{E}[(\Delta X_\nu)^2] \leq \mathrm{P}$

X_ν is distributed according to P^{MB}

F. Kschischang and S. Pasupathy, "Optimal nonuniform signaling for Gaussian channels," *IEEE Trans. Inf. Theory*, vol. 39, pp. 913-929, May. 1993

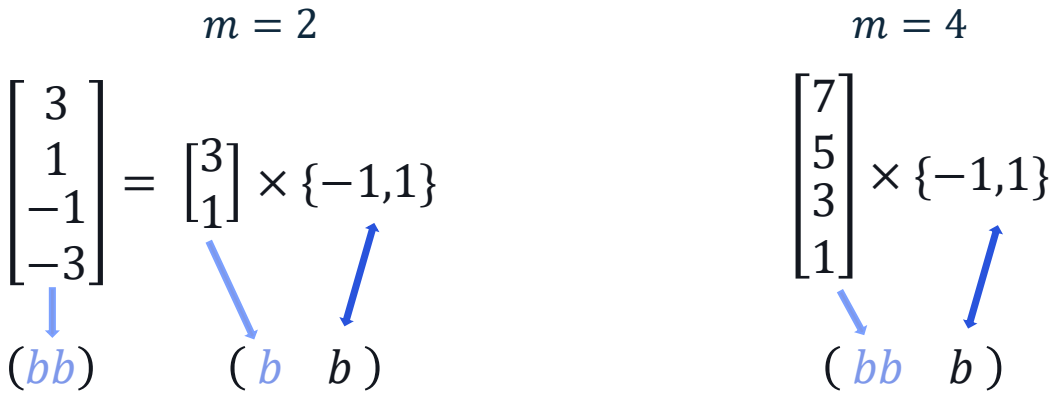
ASK Probabilistic Amplitude Shaping

- Standard ASK: symbols are uniformly distributed



Bits-to-symbol mapping
- Bits are uniform ↔ symbols are uniform

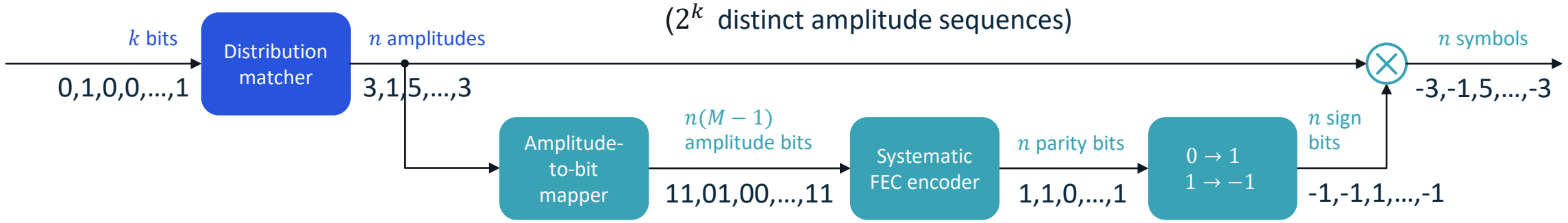
- PAS: Separate sign from magnitude to 'shape' magnitude distribution



Bits-to-sequences mapping
- Induces non-uniform distribution on amplitude sequences
- Symbols → bits used for FEC
- Sign bits as FEC parities

Transmission with Probabilistic Amplitude Shaping

- Transmission chain of a typical PAS architecture

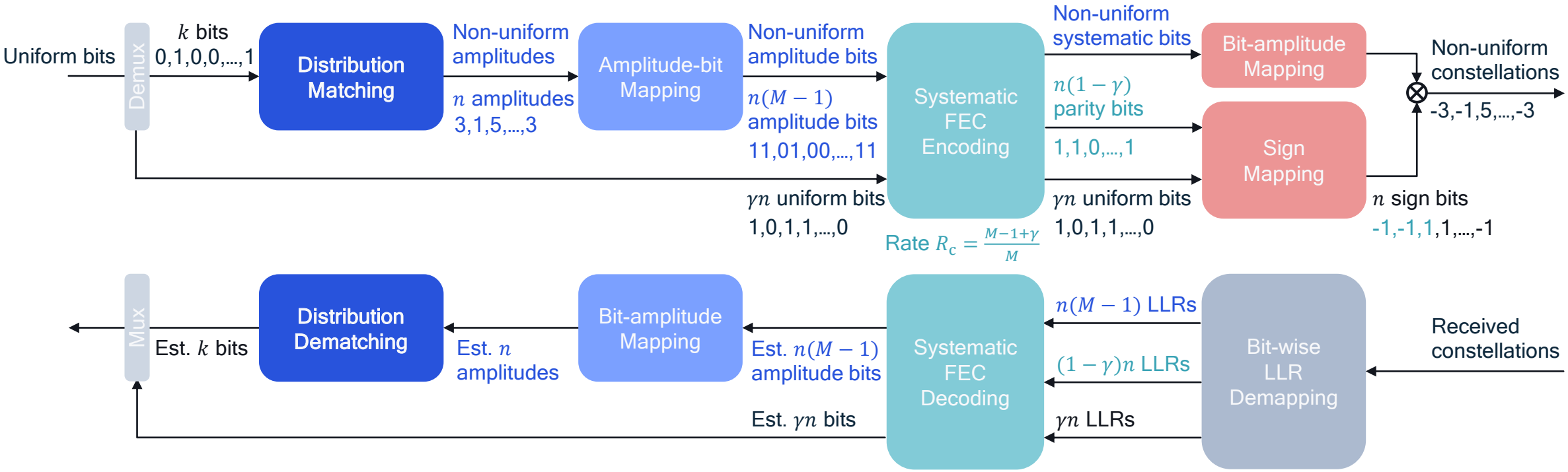


- "Distribution Matching"
  - Sequence space induces non-uniform marginal distribution over symbols $\{\pm 1, \pm 3, \ldots, \pm(2^M - 1)\}$
  - Main idea is that that distribution should be closer to capacity-achieving distribution than uniform, e.g., more Gaussian-like in the AWGN setting

Probabilistic Amplitude Shaping (PAS)

Key application scenario of our study and potential use cases for 6G era

- PAS is a recently introduced technique for coded modulation
  - It combines an outer layer of amplitude shaping with an inner layer of binary forward-error-correction coding



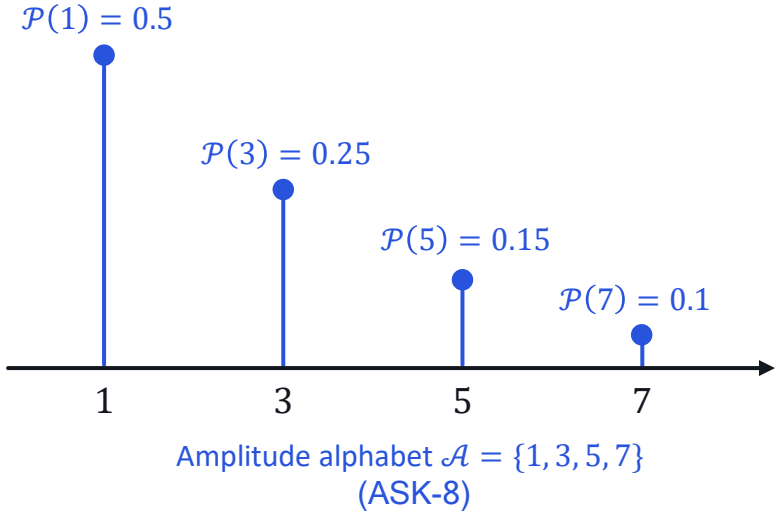
  - It can provide a low-complexity and flexible integration with existing coded modulation schemes
  - It can provide large shaping gain and inherent rate adaptation functionality by the outer layer of shaping

G. Böcherer, F. Steiner, and P. Schulte, "Bandwidth efficient and rate-matched low-density parity-check coded modulation," *IEEE Trans. Commun.*, vol. 63, pp. 4651-4665, Dec. 2015
P. Schulte and G. Böcherer, "Constant composition distribution matching," *IEEE Trans. Inf. Theory*, vol. 62, pp. 430-434, Jan. 2016

Fixed-to-Fixed Distribution Matching

From fixed-length uniform bit sequences to fixed-length non-uniform symbol sequences

- Distribution matching (DM) is a key component in PAS transmission architectures
  - It transforms sequences of uniform bits to sequences of per-dimension amplitudes, aiming at inducing a target probability distribution $\mathcal{P}$ over the underlying amplitude alphabet
  - The transformation needs to be invertible, i.e., the input can be correctly reconstructed given the output
- Fixed-to-fixed DM imposes deterministic lengths for input and output sequences



- Typical design objective
  - Perform low-complexity invertible fixed-to-fixed distribution matching
  - Rate k/n close to entropy $\mathrm{H}(\mathcal{P})$ bits/symbol (e.g., output close to i.i.d. according to $\mathcal{P}$)

Preliminaries

Notation and terminology

- Consider sequences of symbols from an alphabet $\mathcal{A} = \{a_1, a_2, \dots, a_m\}$ of size m
  - Symbol a_i has energy $\mathbf{E}(a_i) = (a_i{}^2\text{-}1)/8$ and symbol energies are nonnegative and mutually distinct
  - ASK-8 ($m = 4$, $\mathcal{A} = \{1, 3, 5, 7\}$) example: $(\mathbf{E}(a_1), \mathbf{E}(a_2), \mathbf{E}(a_3), \mathbf{E}(a_4)) = (0,1,3,6)$.

The **composition** of a sequence $s = (s_1, s_2, \dots, s_n)$

$$\mathbf{k}(s) = \big(k_1(s), k_2(s), \dots, k_m(s)\big)$$

$k_i(s)$ is the number of occurrences of $a_i \in \mathcal{A}$ in the sequence s

The **energy** of a sequence $s = (s_1, s_2, \dots, s_n)$

$$\mathbf{E}(s) = \sum_{i=1}^{n} \mathbf{E}(s_i)$$

Accumulation of symbol energies along the sequence s

- Energy based cardinalities
  - The number of sequences of length n and energy equal to E is denoted by $N(n, E)$

$$N(n, E) = |\mathbf{s}\{s \in \mathcal{A}^n : \mathbf{E}(s) = E\}|$$

  - The number of sequences of length n and energy less than or equal to E is denoted by $N_c(n, E)$

$$N_c(n, E) = |\mathbf{s}\{s \in \mathcal{A}^n : \mathbf{E}(s) \leq E\}|$$

Fixed Length Distribution Matching: MB and CCDM

Energy-constrained sequences can lead to Maxwell-Boltzmann distribution

- MB distribution and maximum entropy principle
  - For a given constellation, the probability distribution over the constellation that maximizes the Shannon entropy, subject to an average energy constraint, is an MB distribution P^{MB}

$$\max_{P}\{\mathrm{H}(P)\} \text{ subject to } \mathbb{E}_{X\sim P}[X^2] = \omega \quad \xrightarrow{\text{optimal solution}} \quad P^{\mathrm{MB}}(x) \propto \mathrm{e}^{-\nu(\omega)x^2}$$

  - Among all sequences having length n and "energy" E, the set of sequences *composed* according to typical realizations of the MB distribution P^{MB} has the largest cardinality

- Constant composition distribution matching (CCDM) and PAS architectures
  - CCDM is a well-known solution to probabilistic shaping
  - It picks amplitude sequences of a single composition according to the MB distribution and typically uses arithmetic coding to selects a particular amplitude sequence
  - It is concatenated with an inner layer of forward-error-correction (FEC) encoder to generate sign bits for the amplitudes: this represents the key components of a typical PAS transmission architecture

F. Kschischang and S. Pasupathy, "Optimal nonuniform signaling for Gaussian channels," *IEEE Trans. Inf. Theory*, vol. 39, pp. 913-929, May. 1993
G. Böcherer, F. Steiner, and P. Schulte, "Bandwidth efficient and rate-matched low-density parity-check coded modulation," *IEEE Trans. Commun.*, vol. 63, pp. 4651-4665, Dec. 2015
P. Schulte and G. Böcherer, "Constant composition distribution matching," *IEEE Trans. Inf. Theory*, vol. 62, pp. 430-434, Jan. 2016

Traditional Shaping (Energy Based) Methods

Complexity considerations on existing sphere shaping approaches

- Traditional approaches include shell mapping and enumerative sphere shaping
  - Both SM and ESS may be viewed as methods for realizing sphere shaping
  - One of the main technical problems in these approaches is ordering and enumerating sequences based on energy, and complexity is a key challenge
  - These approaches assume a fixed and short block length, the shaping performance of which is thus limited
  - However, as Forney already noted earlier:

"As $n \to \infty$, spherical constellations require large table look-ups, and therefore become impractical."

- $N(n, E)$ and $N_c(n, E)$ play a fundamental role in SM and ESS
  - The key challenge is determining such quantities with high efficiency and accuracy for a wide and varying range of n and E
    - A straightforward computation for a value of $N(n, E)$ or $N_c(n, E)$ has a computational complexity quadratic in n
    - Moreover, such a value can have a very large magnitude so that a straightforward tabulation method to accurately and/or approximately store all such values for a wide range of n and E will have a storage complexity that is prohibitively large

G. D. Forney, Jr., "Trellis shaping," *IEEE Trans. Inf. Theory*, vol. 38, no. 2, pp. 281-300, Mar. 1992
F. M. J. Willems and J. J. Wuijts, "A pragmatic approach to shaped coded modulation," in *Proc. IEEE 1st Symp. Commun. and Veh. Technol. in the Benelux*, Delft, The Netherlands, 1993
R. Laroia, N. Farvardin, and S. A. Tretter, "On optimal shaping of multidimensional constellations," *IEEE Trans. Inf. Theory*, vol. 40, no. 4, pp. 1044-1056, Jul. 1994

Three Approaches to MB Distribution

1. We can use Arithmetic Coding (AC) to map input bits into i.i.d. amplitudes distributed according to the MB distribution
   - This leads to variable rate encoding, which has practical drawbacks.
   - We developed a version of this where n is fixed but the number of encoded input bits varies (not the focus of this presentation).

2. CCDM (Schulte and Bocherer, 2015)
   - Pick a fixed amplitude *composition* $(\vec{k}_1, \vec{k}_2, \ldots, \vec{k}_m)$ so that $\vec{k}_i \simeq np_v(a_i)$.
   - Use arithmetic coding to select a particular sequence with the given composition.
   - Asymptotically in n achieves near optimal shaping gain.
   - Not very efficient for small n (too few sequences).
     - Multi-partition DM (MPDM, Fehenberger *et al.*, 2019)

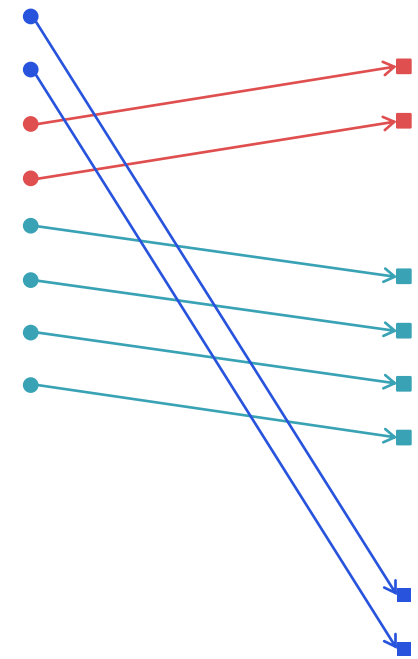
3. Sphere shaping: Use minimum energy sequences
   - Marginal distribution is close to MB.
   - Near optimal shaping gain and minimum energy use for given rate.
   - *"As $N \to \infty$ spherical constellations require large table look-ups, and therefore become impractical."* (Forney, 1992)

Illustration of Some DM Designs

- "CCDM," "MPDM" and "Sphere Shaping"

Bit Sequence		
Bit 1	Bit 2	Bit 3
0	0	0
0	0	1
0	1	0
0	1	1
1	0	0
1	0	1
1	1	0
1	1	1

mapping



"CCDM" "MPDM"

8 sequences with minimal energy

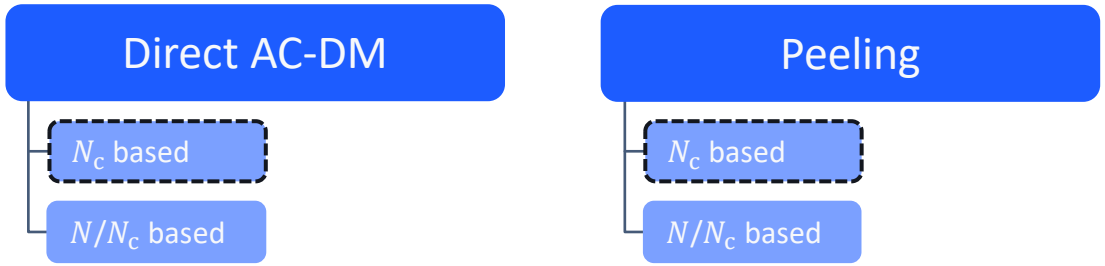
all sequences of energy 13

Symbol Sequence				Composition		Energy
Symbol 1	Symbol 2	Symbol 3	Symbol 4	Index	Description	
1	1	1	1	1	(4, 0)	4
1	1	1	3	2	(3, 1)	13
1	1	3	1			
1	3	1	1			
3	1	1	1			
1	1	3	3	3	(2, 2)	20
1	3	1	3			
1	3	3	1			
3	1	1	3			
3	1	3	1			
3	3	1	1			
1	3	3	3	4	(1, 3)	28
3	1	3	3			
3	3	1	3			
3	3	3	1			
3	3	3	3	5	(0, 4)	36

- Key design aspects
  - How to select compositions to meet target?
  - How to design efficient bit-to-symbol mapping?

Novel DM Designs

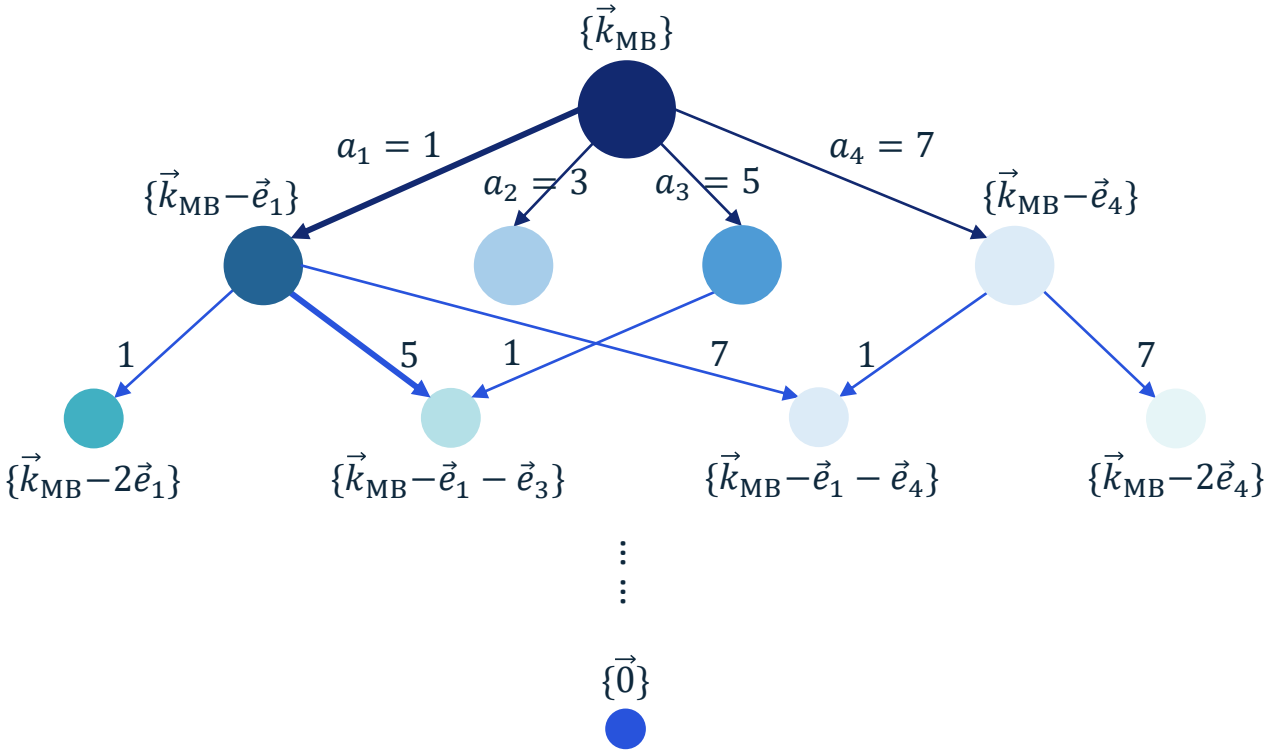
- We present two DM methods that allow for realizing exact sphere shaping
  - Direct AC-DM
  - Peeling
- Approximation of key parameters $N(n, E)$ and $N_c(n, E)$ for implementation.
- Practical implementation



CCDM Arithmetic Coding - Composition Graph

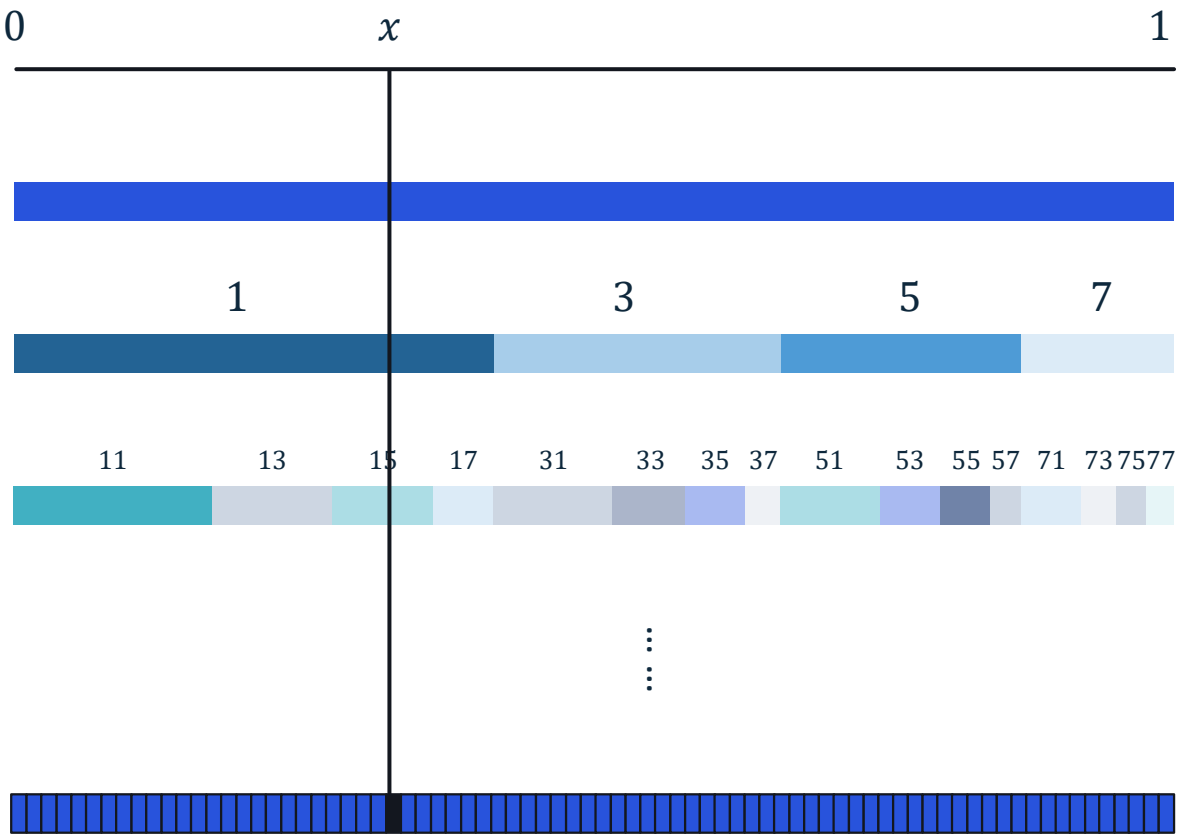
- AC: Map $x \in [0,1)$ to sequence

Partitioning of residual sequence space by composition
Nodes (states) represents composition of residual string
Traverse nodes accumulating string according to x



Terminate on empty composition

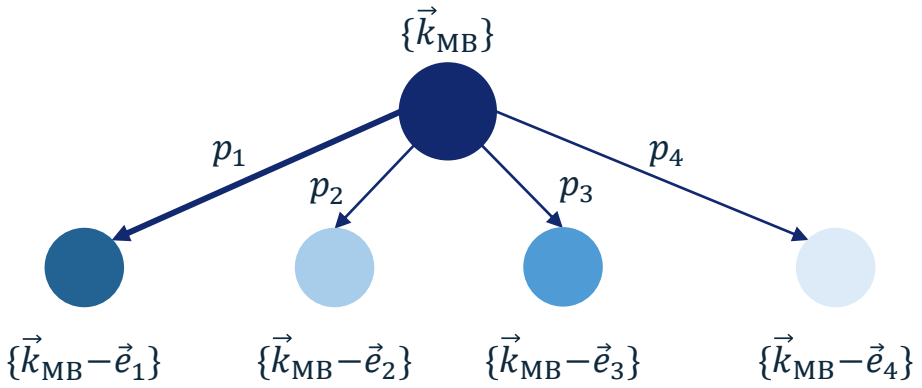
Intervals correspond to accumulated prefix

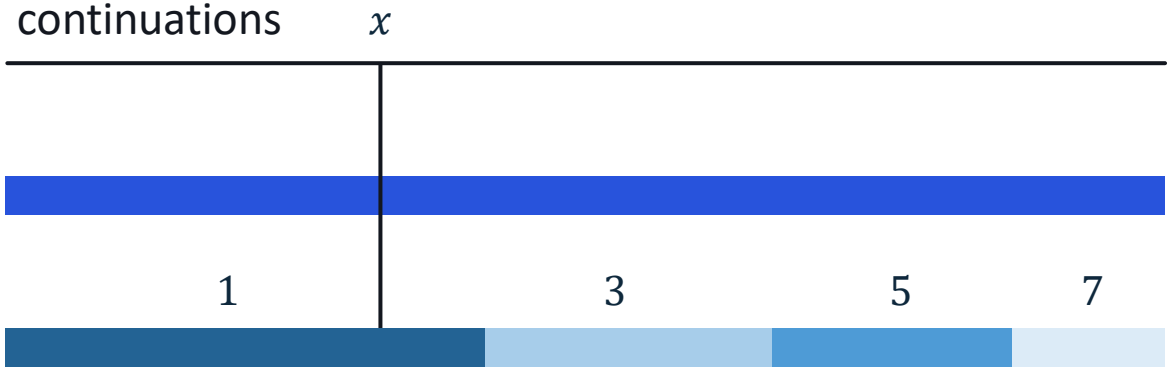


Total probability, all sequences equally probable

CCDM Arithmetic Coding – Transitions on Composition Graph

- AC: Map $x \in [0,1)$ to sequence
  - Each node represents to a disjoint union of its children
  - Transition probabilities given by cardinalities of represented continuations



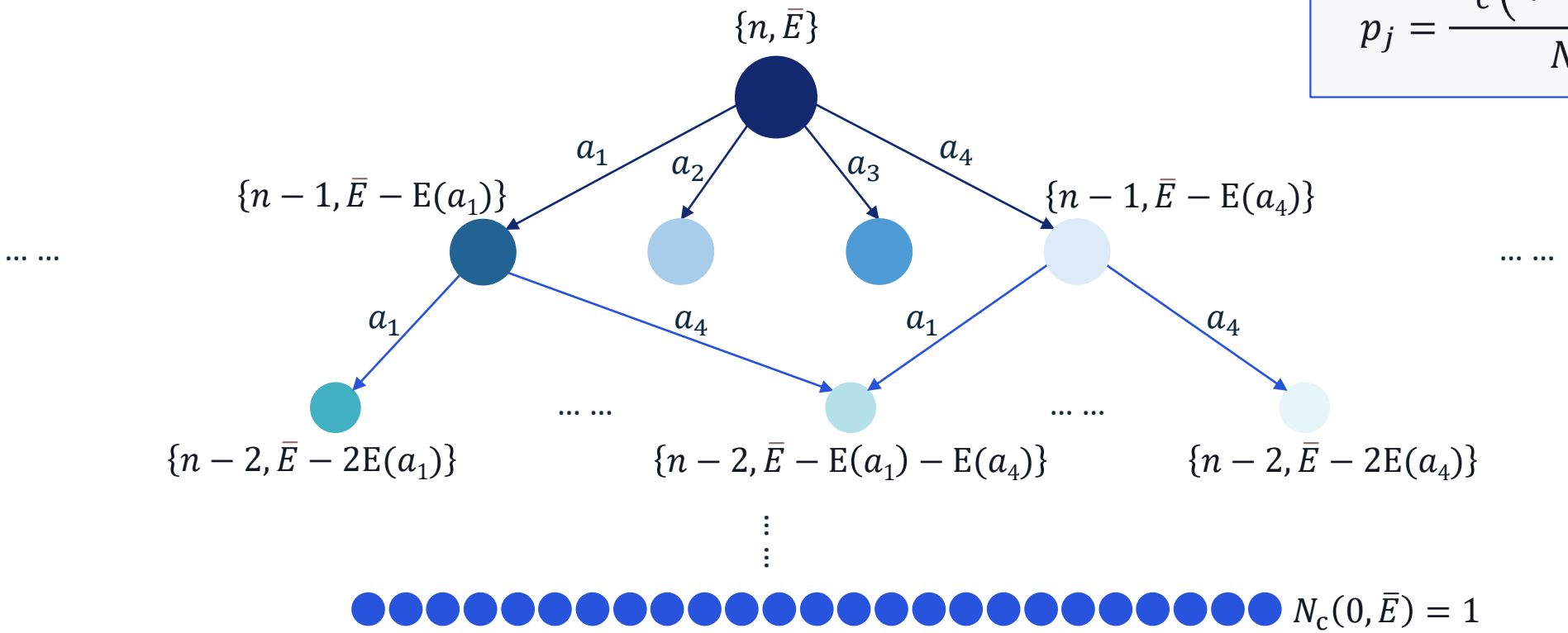


$$p_j = p\left(\vec{k}_{\mathrm{MB}} \to \vec{k}_{\mathrm{MB}} - \vec{e}_j\right) = \frac{\left|\left\{s: \vec{k} = \vec{k}_{\mathrm{MB}} - \vec{e}_j\right\}\right|}{\left|\left\{s: \vec{k} = \vec{k}_{\mathrm{MB}}\right\}\right|} = \frac{\left|\left\{s: \vec{k} = \vec{k}_{\mathrm{MB}} - \vec{e}_j\right\}\right|}{\sum_l \left|\left\{s: \vec{k} = \vec{k}_{\mathrm{MB}} - \vec{e}_l\right\}\right|} = \frac{\binom{n-1}{\vec{k}_{\mathrm{MB}}-\vec{e}_j}}{\binom{n}{\vec{k}_{\mathrm{MB}}}} = \frac{(\vec{k}_{\mathrm{MB}})_j}{n}$$

Energy-Based Sphere Shaping Arithmetic Coding

- Energy graph representation
  - Partitioning of sequence space by length and energy
  - Nodes (states) correspond to residual length and residual (max) energy

$$p_j = \frac{N_c\left(n-1, \bar{E} - \mathrm{E}(a_j)\right)}{N_c(n, \bar{E})}$$



Terminates at $n = 0$ with residual energy ≥ 0

Fundamental Role of $N(n, E)$ in Sphere Shaping

- Energy-based sphere shaping
  - Consider sphere shaping using all sequences $\{s_1^n\}$ of energy at most $\bar{E}$ and length n
  - Ideal sphere shaping samples the sequence space according to the distribution

$$p(s_1^n) = \frac{1}{N_c(n, \bar{E})}$$

  - Given a prefix of length n_1 and energy E_1, the number of sequences where the next symbol has energy E' is given by $N(n - n_1 - 1, \bar{E} - E_1 - E')$, and the fraction of sequences is

$$\frac{N(n - n_1 - 1, \bar{E} - E_1 - E')}{N_c(n - n_1, \bar{E} - E_1)}$$

  - These fractions are used for sphere shaping encoding in arithmetic coding
- Remarks
  - We can apply AC directly to use all sequences of fixed energy or all sequences with at most some fixed energy
  - Our basic approach is to approximate $N(n, E)$ and $N_c(n, E)$ and to determine the loss due to the approximation to give predictable performance

Sphere Shaping by Peeling

- First phase of Peeling
  - Assume m and uniform distribution over strings of length n and energy at most $\bar{E}$

$$p(\vec{k}) = p(\vec{k}_m)p(\vec{k}_{m-1}|\vec{k}_m)\cdots p(\vec{k}_1|\vec{k}_m, \dots, \vec{k}_2)$$

  - Each subsequent factor is equivalent to selecting a maximum energy symbol count for smaller m, n, E

$$p(\vec{k}_{m-j}|\vec{k}_{m-j+1}, \dots, \vec{k}_m; m, n, \bar{E}) = p(\vec{k}_{m-j}|m-j, n_j, E_j)$$

$$n_j = n - \sum_{m'=m-j+1}^{m} \vec{k}_{m'} \quad \text{and} \quad E_j = \bar{E} - \sum_{m'=m-j+1}^{m} \vec{k}_{m'} \mathrm{E}(a_{m'})$$

  - These give rise to the transition probabilities used in the AC process

- Second phase of Peeling
  - This is based on the composition selected in the first phase
  - The continuation of the AC process for constant composition enables the second factor

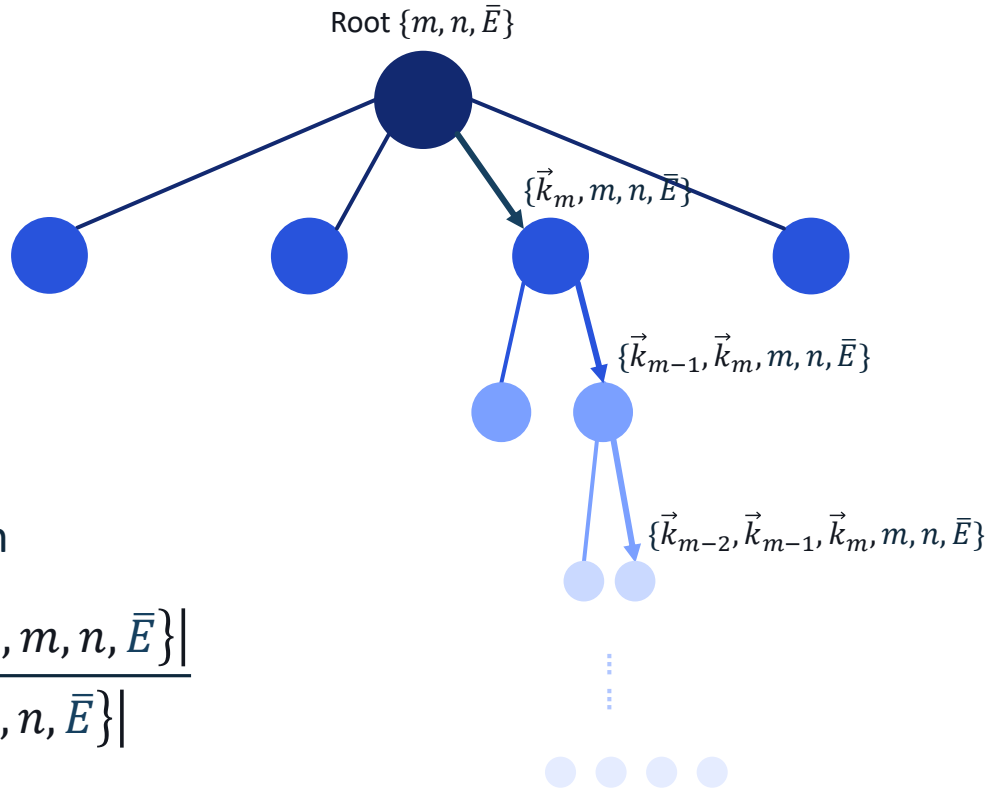
Peeling - Transitions on Energy Tree

- Transitions from parent to child nodes
  - The partitioning at each internal node induces a transition probability distribution over its child nodes
  - A transition probability is determined by the relative cardinality of the set that a child node represents
  - E.g., at the root, transition probabilities are of the form

$$p(\vec{k}_m | m, n, \bar{E}) = \frac{|\{\vec{k}_m, m, n, \bar{E}\}|}{|\{m, n, \bar{E}\}|}$$

"fundamental role of N_c in peeling"

$$= \frac{\binom{n}{\vec{k}_m} N_c^{[m-1]}\left(n - \vec{k}_m, \bar{E} - \vec{k}_m \mathrm{E}(a_m)\right)}{N_c^{[m]}(n, \bar{E})}$$



  - More generally, transitions from a depth j node are of the form

$$p(\vec{k}_{m-j} | \vec{k}_{m-j+1}, \dots, \vec{k}_m; m, n, \bar{E}) = \frac{|\{\vec{k}_{m-j}, \vec{k}_{m-j+1}, \dots, \vec{k}_m, m, n, \bar{E}\}|}{|\{\vec{k}_{m-j+1}, \dots, \vec{k}_m, m, n, \bar{E}\}|}$$

- These transition probabilities induce a probability distribution over the leaf nodes

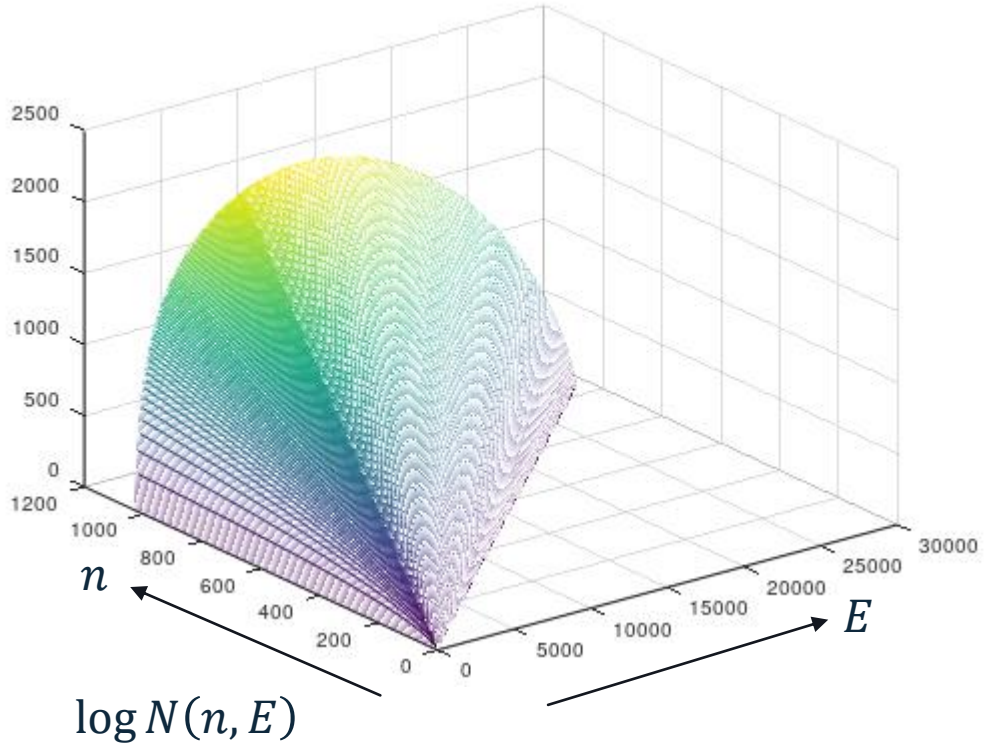
Approximation of $N(n, E)$ and $N_c(n, E)$

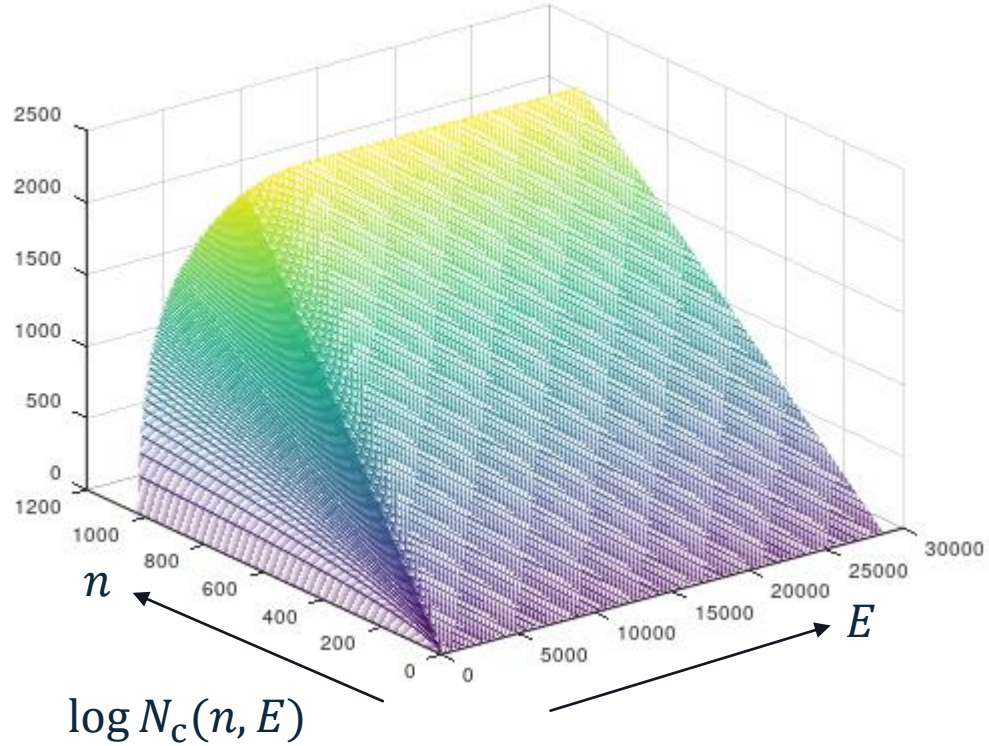
- $N(n, E)$ is the coefficient of x^E in the polynomial $Z_0(x)^n$

$$N(n, E) = |\mathbf{s}\{s \in \mathcal{A}^n : \mathbf{E}(s) = E\}|$$

$$N_c(n, E) = |\mathbf{s}\{s \in \mathcal{A}^n : \mathbf{E}(s) \leq E\}|$$

$$Z_0(x) = \sum_{i=1}^{m} x^{\mathrm{E}(a_i)} \qquad N(n, E) = \mathrm{coeff}\{Z_0(x)^n, x^E\}$$





Hayman Approximation for $N(n, E)$

Hayman approximation

- Hayman method to estimate complex integrals of analytic functions
  - Hayman's method is based on the saddle-point method; in our case we have

$$N(n,E) = \frac{1}{2\pi \mathrm{i}} \oint_{\Gamma} x^{-(E+1)} Z_0(x)^n \mathrm{d}x$$

Choose circular contour near the saddle point of integrand

$$\lambda \frac{Z_0'(\lambda)}{Z_0(\lambda)} = \frac{E}{n} \equiv \omega$$

$$\frac{N(n,\omega n)}{\mathrm{e}^{nH(\omega)}} = \frac{1}{2\pi} \int_{-\pi}^{\pi} \left(\frac{Z_0(\lambda(\omega)\mathrm{e}^{\mathrm{i}\theta})}{Z_0(\lambda(\omega))} \right)^n \mathrm{e}^{-\mathrm{i}\omega n\theta} \mathrm{d}\theta$$

Estimate dominant contribution

  - For fixed ω Hayman's method shows that $N(n,\omega n) = N_{\mathrm{HM}}(n,\omega n)(1 + o_n(1))$ for $\omega n \in \mathbb{N}$

$$N_{\mathrm{HM}}(n,\omega n) = \frac{1}{\sqrt{2\pi n V(\omega)}} \mathrm{e}^{nH(\omega)}$$

The Shannon entropy of distribution p_ω is $H(\omega)$
The variance associated to p_ω is $V(\omega)$

$$p_\omega(k) = \frac{\lambda(\omega)^k}{Z_0(\lambda(\omega))},\ k \in \{\mathbf{E}(a_1), \ldots, \mathbf{E}(a_m)\}$$

- Probabilistic interpretation of result
  - Let $k_{\lambda(\omega)}$ be a random variable distributed according to p_ω and let $k_{\lambda(\omega),n}$ be a random variable that has the same distribution as the sum of n i.i.d. copies of $k_{\lambda(\omega)}$

$$\mathbb{P}\left(k_{\lambda(\omega),n} = \omega n\right) = \frac{N(n,\omega n)}{\mathrm{e}^{nH(\omega)}}$$

This is equal to the inverse Fourier transform of the characteristic function

$$\phi_{k_{\lambda(\omega),n}}(\theta) = \mathbb{E}\left[\mathrm{e}^{\mathrm{i}\theta k_{\lambda(\omega),n}}\right]$$

  - The Hayman approximation in this case is closely related to the Gaussian approximation of the sum $k_{\lambda(\omega),n}$

W. K. Hayman, "A generalization of Stirling's formula," *Journal für die reine und angewandte Mathematik*, vol. 1956, no. 196, pp. 67-95, 1956

Improved Hayman Approximation for $N(n, E)$

Improving Hayman approximation

- High-order (Hayman) approximation of $\log N(n, \omega n)$ as

$$\log N_{\mathrm{HM}}(n, \omega n) = nH(\omega) - \frac{1}{2}\log n - \log\sqrt{2\pi V(\omega)}$$

Terms of order n, $\log n$, and 1

  - For fixed $\omega = E/n$, the Hayman approximation converges as $o_n(1)$
  - The Hayman approximations are less accurate for small E, large E, and small to moderate values of n
    - For $m \geq 3$, the large E region, i.e., for $E \simeq n\mathbf{E}(a_m)$, is fundamentally not amenable to smooth approximation due to the sparseness of larger exponents in the polynomial Z_0; fortunately, this region is generally not needed for our application of interest

- Improving the Hayman approximation of $\log N(n, \omega n)$
  - We determine two smooth functions of ω, R_1 and R_2, such that for fixed ω we have

$$\frac{N(n, \omega n)}{N_{\mathrm{HM}}(n, \omega n)} = 1 + \frac{R_1(\omega)}{n} + \frac{R_2(\omega)}{n^2} + o(n^{-2})$$

Explicit lower-order terms

Terms of order $\frac{1}{n}$ and $\frac{1}{n^2}$

$$G_1(\omega) = \frac{3V(\omega)V''(\omega) - 2V'(\omega)^2}{24V(\omega)}$$

$$G_2(\omega) = -\frac{V(\omega)V''''(\omega)}{48}$$

  - This results in an improved approximation of $\log N(n, \omega n)$ based on

$$\log N(n, \omega n) = \log N_{\mathrm{HM}}(n, \omega n) + \frac{G_1(\omega)}{n} + \frac{G_2(\omega)}{n^2} + o(n^{-2})$$

  - Note that the expansion in $\frac{1}{n}$ can be formally extended to any fixed order larger than 2

Approximation for $N(n, E)$

Improving Hayman approximation

- Special $m = 2$ case for binomial coefficients $\binom{n}{E}$
  - The Hayman approximation in this case is determined exactly by the standard Stirling's approximation
  - But there are further improvements

$$\log N_{\mathrm{HM}}(n, \omega n) + \frac{1}{12}\left(\frac{1}{n} - \frac{1}{E} - \frac{1}{n-E}\right) + c(E)$$

This accounts for an improved accuracy for very small E near 0 and very large E near n

$c(E) = s(E)$ for $E \le n/2$ and $c(E) = s(n-E)$ otherwise

$$s(E) = \log \frac{\sqrt{2\pi} E^{E+\frac{1}{2}} \mathrm{e}^{-E} \mathrm{e}^{\frac{1}{12E}}}{E!}$$

$\frac{G_1(\omega)}{n}$ term: $G_1(\omega) = \frac{1}{12}\left(1 - \frac{1}{\omega} - \frac{1}{1-\omega}\right)$

Can be obtained from the more precise Stirling's formula by Robbins

  - Tabulating $s(E)$ only for $E \le 16$ and adding the corresponding $c(E)$ yield an approximation with smaller than 10^{-6} absolute error for all $n \ge 32$

- Further improving the Hayman approximation of $\log N(n, \omega n)$
  - The $c(E)$ term is also significant for small fixed E for all $m \ge 3$ cases
    - $N(n, E)$ is a polynomial in n of degree E
    - $\binom{n}{E}$ is the highest-order term in n in an enumeration by composition of $N(n, E)$; e.g., $N(n, E) = \frac{n^E}{E!}\left(1 + O(n^{-1})\right)$ for fixed E

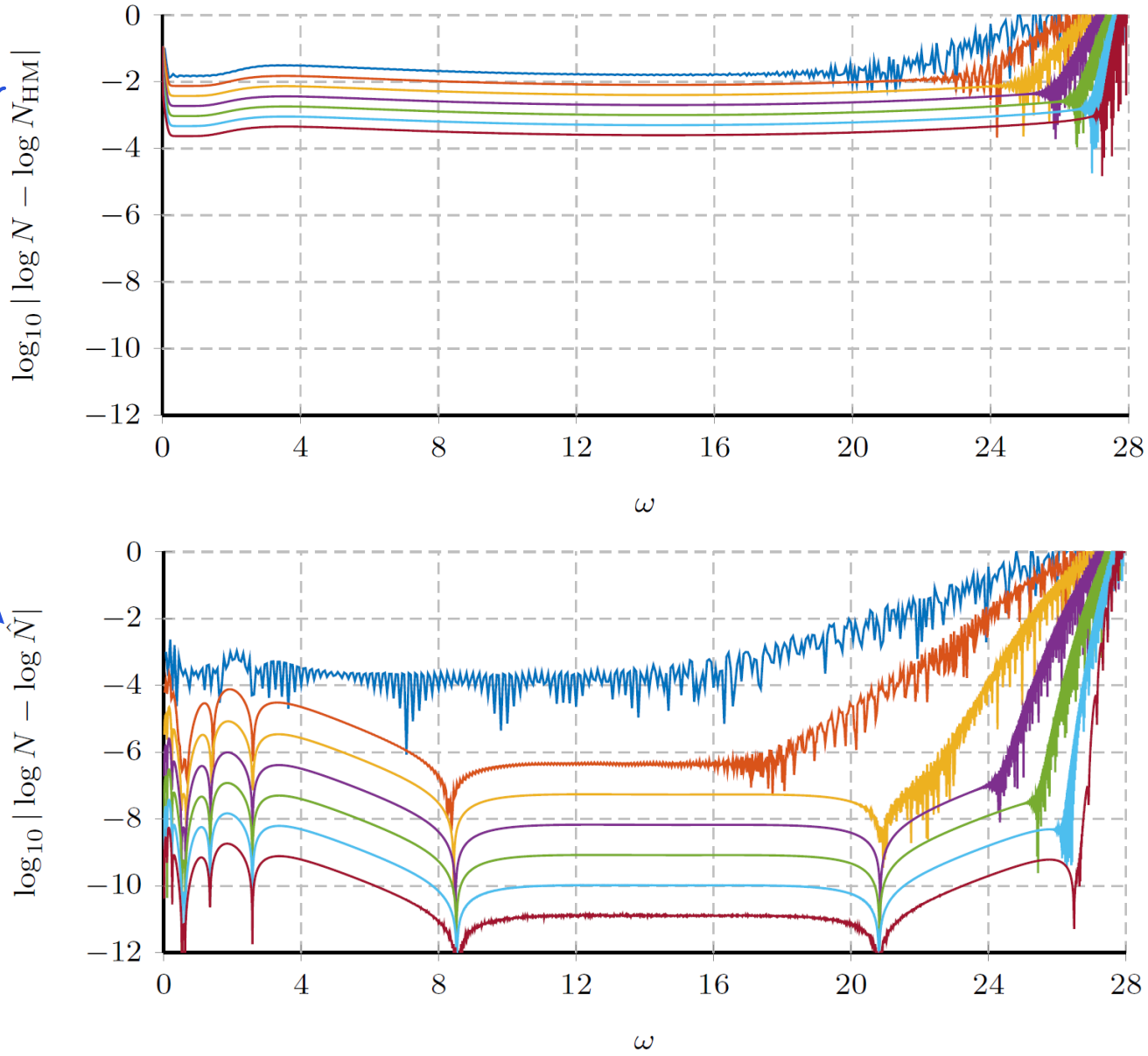
J. Stirling, *Methodus differentialis: sive Tractatus de Summatione et Interpolatione Serierum Infinitarum*. London, UK: G. Strahan, 1730
H. Robbins, "A remark on Stirling's formula," *The American Mathematical Monthly*, vol. 62, no. 1, pp. 26-29, 1955

Approximation for $N(n, E)$

Numerical evaluation for QAM-256/ASK-16

Adding additional terms $\frac{G_1(\omega)}{n}$, $\frac{G_2(\omega)}{n^2}$ and $c(E)$ to the Hayman approximation

- Approximation accuracy for $m = 8$
  - Approximation error for $\log N(n, E)$ below 10^{-3} for all $n > 16$ and uniformly for most region of E
    - Plot of $\log_{10} | \log N(n, \omega n) - \log \widehat{N}(n, \omega n) |$ for $n \in \{2^4, \ldots, 2^{10}\}$
    - The approximation absolute errors are below 10^{-3}outside of the following subregions: $\{(n, E) \mid n \leq 16\}$, $\{(n, E) \mid n \leq 19, E \leq 6\}$, $\{(n, E) \mid 16 < n \leq 400, E \geq 27.17n - 202.02\}$, and $\{(n, E) \mid n > 400, E \geq 27.573n - 348.106\}$.
  - Accuracy improves as n increases



Approximation for $N_{\mathrm{c}}(n, E)$

Local expansion approach

- Hayman's method also gives rise to a local expansion of the form

$$\frac{N(n, \omega n - j)}{N_{\mathrm{HM}}(n, \omega n)} = \lambda(\omega)^j \mathrm{e}^{-\frac{j^2}{2nV(\omega)}}\left(1 + o_n(1)\right)$$

Can be improved to $\frac{1}{\sqrt{n}} L_1\left(\omega, \frac{j}{\sqrt{n}}\right) + \frac{1}{n} L_1\left(\omega, \frac{j}{\sqrt{n}}\right) + o(n^{-1})$
Valid over the range $j = O(\sqrt{n})$

- We use the local expansion to approximate $N_{\mathrm{c}}(n, E)$
  - Cumulate the number of sequences of length n and energy up to E
  - $N_{\mathrm{c}}(n, E)$ is locally determined up to exponentially small corrections

$$N_{\mathrm{c}}(n, E) = \sum_{j=0}^{E} N(n, E - j) \simeq \mathrm{e}^{nH(\omega)} \frac{\Psi(nV(\omega), \lambda(\omega), 0)}{\sqrt{2\pi nV(\omega)}} \qquad E < \omega^{\mathrm{u}} n$$

$$\Psi(\sigma^2, \lambda, k) = \sum_{j=k}^{\infty} \lambda^j \mathrm{e}^{-\frac{j^2}{2\sigma^2}} \ (\lambda < 1)$$

$$N_{\mathrm{c}}(n, E) = m^n - \sum_{j=1}^{n\mathbf{E}(a_m) - E} N(n, E + j) \simeq m^n \left(1 - \mathrm{e}^{n(H(\omega) - \log m)} \frac{\Psi(nV(\omega), \lambda(\omega)^{-1}, 0) - 1}{\sqrt{2\pi nV(\omega)}}\right) \qquad E > \omega^{\mathrm{u}} n$$

  - Integral approximation of Ψ results in an approximation for $N_{\mathrm{c}}(n, E)$

Integral approximation of Ψ

$$\widehat{\Psi}(\sigma^2, \lambda, 0) = \frac{1}{\sqrt{\lambda} + 1} + \sqrt{\sigma^2} \sqrt{\frac{\pi}{2}} \sqrt{\lambda} \frac{\log \lambda}{\lambda - 1} \operatorname{erfcx}\left(-\frac{\sqrt{\sigma^2} \log \lambda}{\sqrt{2}}\right)$$

Approximation for $N_{\mathrm{c}}(n, E)$

Expanded approximation

- Considerations on approximation accuracy and computational complexity
  - For fixed $\omega = E/n$ the above integral approximation converges as $o_n(1)$
  - However, the approximation is less accurate for small n and E, and E in a neighborhood around $\omega^{\mathrm{u}} n$
  - It also involves nested evaluations of functions that may not be practically simple to compute
- An expanded approximation formula for $\log N_{\mathrm{c}}(n, \omega n)$
  - We simplify and approximate the above integral approximation into a practically implementable form that is also amenable to numerical improvement

Order-n term Order-1 terms Order-$\frac{1}{\sqrt{n}}$ term Order-$\frac{1}{n}$ terms

$$\log \widehat{N_{\mathrm{c}}}(n, \omega n) = nH^{\mathrm{sat}}(\omega) + G_0(\omega) + G_0^{\mathrm{s}}(\nu) + \frac{1}{\sqrt{n}} G_{\frac{1}{2}}^{\mathrm{s}}(\nu) + \frac{1}{n} G_1(\omega) + \frac{1}{n} G_1^{\mathrm{s}}(\nu)$$

$$\left(\text{where } \omega = \frac{E}{n} \text{ and } \nu = \sqrt{n}(\omega - \omega^{\mathrm{u}}) \right)$$

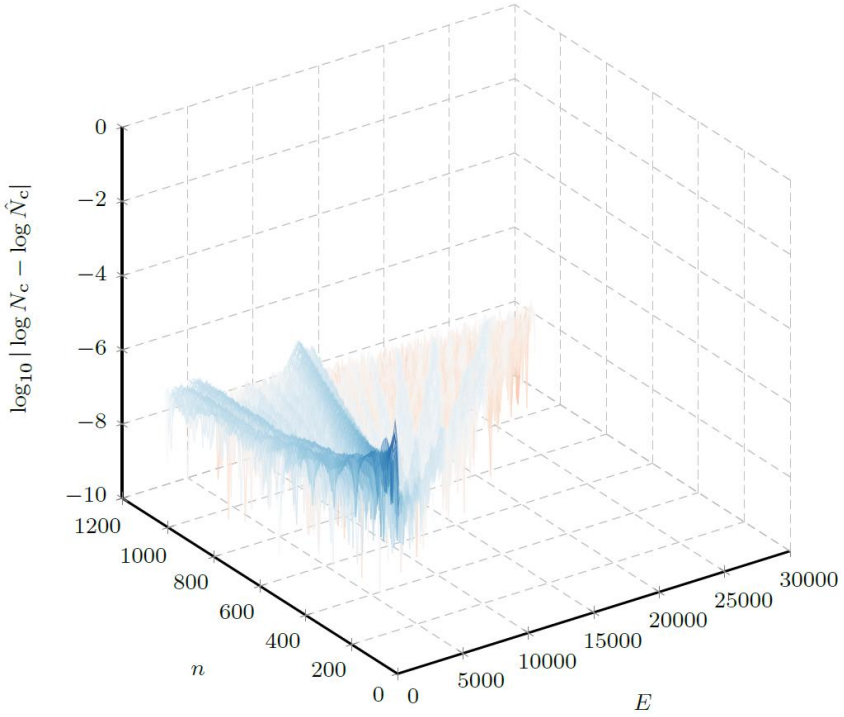
- Formal characteristics
  - Each additive term has an explicit and simple dependence on n
  - Each smooth function depends only on normalized energy $\omega = E/n$ or scaled energy $\nu = \sqrt{n}(\omega - \omega^{\mathrm{u}})$
  - The formula captures a special scaling behavior near ω^{u}

Approximation for $N_c(n, E)$

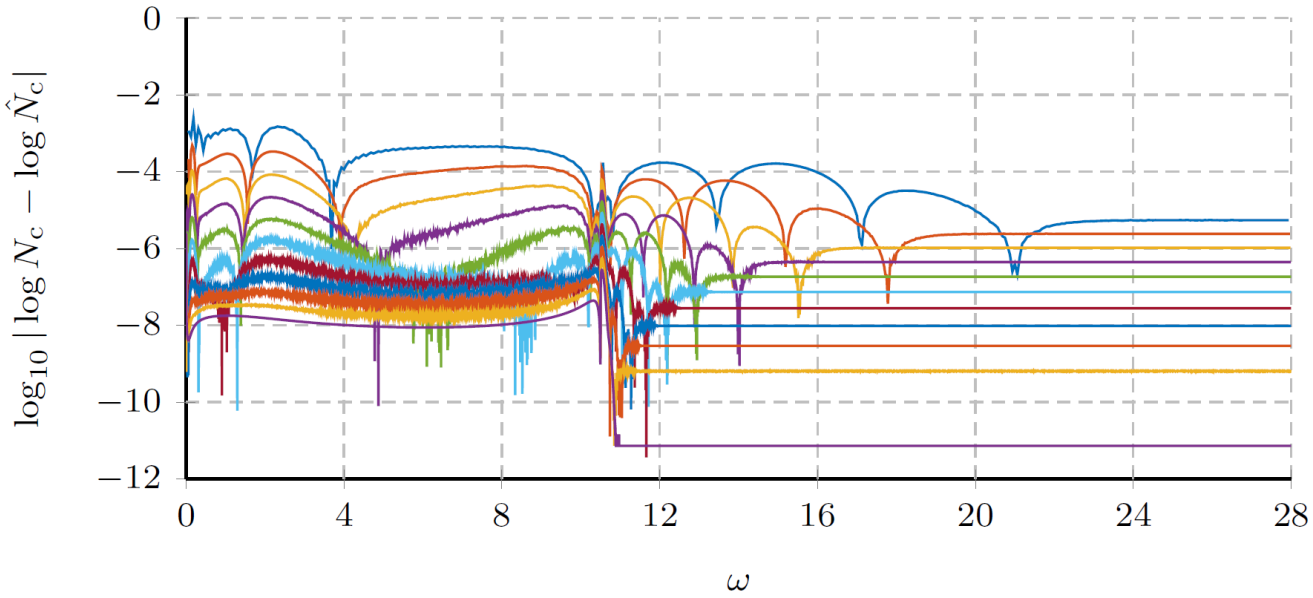
Numerical evaluation for QAM-256/ASK-16

- Approximation accuracy for $m = 8$
  - Approximation error for $\log N_c(n, E)$ below 10^{-3} for all $n > 16$ and uniformly for most region of E
    - The approximation absolute errors are below 10^{-3} outside of the following subregions: $\{(n, E) \mid n \leq 16\}$ and $\{(n, E) \mid n \leq 26, E \leq 26\}$
  - Accuracy improves as n increases

All $n \leq 1024$ and $E \leq n\mathbf{E}(a_m)$



$n \in \{2^4, 2^5, \ldots, 2^{10}\}$ and $E \leq n\mathbf{E}(a_m)$



Practical Implementation of Approximation

Function fitting and term selection

- Smooth functions of ω or ν are further approximated using simpler alternatives
  - We use a piecewise polynomial to approximate each such smooth function
    - Polynomials can be easily described, e.g., only their coefficients and degrees need be stored
    - Evaluation computations are relatively easy, e.g., elementary addition and multiplication

Piecewise polynomials

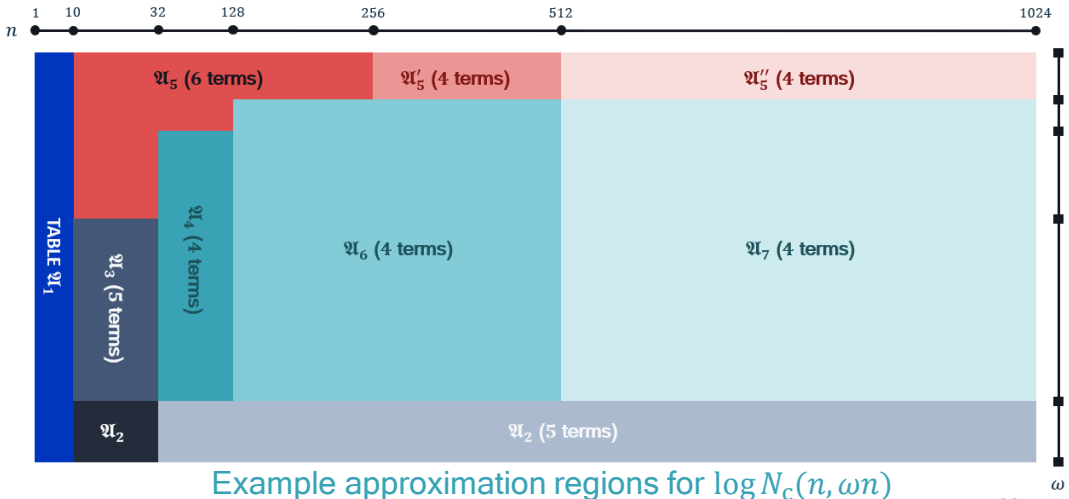
$$\log N(n,\omega n): n\widehat{H}(\omega), \widehat{\log V}\,(\omega), \frac{1}{n}\hat{G}_1(\omega)$$

Piecewise polynomials

$$\log N_c\,(n,\omega n): n\widehat{H}^{\mathrm{sat}}(\omega), \hat{G}_0(\omega), \hat{G}_0^{\mathrm{s}}(\nu), \frac{1}{\sqrt{n}}\hat{G}_{\frac{1}{2}}^{\mathrm{s}}(\nu), \frac{1}{n}\hat{G}_1(\omega), \frac{1}{n}\hat{G}_1^{\mathrm{s}}(\nu)$$

  - Different n dependence gives different accuracy requirements
    - E.g., approximation error in $H(\omega)$ gives error increasing in n whereas $G_1(\omega)$ gives decreasing error
    - This gives rise to a tradeoff between accuracy and storage complexity

- Term selection and localization
  - Based on target accuracy, certain terms can be eliminated
    - E.g., when n is large, terms scaled by $\frac{1}{n}$ may be neglected
  - Polynomial fitting can be localized to specific ranges of n

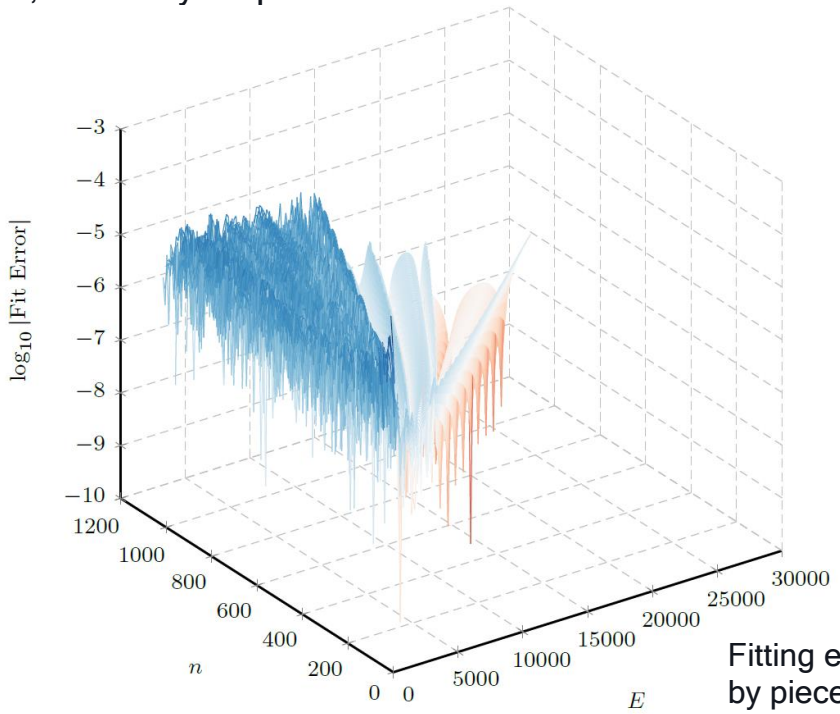


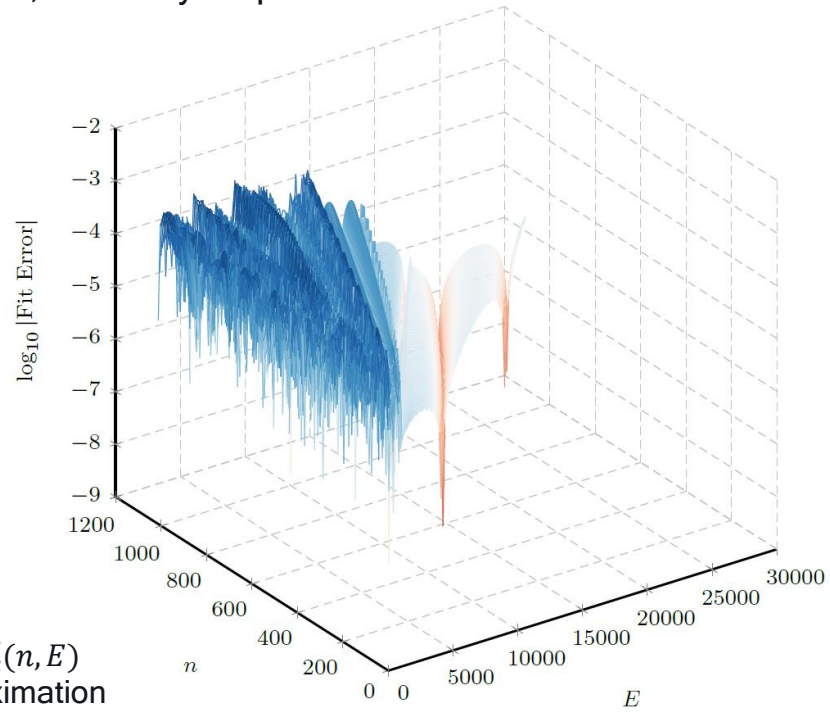
Example approximation regions for $\log N_c(n,\omega n)$ for $m = 8$, $\omega < \omega^{\mathrm{u}}$ and $1 \le n \le 1024$

Practical Implementation of Approximation

Numerical evaluation for piecewise polynomial approximation of $\log N_{\mathrm{c}}\,(\boldsymbol{n}, \boldsymbol{E})$

- High-accuracy example for $m = 8$
  - Left figure shows a high-accuracy example with a worst-case absolute error 0.0007 for $32 \leq n \leq 1024$
    - The total number of polynomial pieces is 152
    - All polynomials are of degree 3
    - The fixed storage of quantized polynomial coefficients are 1824 bytes, with 3 bytes per coefficient
- Low-accuracy example for $m = 8$
  - Right figure shows a low-accuracy example with a worst-case absolute error 0.0014 for $32 \leq n \leq 1024$
    - The total number of polynomial pieces is 67
    - All polynomials are of degree 3
    - The fixed storage of quantized polynomial coefficients are 804 bytes, with 3 bytes per coefficient





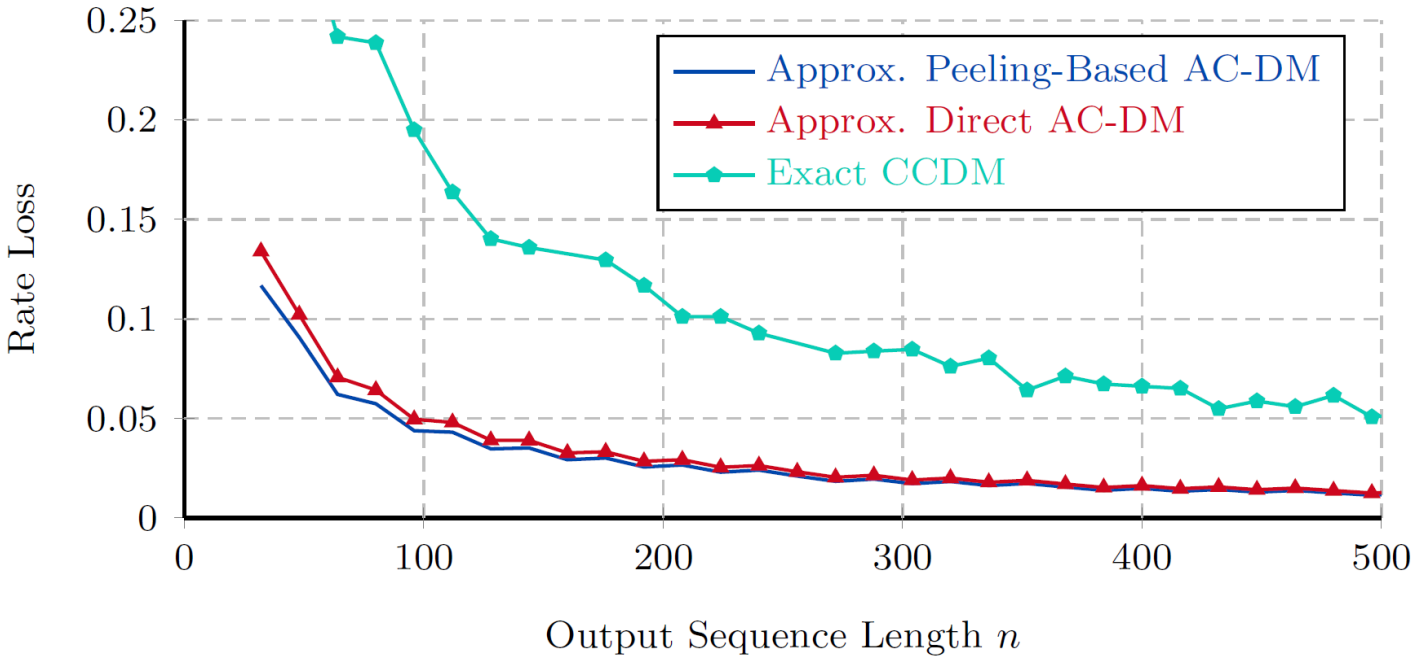
Fitting error $\log N_{\mathrm{c}}(n, E) - \log \widehat{N_{\mathrm{c}}}(n, E)$ by piecewise polynomial approximation

Performance Evaluation

Rate loss comparison for a target MB over $\mathcal{A} = \{1, 3, 5, 7, 9, 11, 13, 15\}$ (ASK-16/QAM-256)

- Rate loss comparison with CCDM
  - We use rate loss as an indicator for performance evaluation
  - Similar as the previous case, our methods show much smaller rate loss compared with CCDM, especially at small-to-medium lengths

$$R_{\text{loss}} = \mathrm{H}\left(P^{\mathrm{MB}}\right) - \frac{k}{n}$$



Target MB distribution has parameter $\nu = 0.013$ with

$$P^{\mathrm{MB}}(1) = 0.2586$$
$$P^{\mathrm{MB}}(3) = 0.2326$$
$$P^{\mathrm{MB}}(5) = 0.1881$$
$$P^{\mathrm{MB}}(7) = 0.1369$$
$$P^{\mathrm{MB}}(9) = 0.0896$$
$$P^{\mathrm{MB}}(11) = 0.0527$$
$$P^{\mathrm{MB}}(13) = 0.0279$$
$$P^{\mathrm{MB}}(15) = 0.0136$$

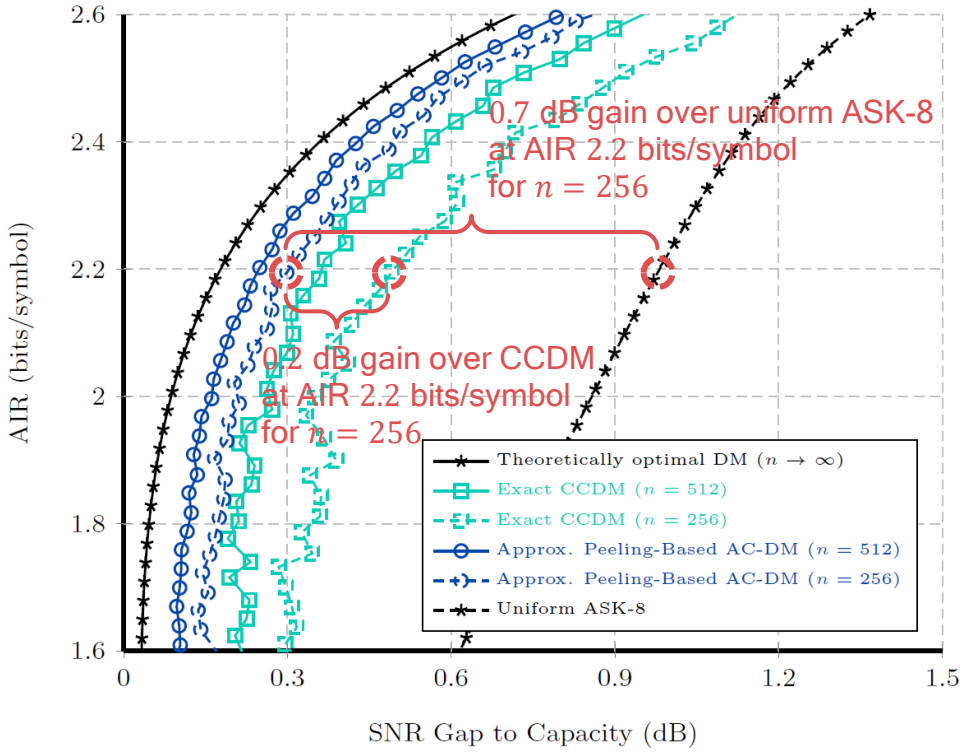
P. Schulte and G. Böcherer, "Constant composition distribution matching," *IEEE Trans. Inf. Theory*, vol. 62, pp. 430-434, Jan. 2016

Performance Evaluation

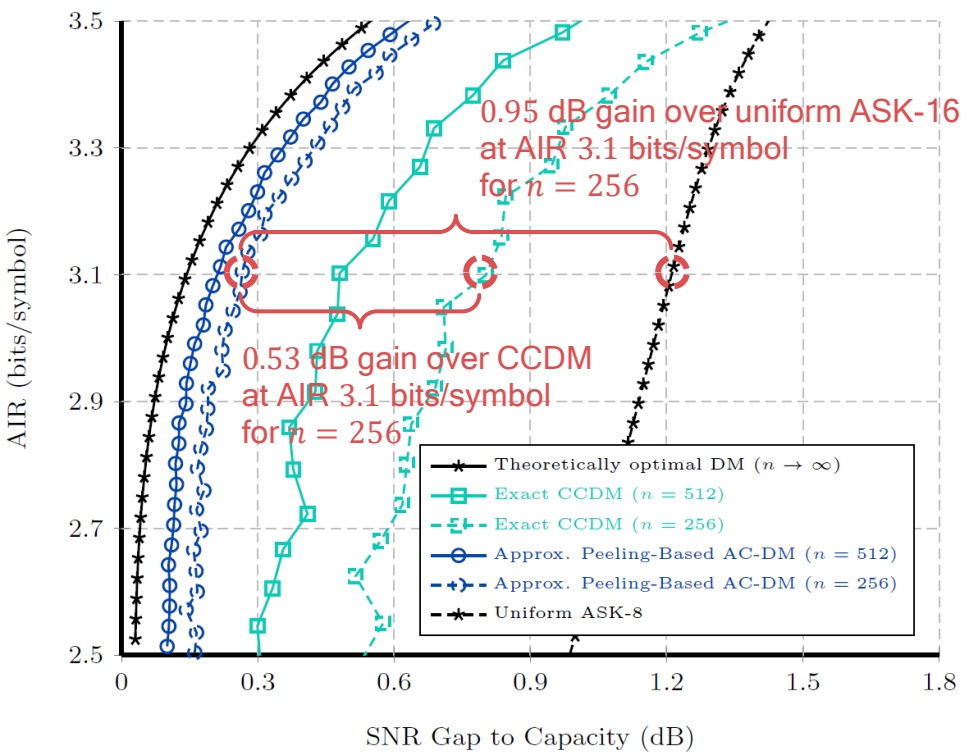
Achievable information rate to capture DM rate loss

- Achievable information rate (AIR) comparison for the AWGN channel
  - The figures show SNR gap to capacity as a function of AIR for ASK-8 and ASK-16 ($n \in \{256, 512\}$)

ASK-8



ASK-16

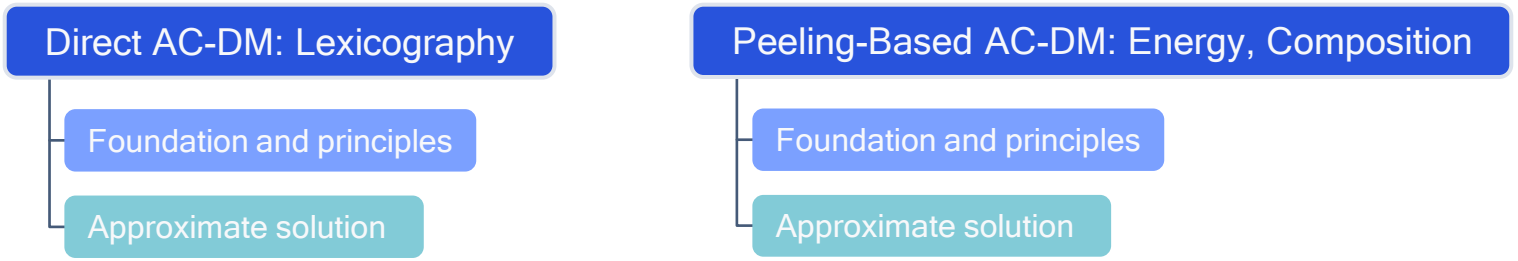


G. Böcherer, F. Steiner, and P. Schulte, "Bandwidth efficient and rate-matched low-density parity-check coded modulation," *IEEE Trans. Commun.*, vol. 63, pp. 4651-4665, Dec. 2015

Conclusion

- We introduced two classes of energy-based arithmetic coding (AC) methods for PAS, respectively termed direct AC-DM and peeling-based AC-DM
- We equipped such methods with efficient approximation of key involved energy-based quantities N and N_c
- We introduced means for determining the number of uniquely encodable bits and calibrating the rate loss

Two classes of energy-based AC methods



- Our methods establish a theoretical and algorithmic foundation for performing fixed-to-fixed invertible DM for PAS systems of varying sizes and practically realizing nearly-optimal shaping gain at low complexity

Thank you

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