A graph expansion-contraction method for estimating error floors of LDPC codes

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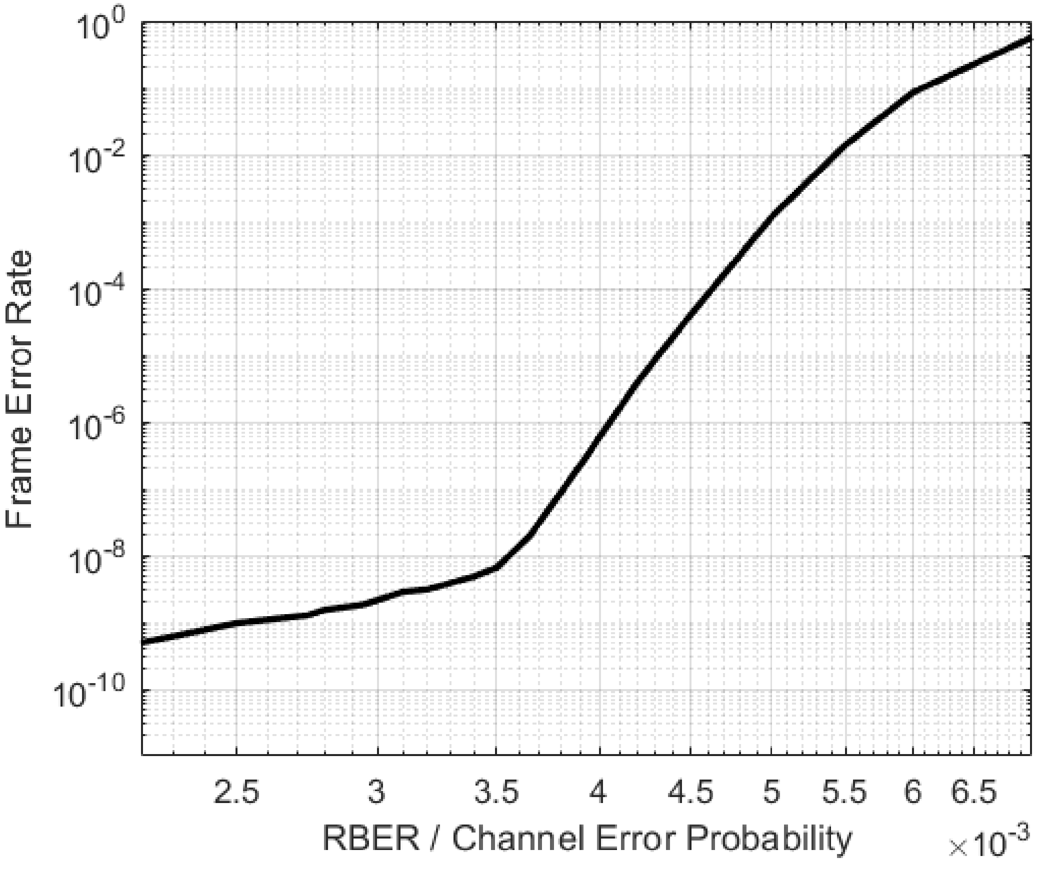
Error floor

- An abrupt degradation of FER at low RBER caused by a failure of an iterative decoder to converge to a codeword



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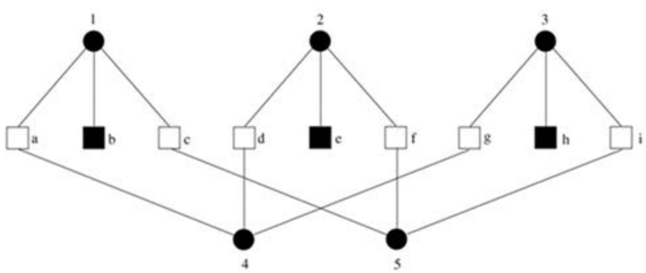
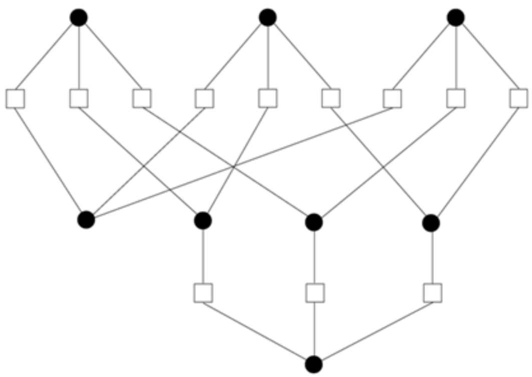
Trapping sets

- Error floor is attributed to dense subgraphs present in the Tanner graph – trapping sets
- <u>An (A, B) trapping set</u>: a set of not eventually correct variabe nodes of size A, inducing a subgraph of the B odd degree check nodes.

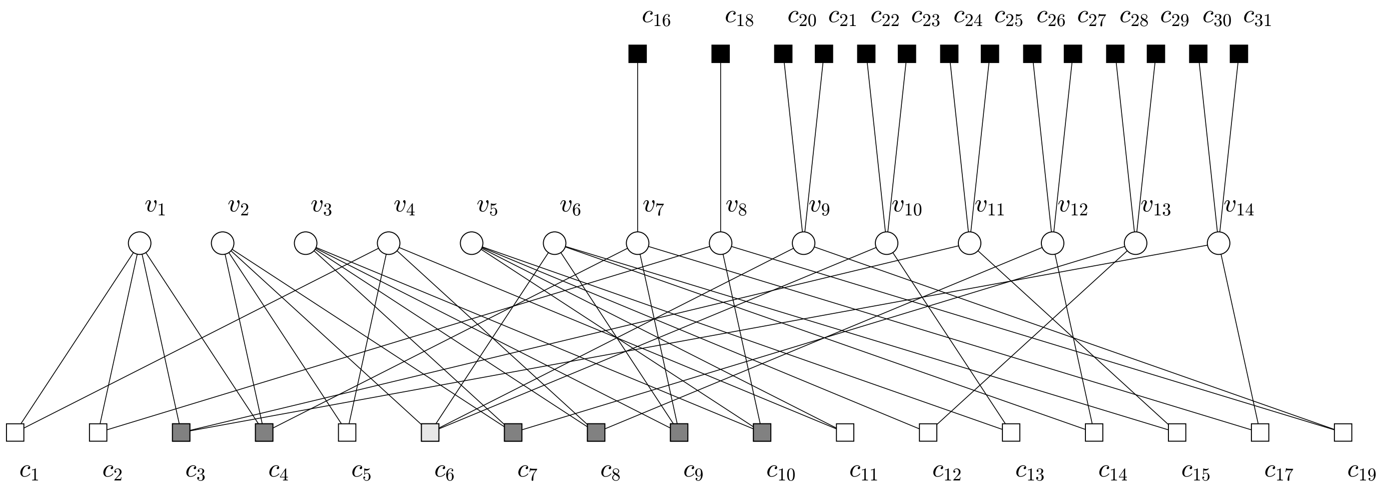


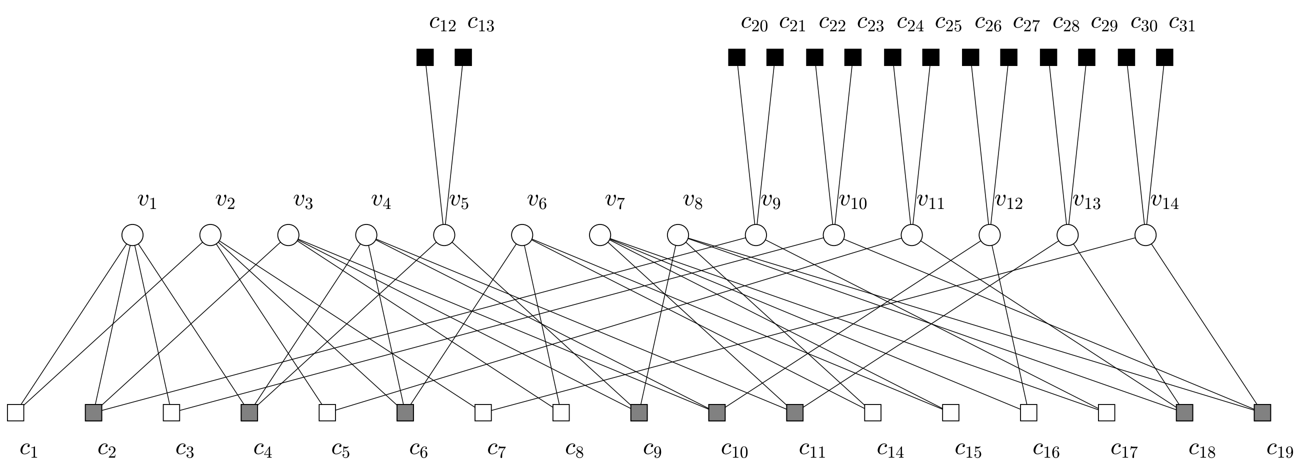
(5,3) trapping set



(8,0) Trapping set

Some large trapping sets





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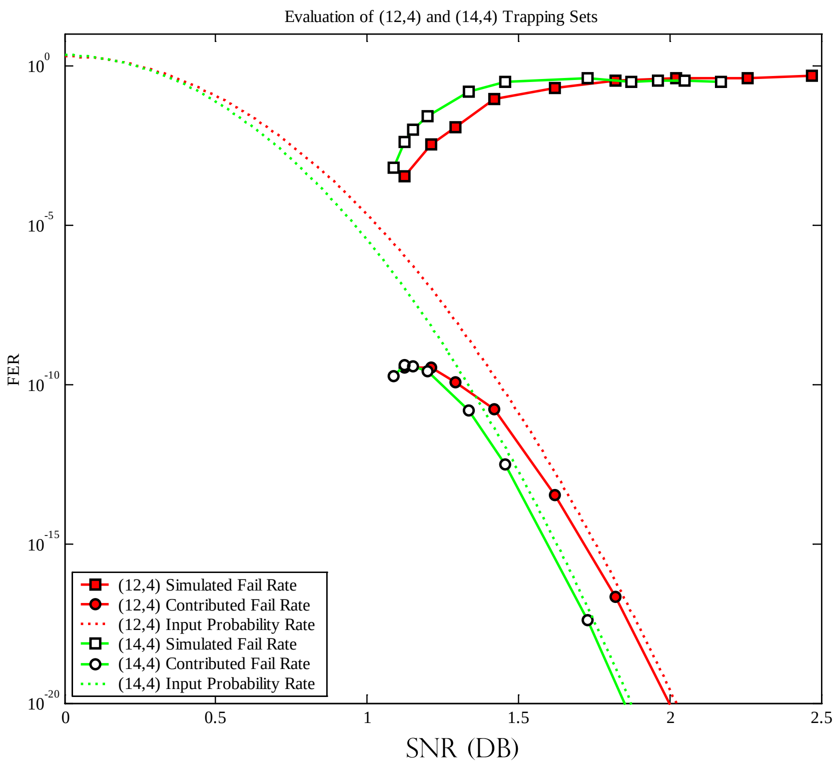
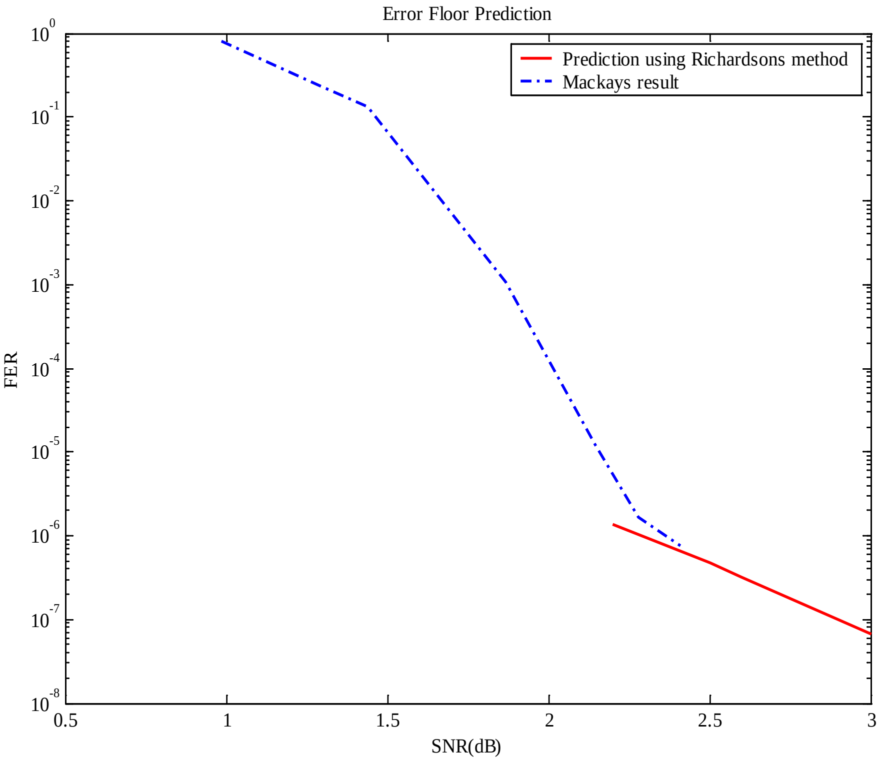
FER estimation using importance sampling

- Tom Richardson (Allerton 2003)
  - An experimental evidence of error floors of LDPC codes, and makes a connection with trapping set
  - Introduced the importance sampling to estimate error floor





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Importance sampling

- Input:
  - Tanner graph G and the decoding algorithm $\mathcal{D}$
  - Collection of subgraphs $\mathcal{T}_1, \mathcal{T}_2, \ldots, \mathcal{T}_T$ that are believed to be harmful to $\mathcal{D}$
- Algorithm:
  - Find positions of variable nodes of each of the harmful subgraphs in G (sensitive variable nodes)
  - In Monte-Carlo simulations, corrupt sensitive variable nodes in G, run $\mathcal{D}$, and record if an error patterns that lead to failure of $\mathcal{D}$
  - Obtain the contributions $\mathrm{FER}_{\mathcal{T}_1}, \mathrm{FER}_{\mathcal{T}_2}, \ldots, \mathrm{FER}_{\mathcal{T}_T}$ of each subgraph to the FER and reweight them by occurrence frequencies of subgraphs $\mathcal{T}_1, \mathcal{T}_2, \ldots, \mathcal{T}_T$ in G

$$FER = \sum_{\mathcal{T}} w_{\mathcal{T}} \mathrm{FER}_{\mathcal{T}}$$

Harmfulness

- Major flaw of importance sampling – "subgraphs that are believed to be harmful"
- Vasić (Allerton 2005, ICC 2006)
  - Harmfulness defined based on uncorrectable error patterns
  - Defined the critical number C and strength S as a measure of harmfulness of a trapping set
  - No assumptions on a trapping set topology were made!
  - A simple formula for calculation error floor from harmfulness of trapping sets
  - Sufficient conditions for failure of Gallager B and bit-flipping algorithms on column weight three codes (γ=3)
- Vasić (Allerton 2009): trapping set ontology - a database of topological relationship of trapping sets of simple decoders for column weight-three code

Harmfulness for stronger decoding rules

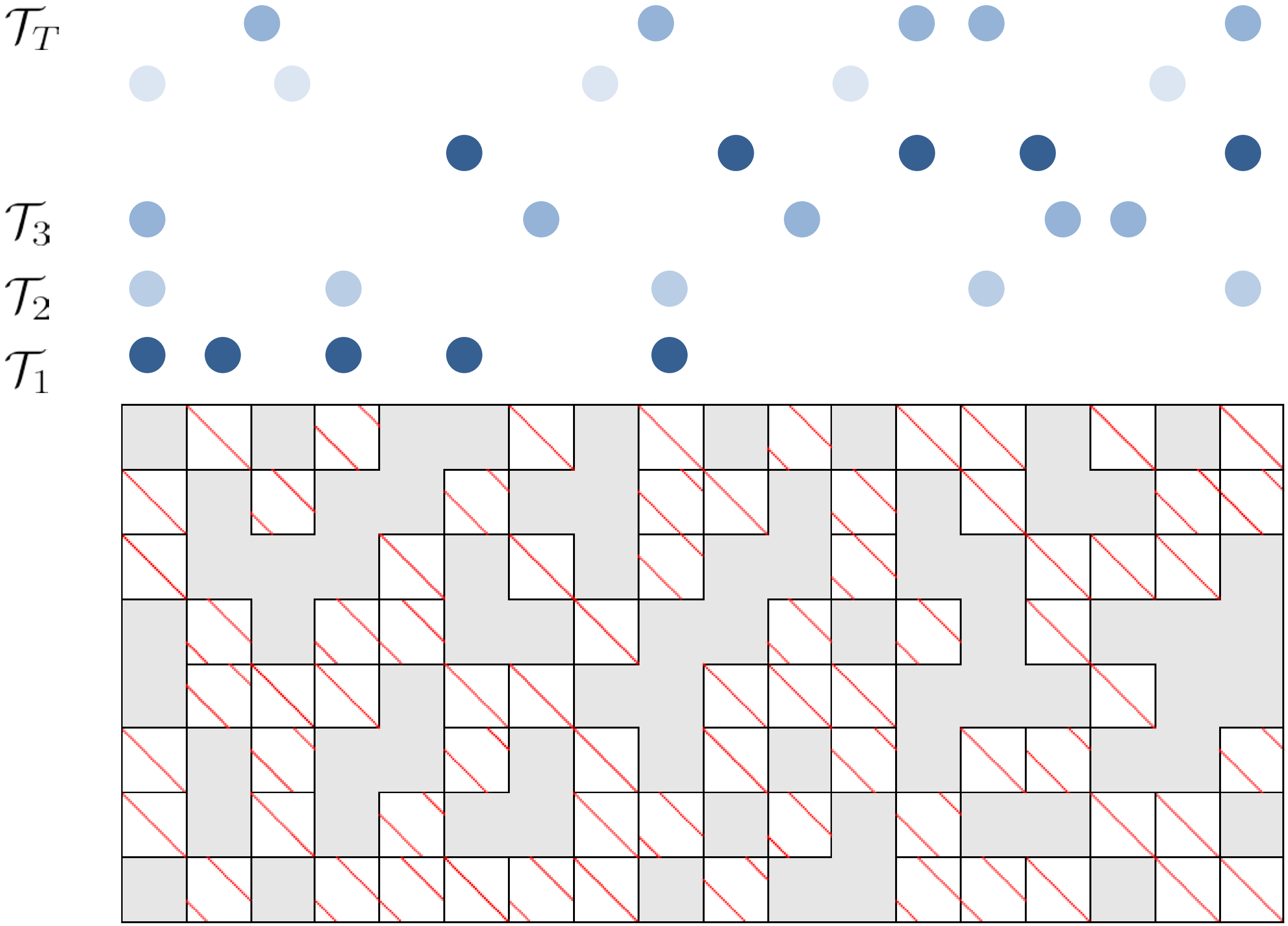
- Many papers on combinatorial characterization and search of trapping sets
  - Measuring harmfulness based solely on the value of (A,B) parameters is and their relation is wrong!
  - No assumptions on a trapping set topology must be made!
  - The only theory-supported indicators of harmfulness of a trapping set are its expansion (or "density") and cycle profile
- Whether a trapping set (subgraph) is harmful depends on a **decoder** and its **neighborhood** in the Tanner graph
  - For simple decoders, trapping sets can be treated isolated from the rest of the graph
  - For decoders of interest (offset min-sum, FAID), an isolated trapping set is not sufficient to predict its impact on error floor

FER of higher column-weight codes

- Even for γ=4, the number of nonisomorphic dense subraphs is enormous
- It is impractical to create a trapping set ontology as we did for for γ=3, instead harmfull subraphs must be searched for in the specific Tanner graph
- For accuracy of FER estimation, we cannot afford to miss any harmful trapping set, thus verification of harmfulness must be exhaustive

- But which graphs are harmful?
- How do we estimate error floor based on harmfulness?

Trapping set positions in the QC-LDPC code



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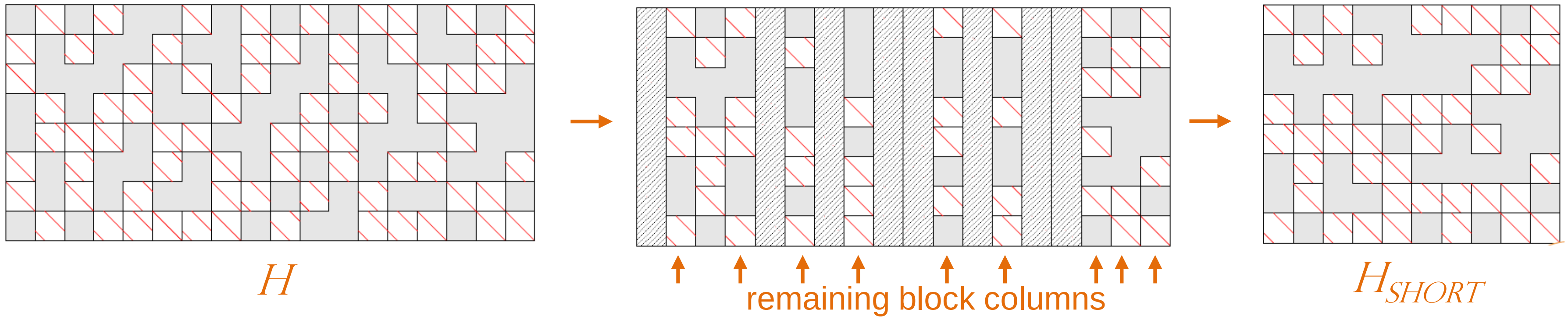
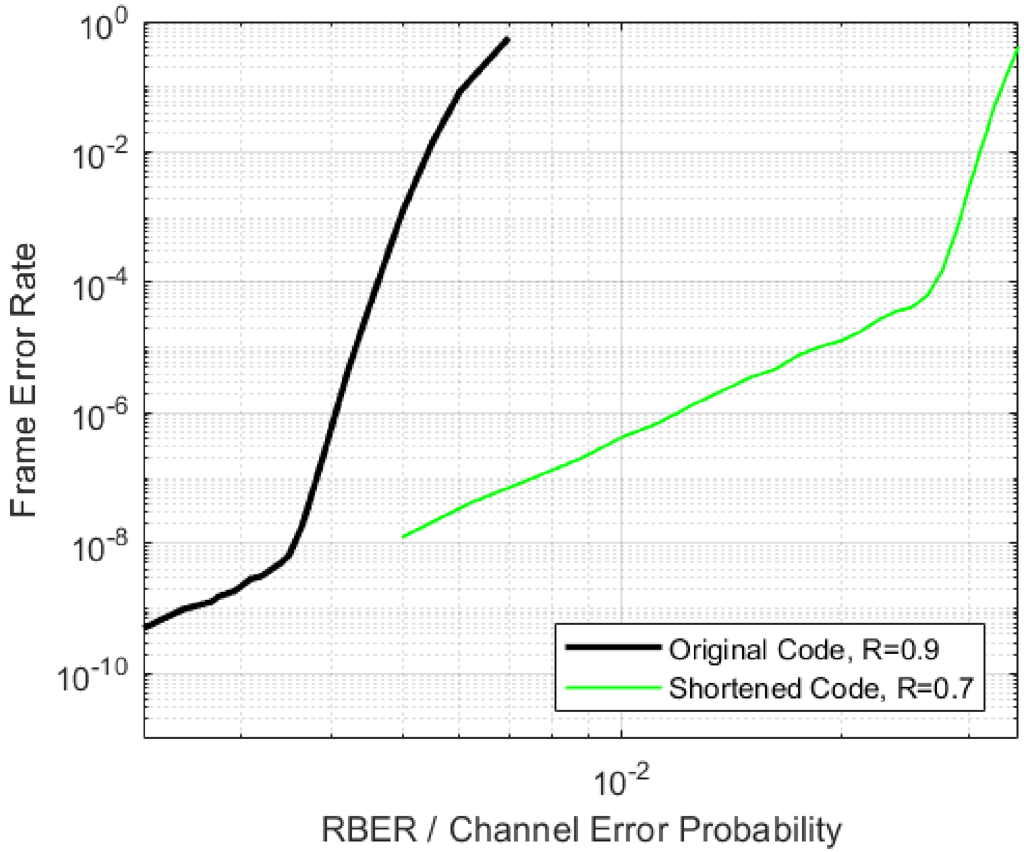
Importance sampling through code shortening

- Create a shortened code H_{SHORT} which contains the same most harmful trapping sets as the original code H, and run Monte Carlo simulations on H_{SHORT} to detect its error floor.



Decoding on the shortened code

- If the decoder fails on the same structures as in H, the error floor will have the same slope, but since H_{SHORT} has lower rate, the error floor will appear at a higher FER, resulting in computational savings



Reasons for accuracy

- Each trapping set is not treated as an isolated graph but in its “natural surrounding”
- Message from the variables outside the trapping set are realistic (not considered to be saturated)

Code shortening constrains

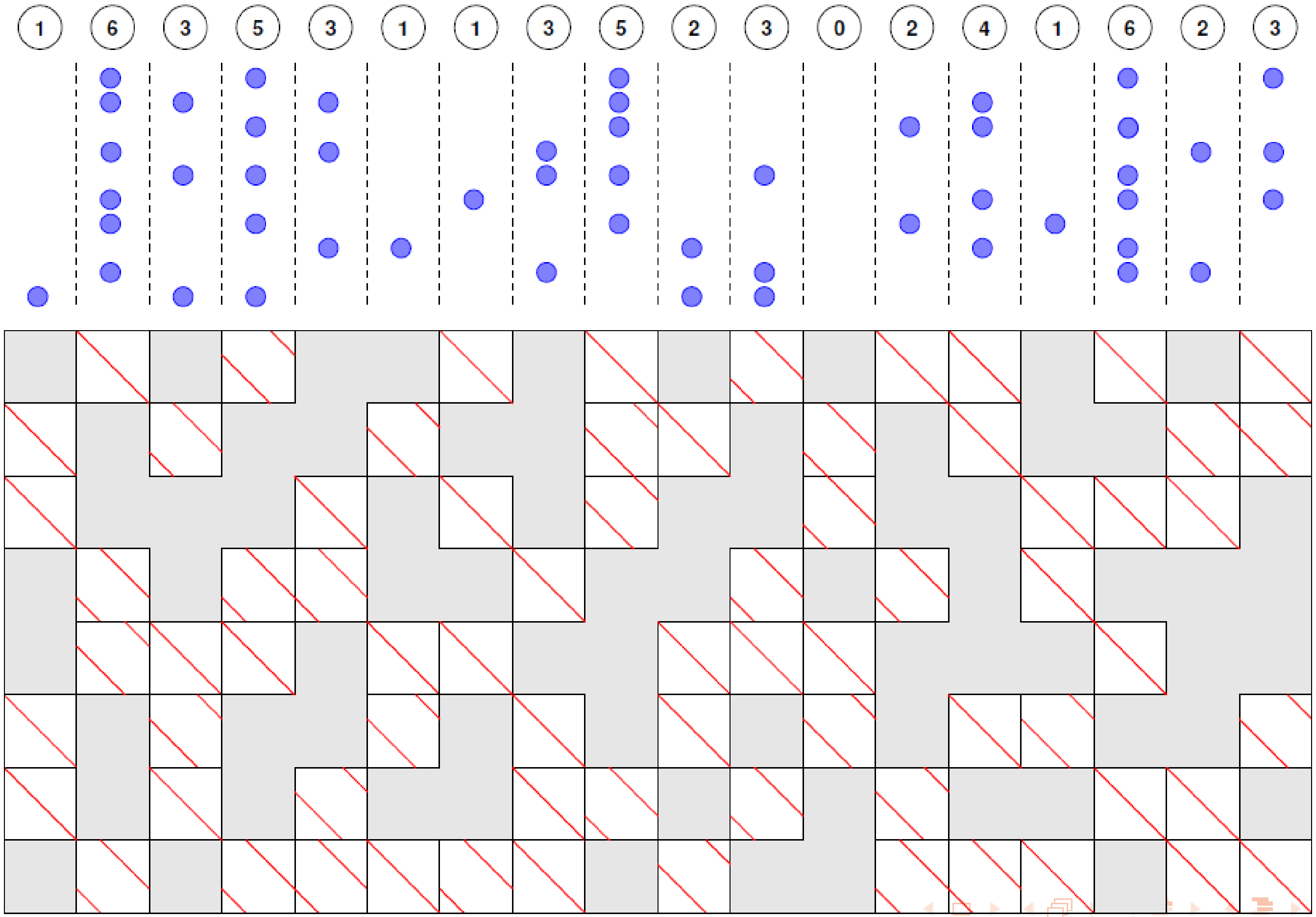
- Choosing properly which block columns to keep is critical for the efficiency
  - the less shortening - the easier to ensure that the harmful trapping sets of H will remain in H_{SHORT}, but smaller computational saving
  - with too much shortening, there is a chance that H_{SHORT} does not contain the harmful trapping sets any more, resulting in an erroneous prediction of the error floor

Optimization problem

- Design the shortest, worst possible shortened code
- Minimize the number of block columns kept, while still ensuring that the most harmful trapping sets are present in H_{SHORT}
- This optimization problem is closely related to the well studied weapon-target assignment problem and the hypergraph demand matching problem

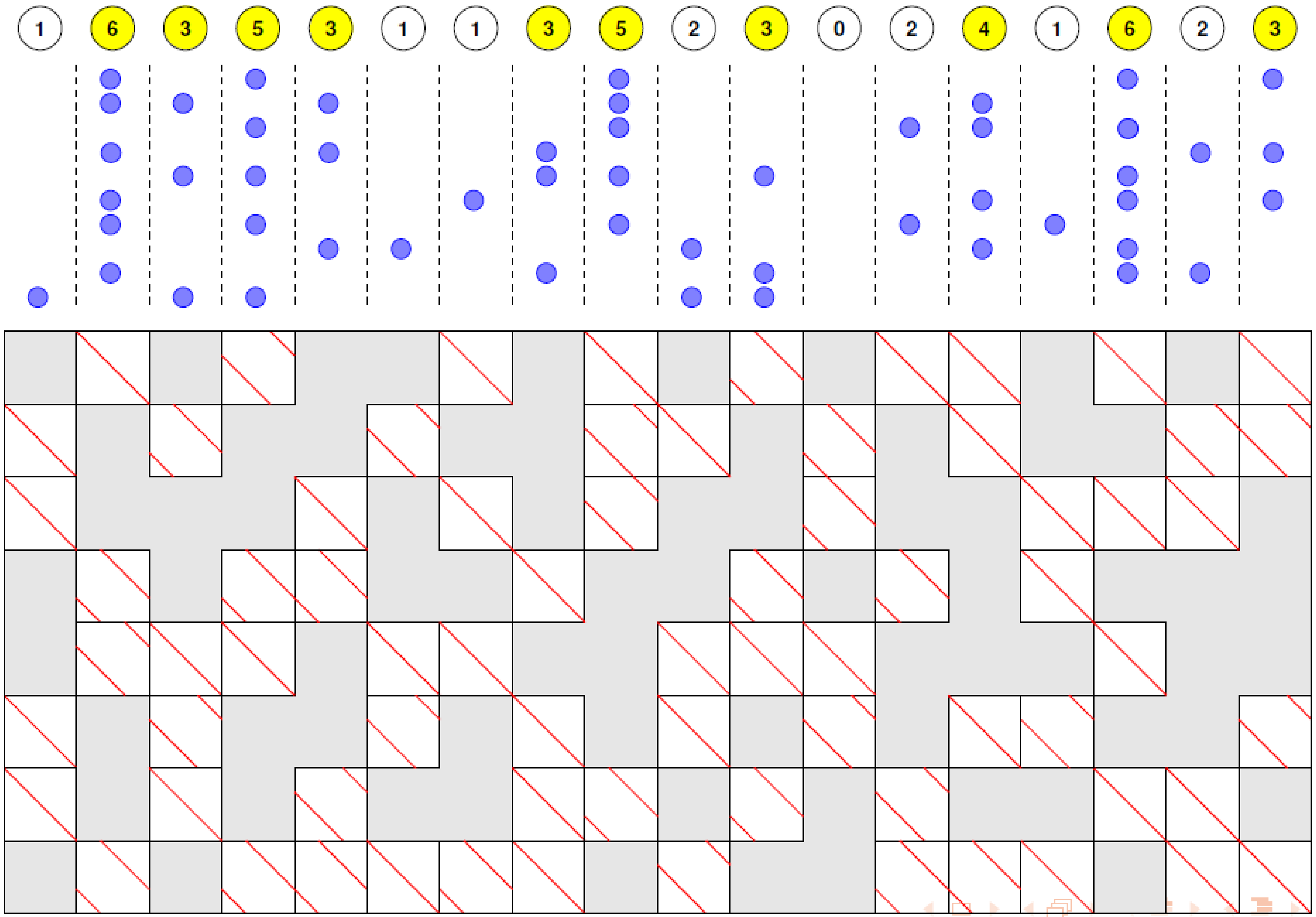
Step 1

- The l-th block harmfulness = sum of harfmulnesses of all trapping sets having a variable node in it



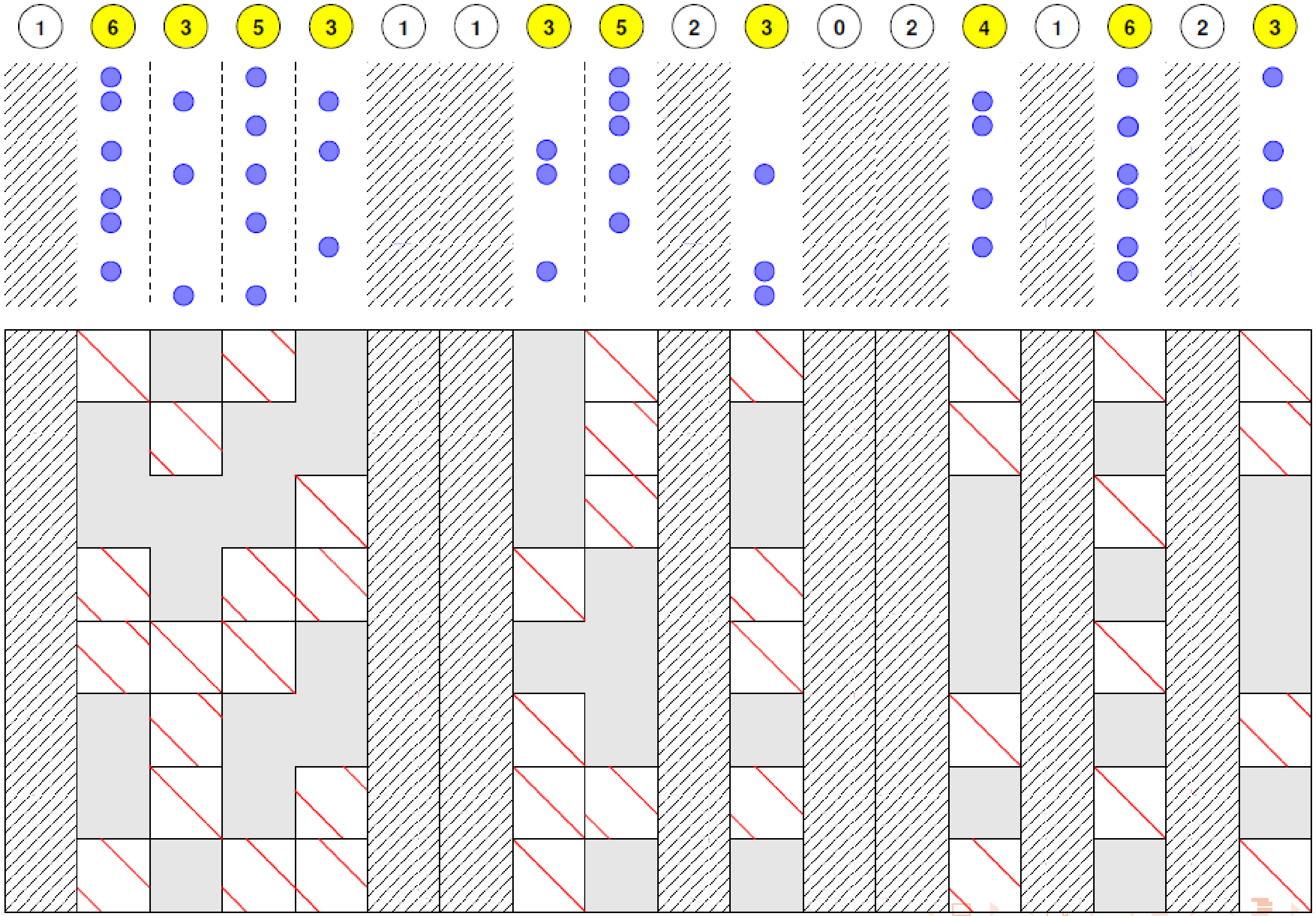
Step 2

- Select the block-columns which have total harmfulness weight $\mathbf{H}$ greater than a threshold T



Step 3

- Build a shortened version of the code, with $N_{SHORT}=9$ block-columns



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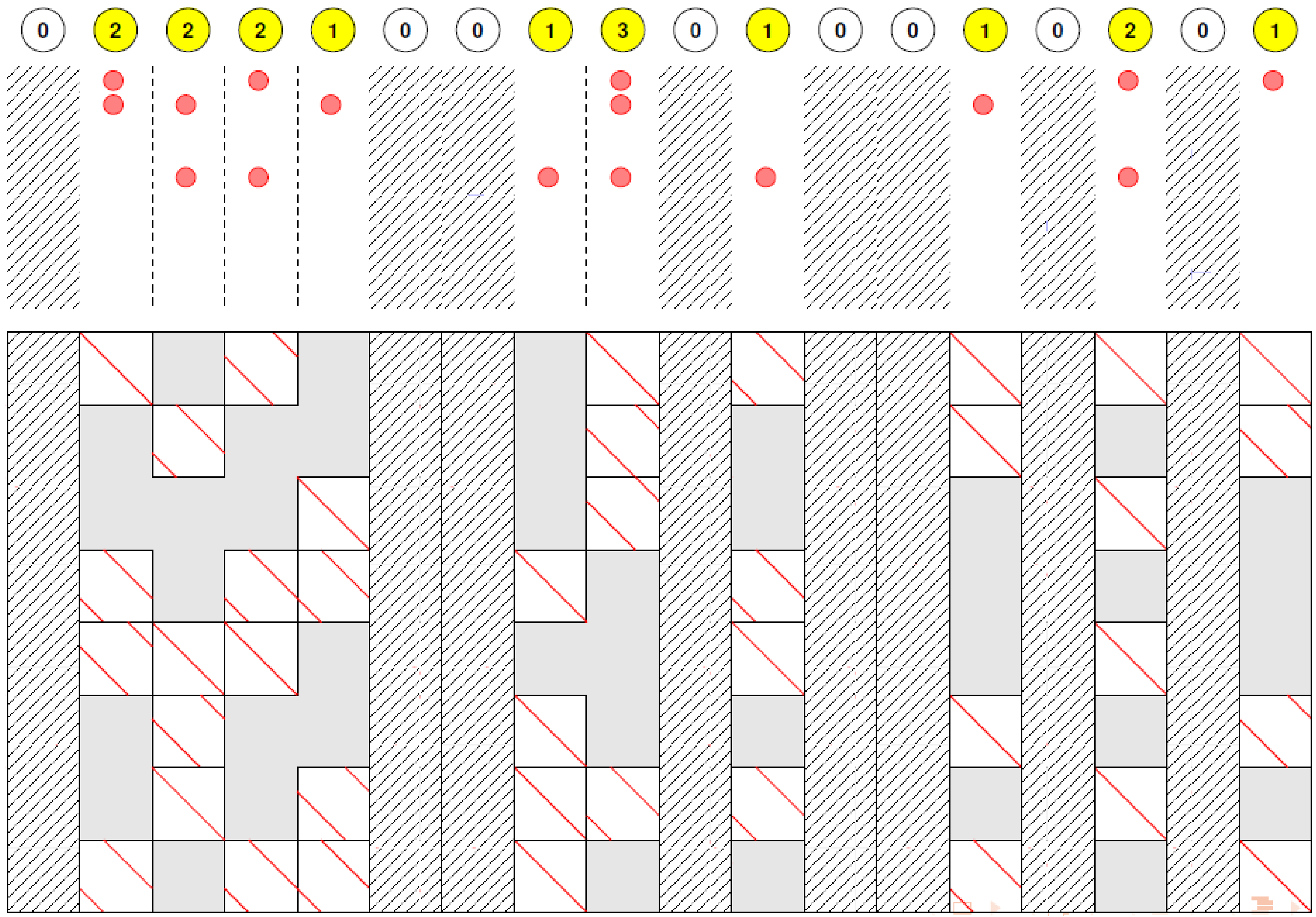
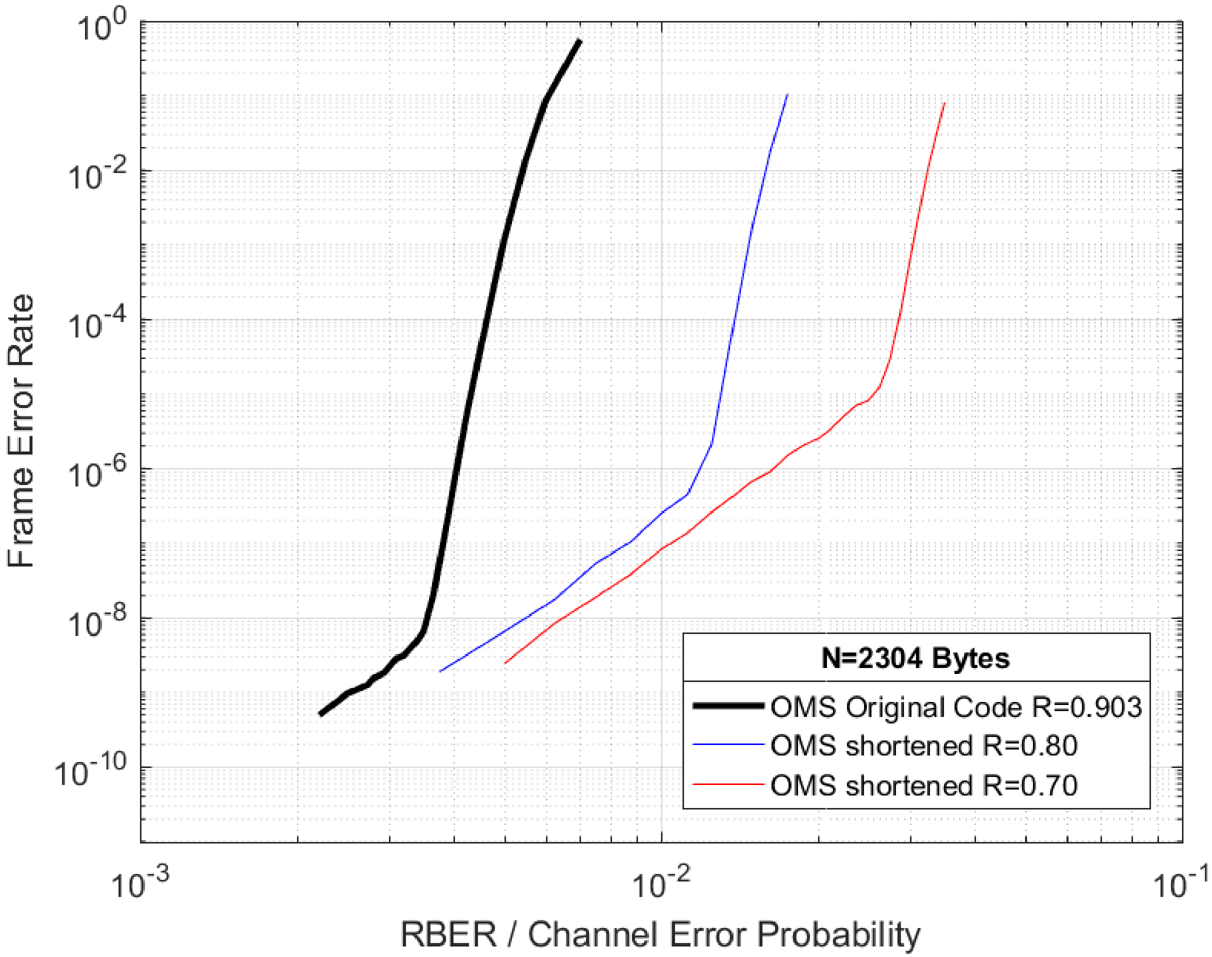
Step 4

- Correction factor - the ratio between remaining harmfulness weight and total harmfulness weight.



Direct simulation of the shortened codes



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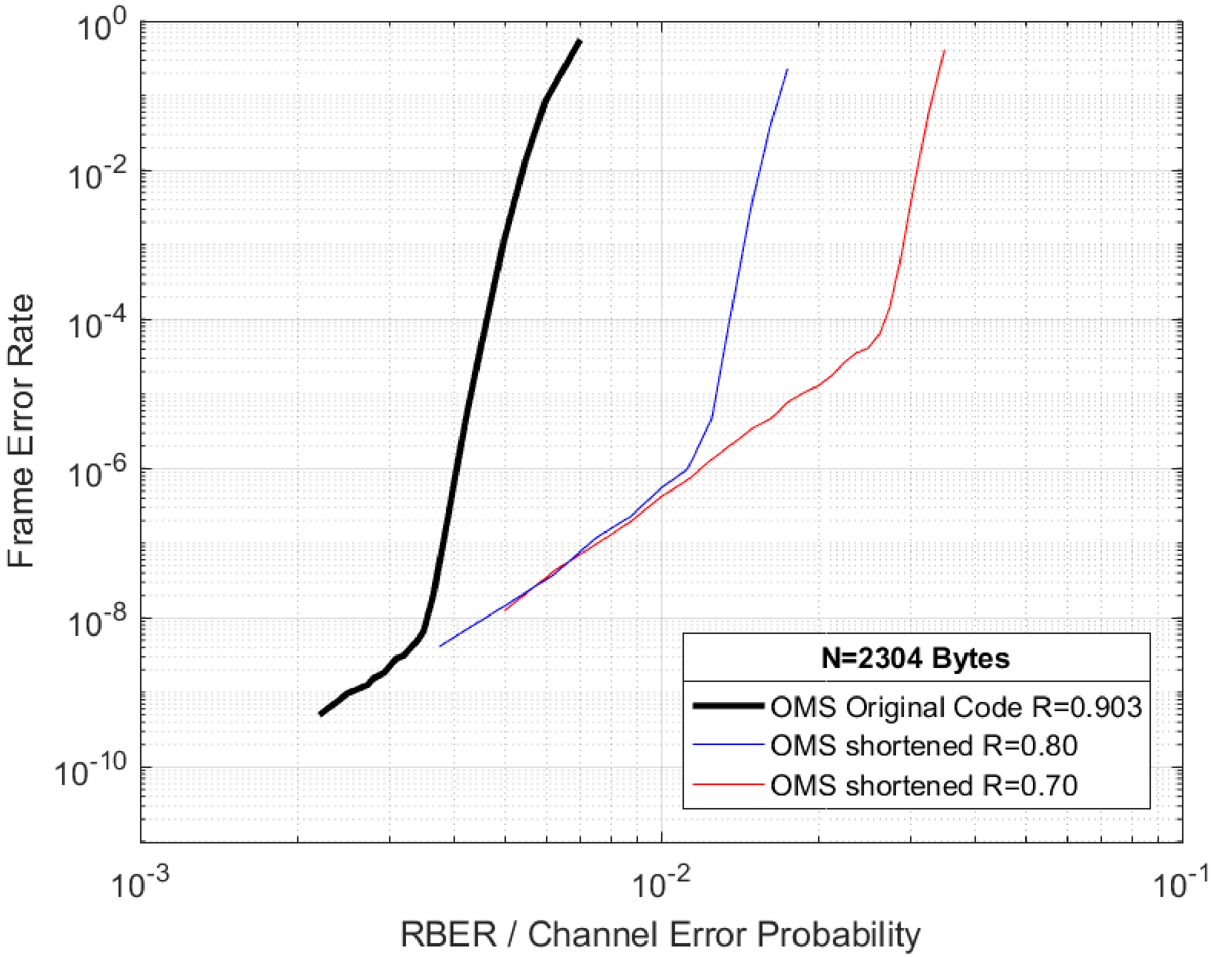
Results with the correction factor



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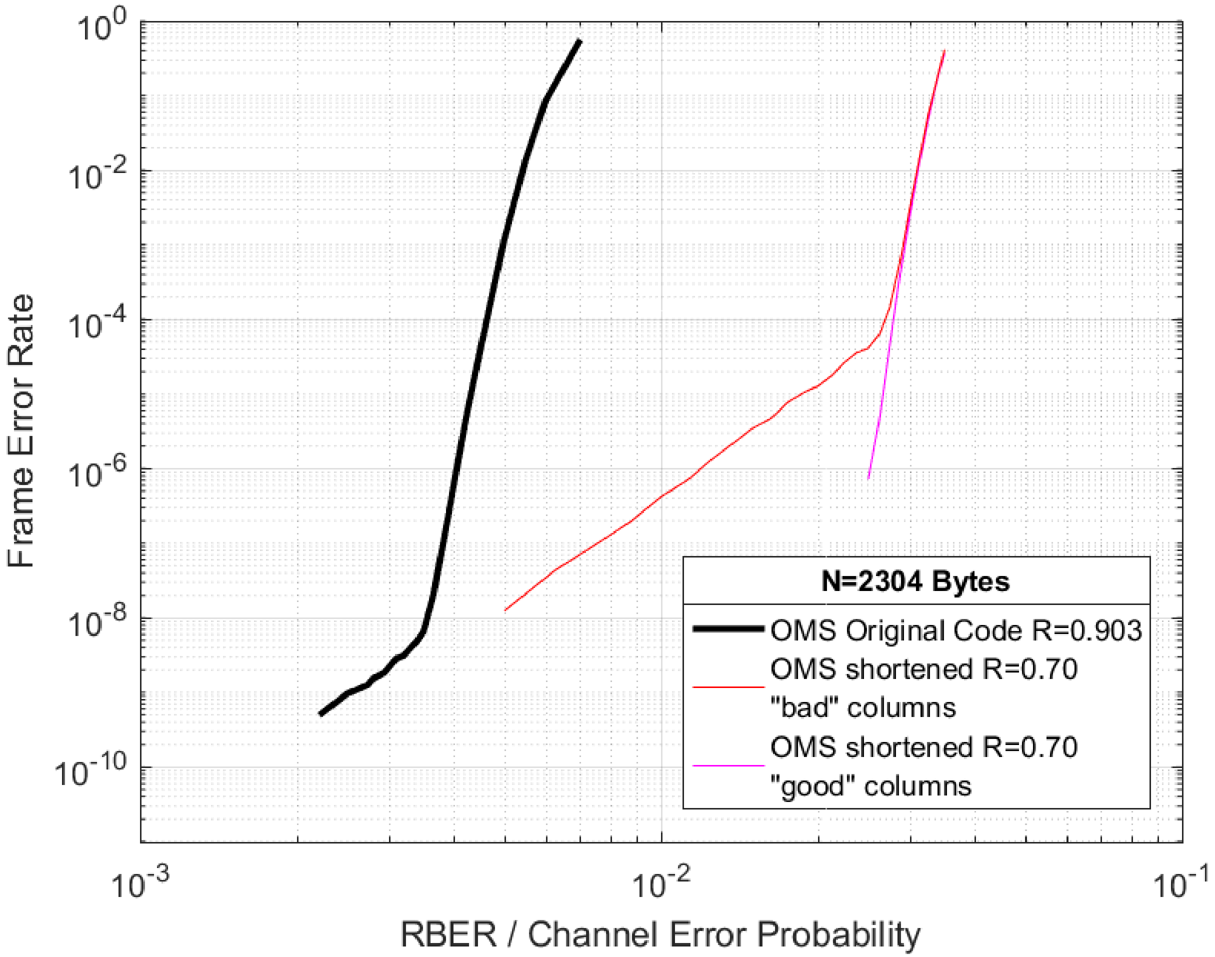
Selecting right block columns is critical



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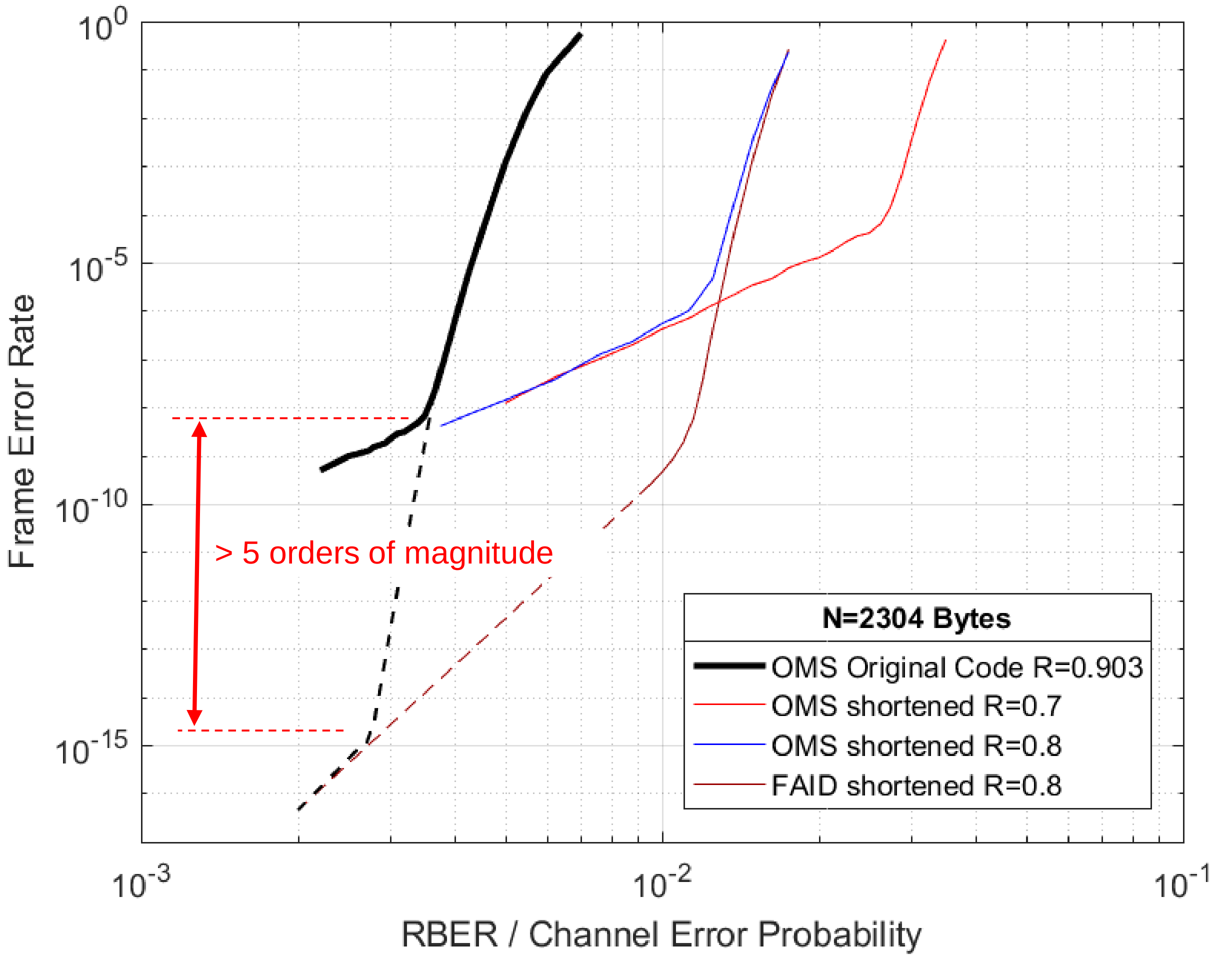
Results for stronger decoders (FAID)

- Our prediction method is valid for ANY decoder



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Harmfulness

- Harmfulness of a trapping set is determined by its critical number. Relative harmfulness of two trapping sets with equal critical numbers C is a ratio of their strengths

Basic terminology

- *Failure inducing set* is a set of variable nodes that have to be initially in error for the decoder $\mathcal{D}$ to fail
- The *critical number* C of a trapping set is the minimal number of variable nodes that have to be initially in error for the decoder to end up in that trapping set
- *Strength* S of a trapping set with critical number C is the number of inducing sets of cardinality C (the number of weight-C error patterns on variable nodes in the trapping set)

S. K. Chilappagari, D. V. Nguyen, B. Vasić, and M. W. Marcellin, "Error correction capability of column-weight-three LDPC codes under the Gallager A algorithm - Part II," *IEEE Trans. Information Theory,* June 2010.

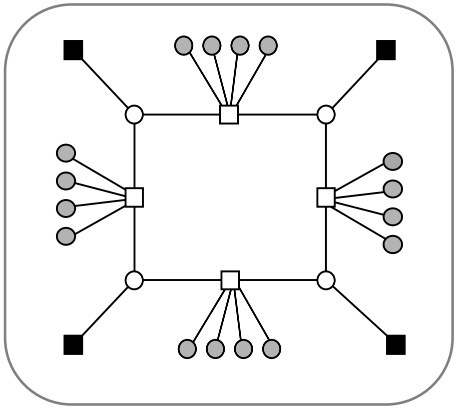
S. K. Chilappagari, D. V. Nguyen, B. Vasić, and M. W. Marcellin, "On Trapping Sets and Guaranteed Error Correction Capability of LDPC codes and GLDPC Codes," *IEEE Trans. Information Theory,* Apr. 2010.

B. Vasić, D.V. Nguyen, and S. K. Chilappagari, "Chapter 6 - Failures and Error Floors of Iterative Decoders ," *Academic Press Library in Mobile and Wireless Communications ,* 2014.

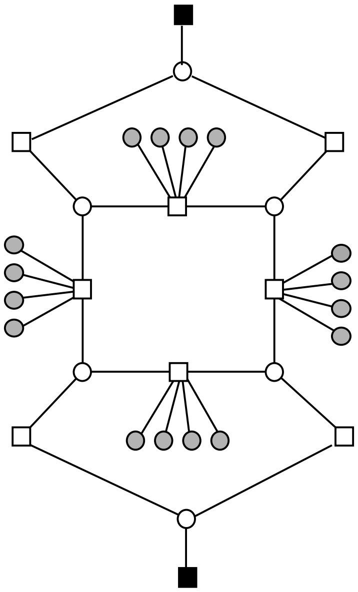
The expansion-contraction method

- In G find all **<u>dense</u>** subgraphs $\mathcal{G}$ up to a_{max} variable nodes that expand up to b_{max} check nodes
- The graphs $\mathcal{G}$ are not necessarily trapping sets
- Whether $\mathcal{G}$ contains a failure inducing set of variable nodes depends on its neighborhood in G
- Expand each $\mathcal{G}$ by including neighbors of degree-one check nodes up to certain depth – this creates a possibly large expanded graph $\mathcal{G}_{\mathrm{exp}}$
- Find the critical number C and *<u>all</u>* inducing sets $E_1, E_2, \ldots, E_s$ in $\mathcal{G}_{\mathrm{exp}}$ - the contracted graph induced by the variable nodes $\cup_i E_i$ is a true trapping set (with strength S)

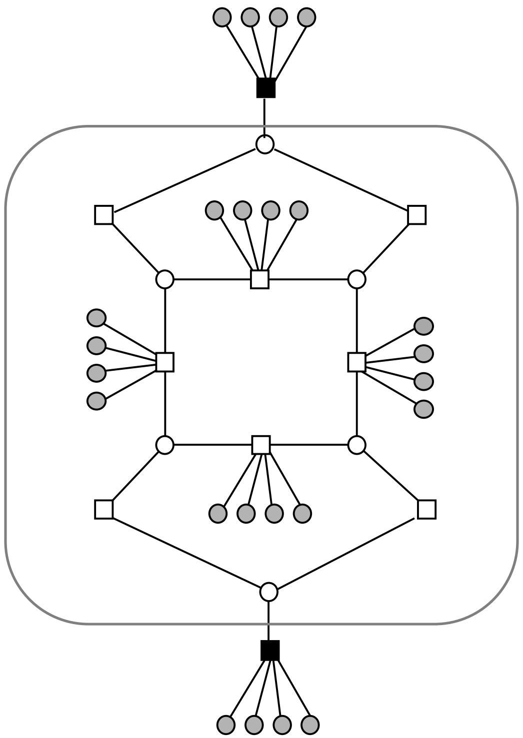
Depth-0



Expansion



Depth-1



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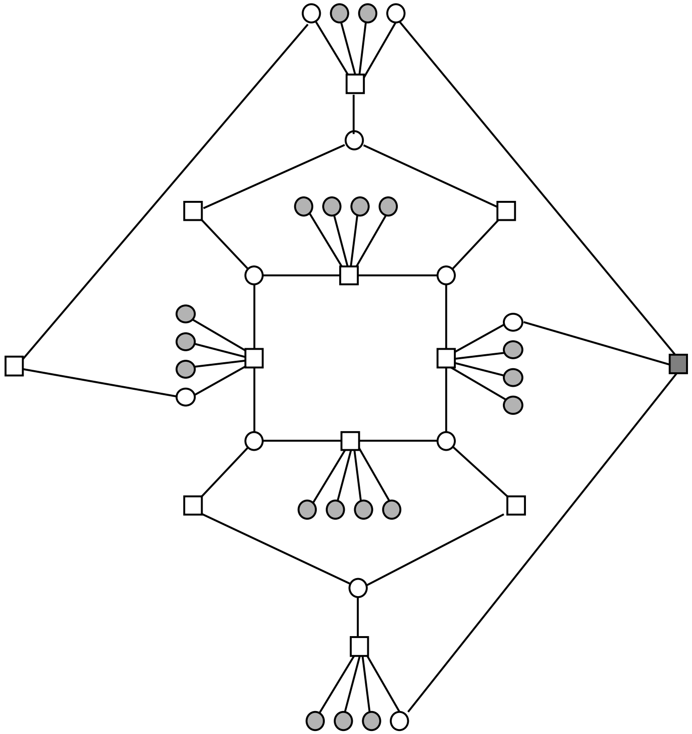
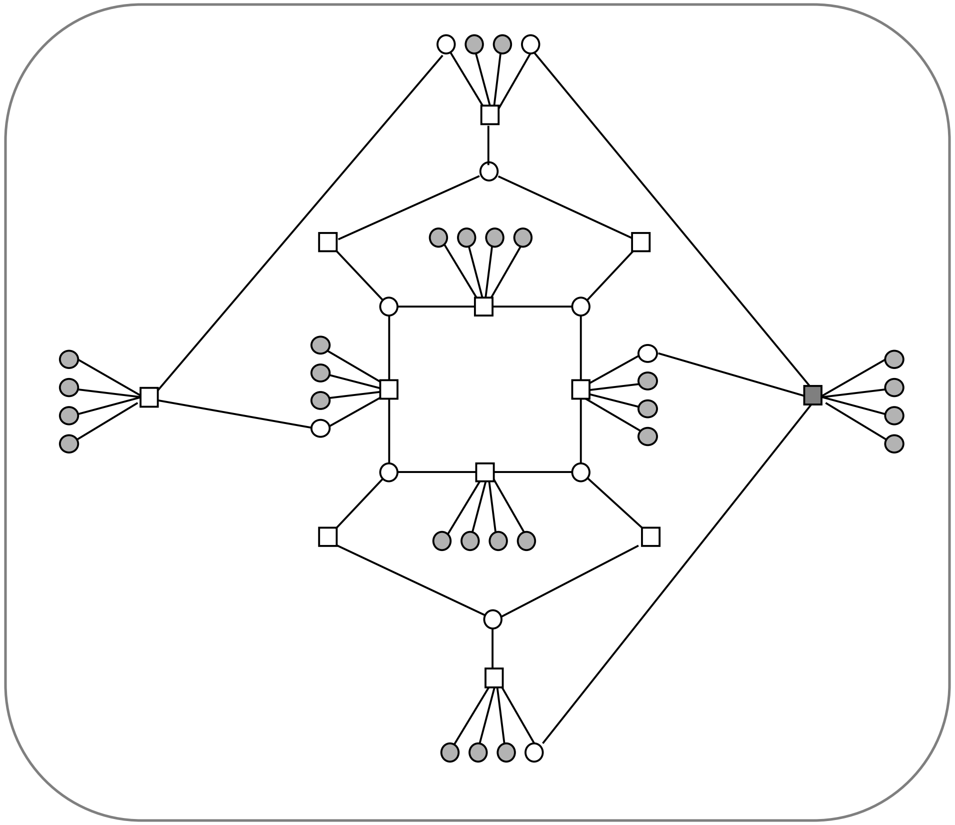
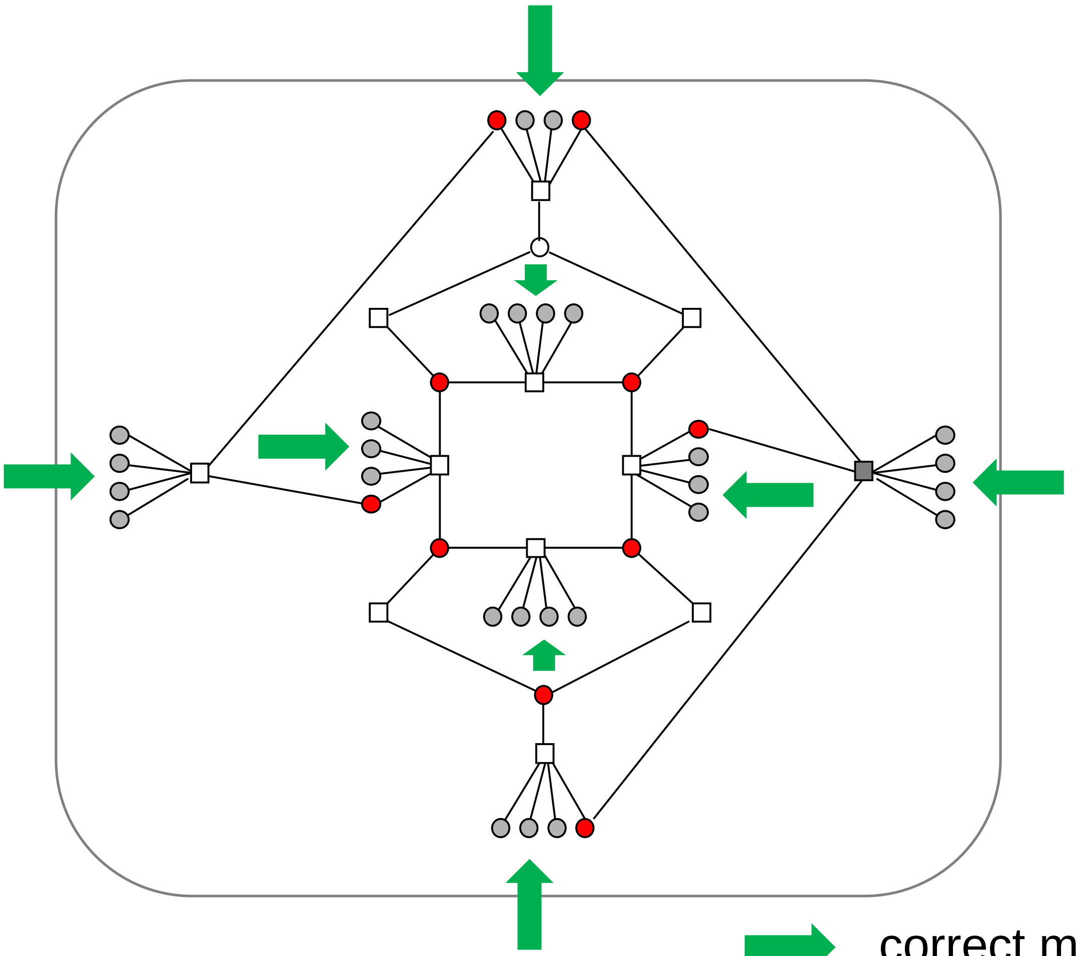
Expansion



Depth-2



Finding failure inducing sets • • • • •



correct messages

possibly corrupt variables

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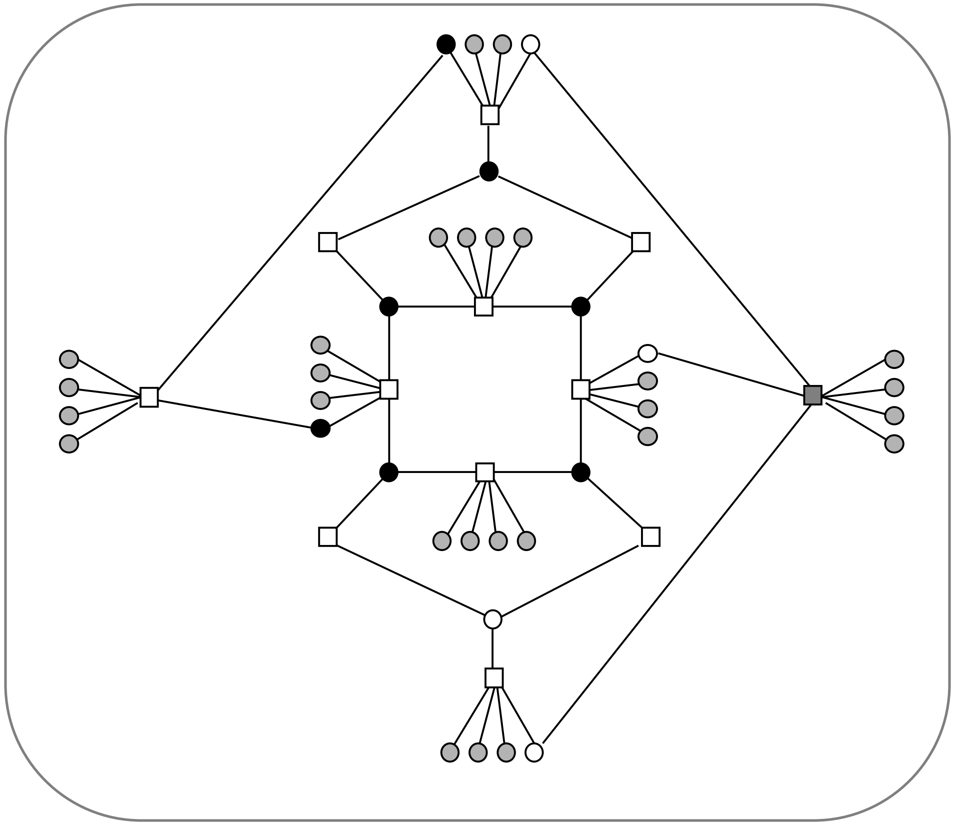
Contraction



- variables appearing in at least one inducing set

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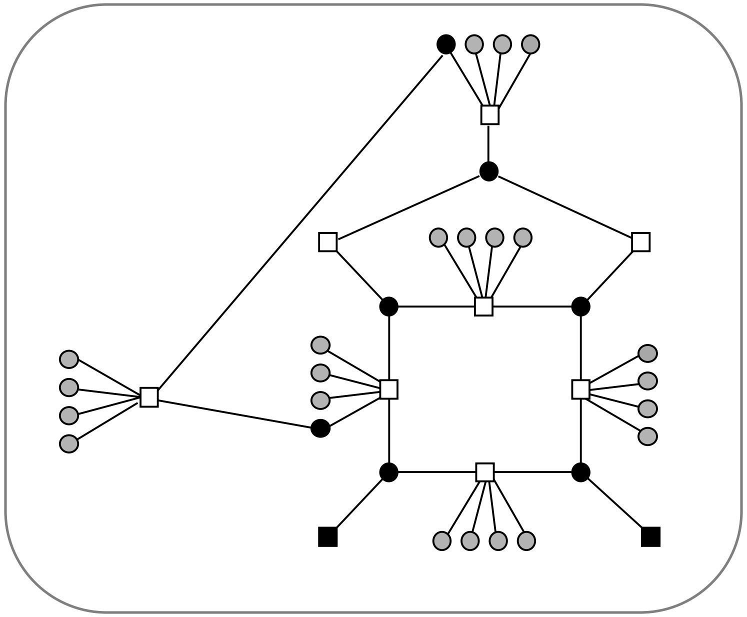
Contracted graph – the true trapping set



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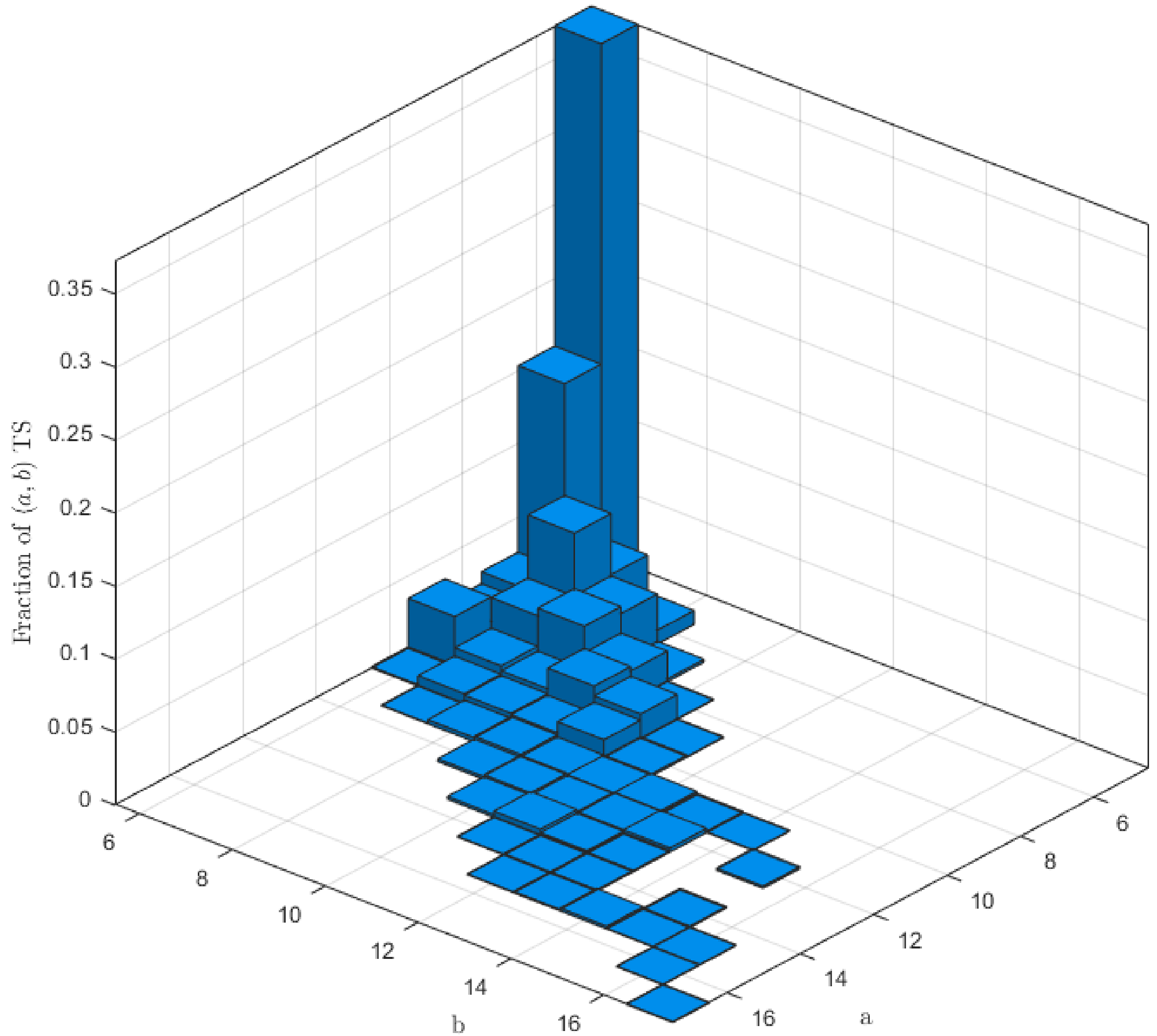
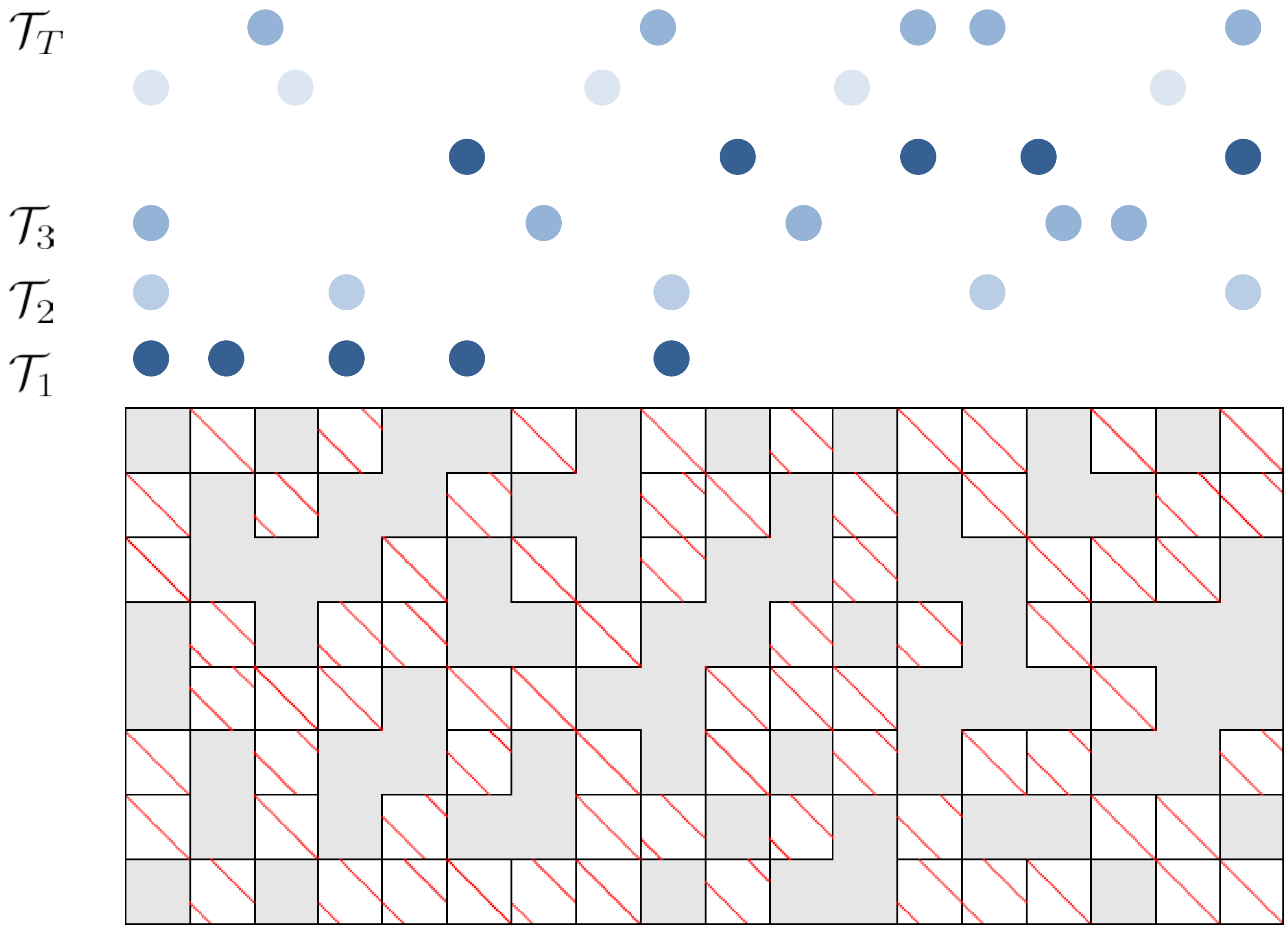
Distribution of trapping sets in a 2kB code



Identify harmful block columns and shorten



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Summary

- A computationally efficient method for estimating error floor of QC LDPC codes over the BSC channel
  - Arbitrary message update rule
  - Applicable to regular and irregular codes
  - Extendable to quantized output channels
- Graphs of small expansion in the Tanner graph are exhaustively expanded and contracted to obtain subgraphs that are true trapping sets
- Based on harmfulness of trapping sets code is shortened but in a way that it still contains most harmful trapping sets
- Allows fast optimization of decoders, and code optimization by removal of true trapping sets

Thank you!

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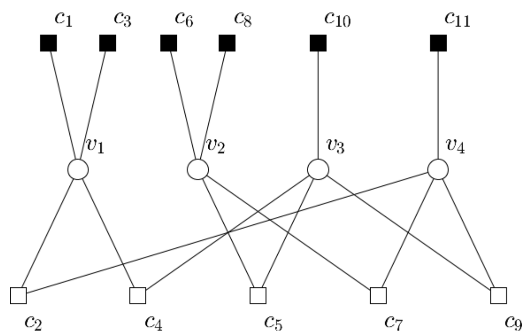
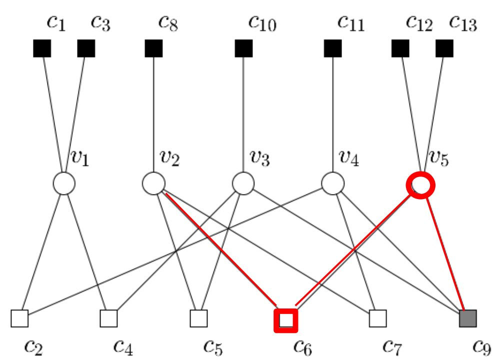
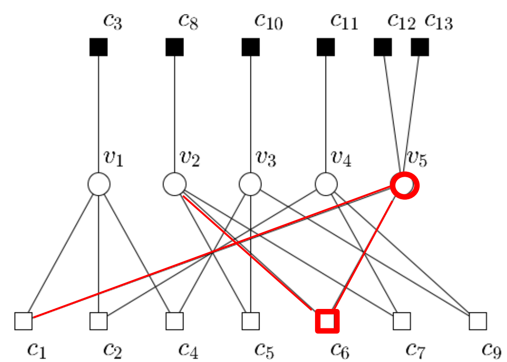
Expansion

(4,6) 0-2-1



(5,7) 0-4-1-2





(5,6) 0-2-3-2

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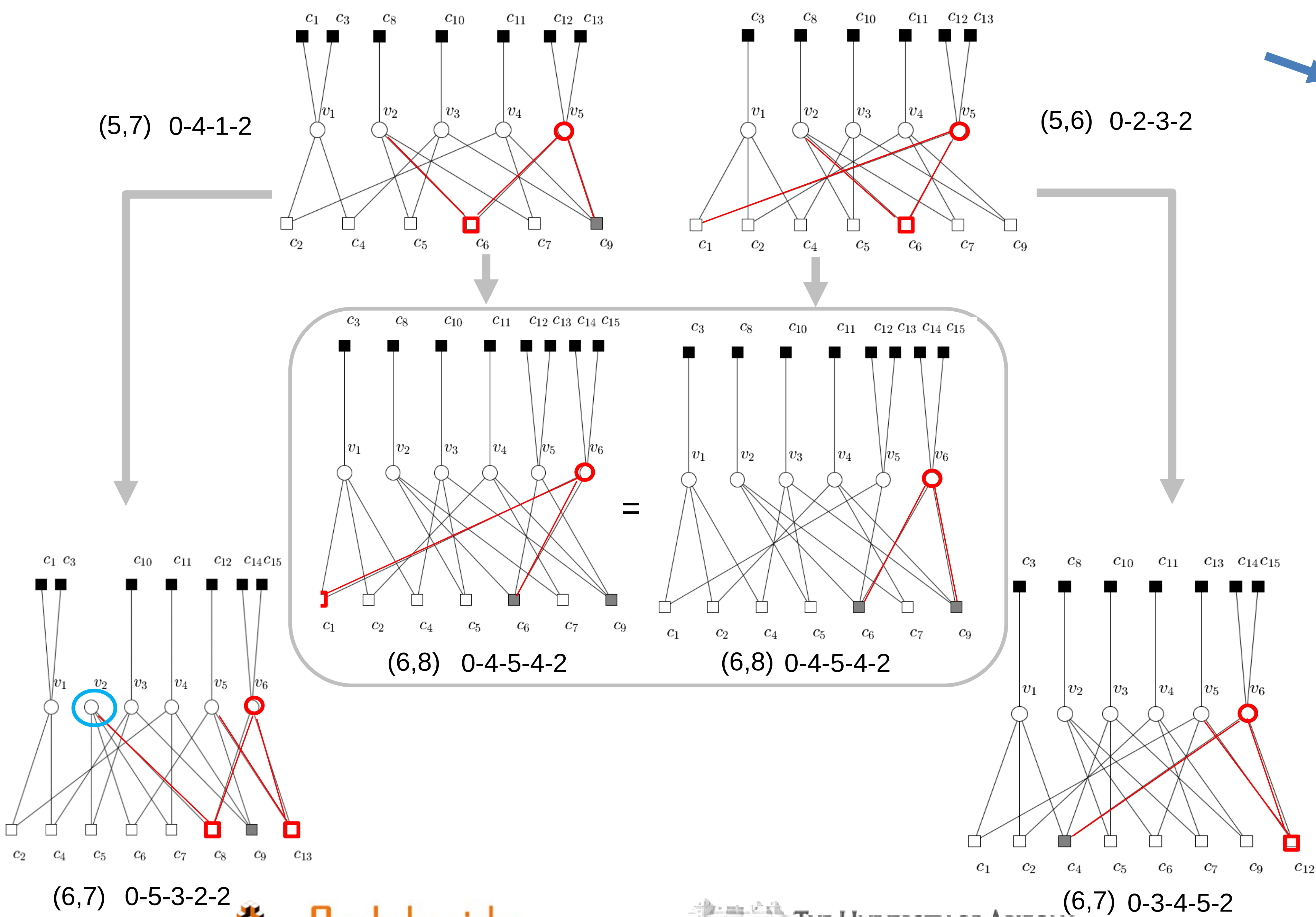

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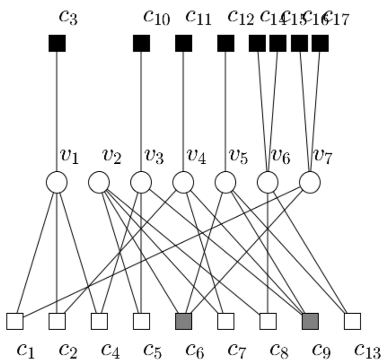
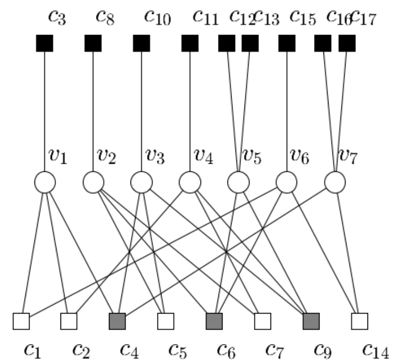
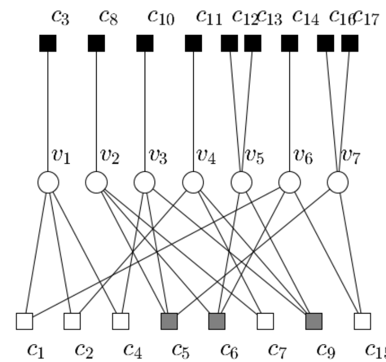
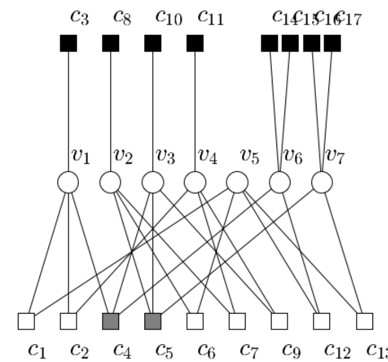
Trapping sets with 7 variable nodes

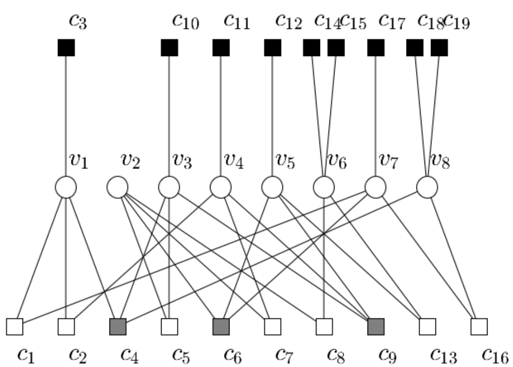


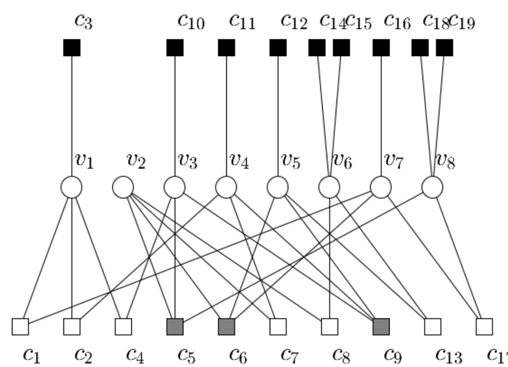


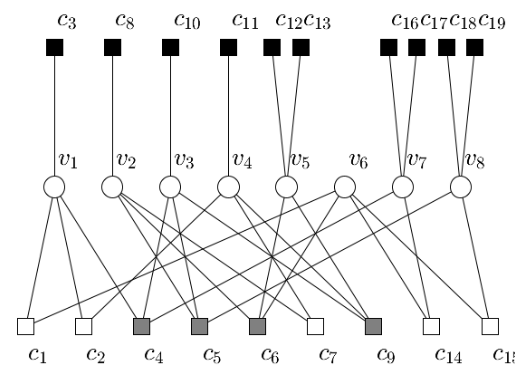




Trapping sets with 8 variable nodes







Trapping sets with 9 variable nodes