EIRP Constraints for Massive MIMO Arrays

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What is EIRP?

Definition: Equivalent Isotropically Radiated Power (EIRP)

EIRP is the amount of power that a theoretical isotropic antenna would emit to produce the peak power density observed in the direction of maximum antenna gain

- EIRP is constrained to protect health

$$\mathrm{EIRP}(\boldsymbol{x}) = \max_{\substack{\theta\in[0,\pi]\\ \varphi\in[0,2\pi]}} \mathrm{E}\left[\left|r(\theta,\varphi,\boldsymbol{x})\right|^2\right] \tag{1}$$

$r(\theta,\varphi,\boldsymbol{x})$: signal at (θ,φ) on a sphere around antenna array
$\boldsymbol{x}$: the transmitted signals at the antennas

$\mathrm{E}[\bullet]$: Expectation/averaging over time interval t

- t very short $\Rightarrow$ EIRP per signal (no expectation needed)
- $t \sim$ Duration of Precoder $\boldsymbol{w} \Rightarrow$ replace $\boldsymbol{x}$ by $\boldsymbol{w}$ (assuming normalized symbols)
- t longer $\Rightarrow$ for sufficient randomness isotropic antenna is approached

Objective of the maximization is not concave!
$\Rightarrow$ Simple solution: Sample at sufficiently many positions on sphere

Goal of This (Ongoing) Work

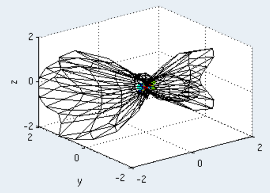
Calculate, bound and/or estimate the EIRP analytically

Is EIRP relevant for massive MIMO?

Example:

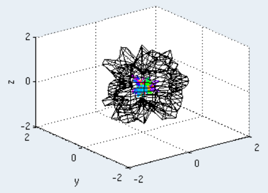
- 20 receivers with single antennas
- Zero-forcing beamforming

20 Tx Antennas



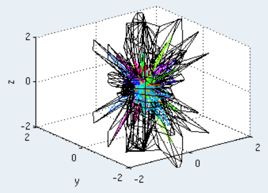
In the fully loaded system broad beams lead to high EIRP

40 Tx Antennas



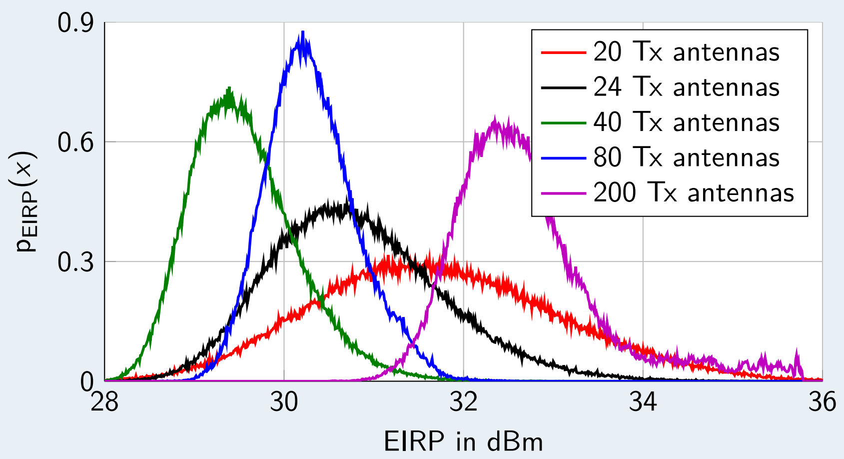
Twice as many Tx antennas seems to work well

200 Tx Antennas



Narrow beams cause high EIRP (one can control this!)

- EIRP depends on transmit power and number of antennas



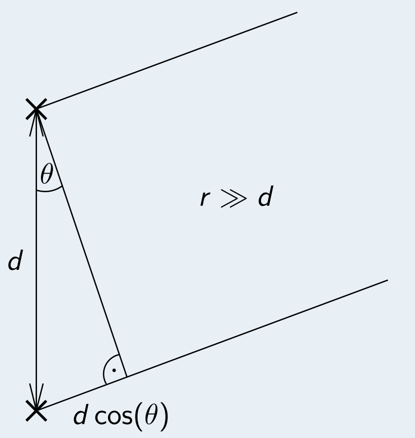
Transmitter Parameters

- uniform linear array (ULA) transmitter with M antennas
- isotropic radiators (point sources)
- antenna spacing of $d = k\lambda = \lambda/2$
- no mutual coupling between antennas

Phase Difference Between Two Antennas of an ULA

- Far field assumption

$\Rightarrow$ Rotational symmetry for the azimuthal angle



$$2\pi\frac{d\cos(\theta)}{\lambda} = 2\pi k\cos(\theta) \tag{2}$$

Calculating the EIRP of an ULA

The signal on the sphere is

$$r(\theta,\boldsymbol{x}) = \sum_{m=1}^{M} x_m e^{j2\pi(m-1)k\cos(\theta)} \tag{3}$$

$$\mathrm{EIRP}(\boldsymbol{x}) = \max_{\theta\in[0,\pi]} \mathrm{E}\left[\left|r(\theta,\boldsymbol{x})\right|^2\right] = \max_{\theta\in[0,\pi]} \mathrm{E}\left[\left|\sum_{m=1}^{M} x_m e^{j2\pi(m-1)k\cos(\theta)}\right|^2\right]$$
$$= \max_{\theta\in[0,\pi]} \mathrm{tr}\left(\boldsymbol{R}(\theta)\boldsymbol{S}\right) \tag{4}$$

where $\omega^m = e^{j2\pi mk\cos(\theta)}$ and

$$\boldsymbol{R}(\theta) = \begin{bmatrix} \omega^0 & \omega^1 & \dots & \omega^{M-2} & \omega^{M-1} \\ \omega^{-1} & \omega^0 & \dots & \omega^{M-3} & \omega^{M-2} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \omega^{-M+2} & \omega^{-M+3} & \dots & \omega^0 & \omega^1 \\ \omega^{-M+1} & \omega^{-M+2} & \dots & \omega^{-1} & \omega^0 \end{bmatrix} \tag{5}$$

$$\boldsymbol{S} = \mathrm{E}\left[\boldsymbol{x}\boldsymbol{x}^{\mathrm{H}}\right] \tag{6}$$

The trace of $\boldsymbol{AB}$ is the sum of all the elements of $\boldsymbol{A}\odot\boldsymbol{B}^{\mathrm{T}}$ [1]

$$\mathrm{tr}(\boldsymbol{AB}) = \boldsymbol{1}^{\mathrm{T}}\left(\boldsymbol{A}\odot\boldsymbol{B}^{\mathrm{T}}\right)\boldsymbol{1} \tag{7}$$

For $a_m \in \mathbb{C}$ and $\boldsymbol{\alpha} = [\alpha_1,\dots,\alpha_{M-1}] \in \mathbb{R}^{M-1}$ we bound

$$a_m e^{j\alpha_m} + a_m^* e^{-j\alpha_m} = 2\,\mathrm{Re}\left\{a_m e^{j\alpha_m}\right\} \le 2\left|a_m e^{j\alpha_m}\right| = 2\left|a_m\right| \tag{8}$$

$$\Rightarrow \max_{\alpha\in\mathbb{R}^{M-1}} \sum_{m=1}^{M-1}\left(a_m e^{j\alpha_m} + a_m^* e^{-j\alpha_m}\right) \le 2\sum_{m=1}^{M-1}\left|a_m\right|, \tag{9}$$

Proposed EIRP Upper Bound for an ULA

$$\mathrm{EIRP}(\boldsymbol{x}) = \max_{\theta\in[0,\pi]} \mathrm{tr}\left(\boldsymbol{R}(\theta)\boldsymbol{S}\right) = \max_{\theta\in[0,\pi]} \boldsymbol{1}^{\mathrm{T}}\left(\boldsymbol{R}(\theta)\odot\boldsymbol{S}^{\mathrm{T}}\right)\boldsymbol{1} \tag{10}$$
$$= \mathrm{tr}(\boldsymbol{S}) + \max_{\theta\in[0,\pi]} \sum_{m=1}^{M-1} \omega^{-m}\sum_{l=1}^{M-m}[\boldsymbol{S}]_{l,l+m} + \omega^{m}\sum_{l=1}^{M-m}[\boldsymbol{S}]_{l+m,l}$$
$$= \mathrm{tr}(\boldsymbol{S}) + \max_{\theta\in[0,\pi]} \sum_{m=1}^{M-1} \omega^{m}\sum_{l=1}^{M-m}[\boldsymbol{S}]_{l,l+m} + \omega^{-m}\sum_{l=1}^{M-m}[\boldsymbol{S}]^*_{l,l+m}$$
$$\le \mathrm{tr}(\boldsymbol{S}) + 2\sum_{m=1}^{M-1}\left|\sum_{l=1}^{M-m}[\boldsymbol{S}]_{l,l+m}\right| \tag{11}$$

The EIRP of an ULA is upper bounded by the sum of the absolute values of the sums of the (off-)diagonals of matrix $\boldsymbol{S} = \mathrm{E}[\boldsymbol{x}\boldsymbol{x}^{\mathrm{H}}]$

IDFT Simplifies Calculating the EIRP of an ULA

$$\mathrm{EIRP}(\boldsymbol{x}) = \lim_{K\to\infty}\left\|\boldsymbol{F}_{K\times 3}\boldsymbol{x}\right\|_\infty^2 \tag{12}$$

where $\boldsymbol{F}_{K\times 3}$ is the length K IDFT matrix [2]

- for $K < \infty$ this is a smart way of sampling
- EIRP of an ULA is similar to PAPR of OFDM

$\Rightarrow$ Proposed upper bound bounds PAPR of OFDM

EIRP/PAPR Upper Bounds

PAPR is upper bounded with the maximum sampled value $|r_{\max}|$ as

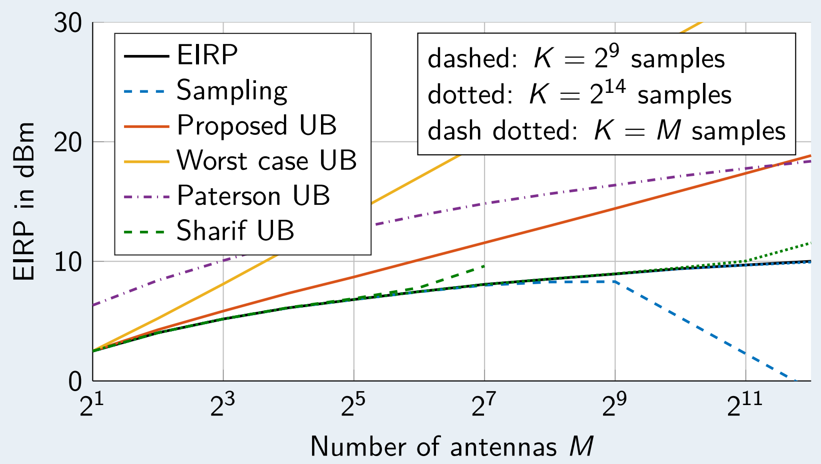
- Worst case: $\mathrm{EIRP}(\boldsymbol{x}) \le \|\boldsymbol{S}\|_1$
- Paterson et al. [3] ($K = M$):

$$\mathrm{EIRP}(\boldsymbol{x}) \le \left(\frac{2}{\pi}\log_e(2M) + 2\right)\left|r_{\max}\right|^2 \tag{13}$$

- Sharif et al. [4] (for $\frac{K}{M} > \frac{pi}{\sqrt{2}}$):

$$\mathrm{EIRP}(\boldsymbol{x}) \le \frac{\frac{K^2}{M^2}}{\frac{K^2}{M^2} - \frac{\pi^2}{2}}\left|r_{\max}\right|^2 \tag{14}$$

Proposed Upper Bound is Close for Few Antennas



$$\boldsymbol{x} \sim \mathcal{CN}\left(\boldsymbol{0}, \frac{1}{M}\boldsymbol{I}\right)$$

Transmission to One Single Antenna Receiver

The received signal is

$$y = \boldsymbol{h}^{\mathrm{H}}\boldsymbol{x} + z \tag{15}$$

where $z \sim \mathcal{CN}(0,\sigma^2)$ and $\boldsymbol{h} \sim \mathcal{CN}(\boldsymbol{0},\boldsymbol{I})$.
Sum power constraint is

$$\|\boldsymbol{x}\|_2^2 = \mathrm{tr}(\boldsymbol{S}) \le P. \tag{16}$$

EIRP constraint is

$$\mathrm{EIRP}(\boldsymbol{x}) \le P_{\mathrm{EIRP}}. \tag{17}$$

For proper complex Gaussian noise capacity is

$$C = \max_{f_{\boldsymbol{x}}(\boldsymbol{x})} I(y,\boldsymbol{x}) = \max_{f_{\boldsymbol{x}}(\boldsymbol{x})} h(y) - h(y|\boldsymbol{x})$$
$$= \max_{f_{\boldsymbol{x}}(\boldsymbol{x})} h(y) - \log_2\left(e\pi\sigma^2\right) \tag{18}$$
$$\text{s.t.:}\quad \mathrm{tr}(\boldsymbol{S}) \le P \quad\text{and}\quad \mathrm{EIRP}(\boldsymbol{x}) \le P_{\mathrm{EIRP}}$$

The differential entropy $h(y)$ is maximized, if and only if, y is proper complex Gaussian distributed
$\Rightarrow$ $\boldsymbol{x}$ has to be proper complex Gaussian distributed?

$$\Rightarrow h(y) = \log_2\left(e\pi\sigma_y^2\right) = \log_2\left(e\pi\boldsymbol{h}^{\mathrm{H}}\boldsymbol{Sh}\right) \tag{19}$$
$$\Rightarrow C = \max_{\boldsymbol{S}} \log_2\left(1 + \frac{1}{\sigma^2}\boldsymbol{h}^{\mathrm{H}}\boldsymbol{Sh}\right) \tag{20}$$
$$\text{s.t.:}\quad \mathrm{tr}(\boldsymbol{S}) \le P \quad\text{and}\quad \mathrm{EIRP}(\boldsymbol{x}) \le P_{\mathrm{EIRP}}$$

$\Rightarrow$ Linear precoding is optimal

- Without EIRP constraint optimal $\boldsymbol{S}$ is found by applying eigenvalue decomposition and water filling
- We do not know how to find optimal $\boldsymbol{S}$ with EIRP constraint

$\Rightarrow$ We use maximum ratio transmission and a scaling to comply with the sum power constraint and the EIRP constraint

$$\boldsymbol{w}_{\mathrm{MRT}} = \frac{\boldsymbol{h}}{\max\left(\|\boldsymbol{h}\|_2, \sqrt{\mathrm{EIRP}(\boldsymbol{h})}\right)} \tag{21}$$

$\Rightarrow$ The EIRP upper bound (11) consists of a sum of absolute values of the sums of products of proper complex Gaussian random variables

Conclusions

- EIRP constraints constraint precoding in MIMO
- We present an upper bound which is close for few antennas

References

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[2] C. Vithanage, Y. Wang, and J. Coon, "Transmit beamforming methods for improved received signal-to-noise ratio in equivalent isotropic radiated power-constrained systems," *IET Communications*, vol. 3, no. 1, pp. 38–47, Jan. 2009.

[3] K. G. Paterson and V. Tarokh, "On the existence and construction of good codes with low peak-to-average power ratios," *IEEE Trans. Inf. Theory*, vol. 46, no. 6, pp. 1974–1987, Sep. 2000.

[4] M. Sharif, M. Gharavi-Alkhansari, and B. H. Khalaj, "On the peak-to-average power ratio of OFDM signals based on oversampling," *IEEE Trans. Commun.*, vol. 51, no. 1, pp. 72–78, Jan. 2003.

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