

OWHTC 2019

Munich, Germany

Feb. 27, 2019



National Research
Council of Italy



Institute of
Electronics
Computer and
Telecommunication
Engineering

Absorbing Sets of Generalized LDPC codes

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Introduction

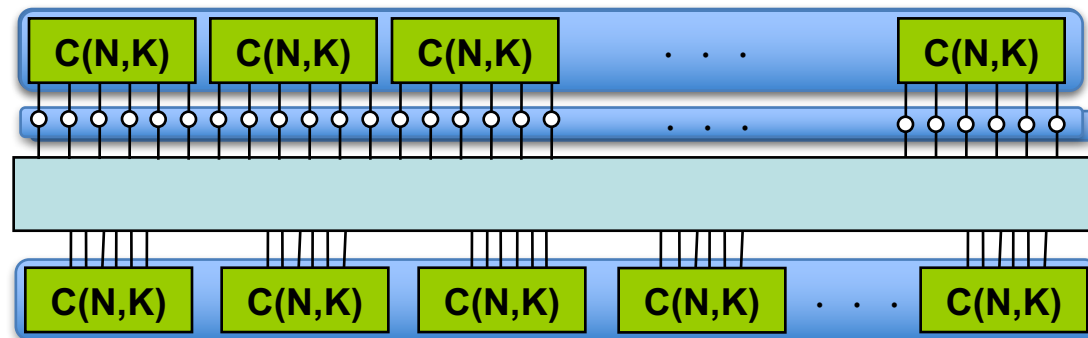
- High data rate ECCs need low error floors.
- Generalized LDPC codes rediscovered from [Tan81] after LDPC codes, as a compromise between LDPC and Turbo codes [LenZig99] [BouPotZem99].
- Floorless concatenated codes, good spectral shape behavior.
- Trapping sets for the BEC and the BSC [MilFos08].
- Absorbing Sets (AS) [Dol.et.al.10]: topological substructures of the Tanner graph, that describe the dominant decoding failures of soft LDPC decoders over AWGN channels.
- *Elementary AS* enable a State-Space linear model for local analysis of iterative LDPC decoding [SunTakFitz04], [SchleZha10-13], [ButSie11-14].

Introduction (2)

- Equilibria of the State-Space linear model with saturation [TomTCOM17]
- *AS threshold*: discriminates the existence/non-existence of misleading equilibria [TomTCOM17-FerITW17].
- This talk: AS for GLDPC codes decoding failures of Max-Log GLDPC decoders [FerIzs18].
- State-space linear model similar to the one used in [TomTCOM17-FerITW17].
- Focus on VN degree-2 GLDPC codes (rate maximized, complexity minimized).

System model and notation

- Single component code $C(N,K)$ of rate $R_c = K/N \rightarrow$ GLDPC code of rate $R = 2R_c - 1$
- Tanner graph uniquely described using a permutation matrix π



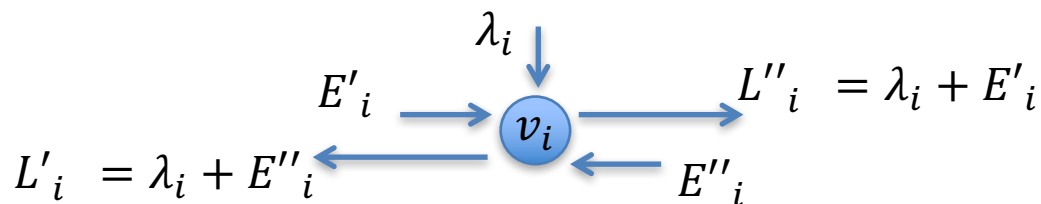
- Iterative decoding = iterative activation of nodes in the Tanner graph.
- At each activation, each node computes extrinsic messages for the other nodes, based on the received messages.

System model and notation (2)

- MAP SISO decoder of $C(N,K)$, activated with input L_k , ($k = 1 \dots N$), computes extrinsic messages E_j , ($j = 1 \dots N$)

$$L_k \rightarrow \boxed{C(N,K)} \rightarrow E_j = \log \left(\frac{\sum_{\{c \in C: c_j = +1\}} \prod_{k=1}^N \exp\left(\frac{c_k L_k}{2}\right)}{\sum_{\{c \in C: c_j = -1\}} \prod_{k=1}^N \exp\left(\frac{c_k L_k}{2}\right)} \right) - L_j$$

- VNs v_i ($i = 1 \dots N_v$), activated with inputs λ_i from the channel and E'_i from the 1st set of CNs, send messages L''_i to the 2nd set of CNs and v.v.



- Final decisions on v_i made based on the a posteriori LLRs

$$O_i = \lambda_i + E'_i + E''_i$$

Absorbing Sets of GLDPC codes

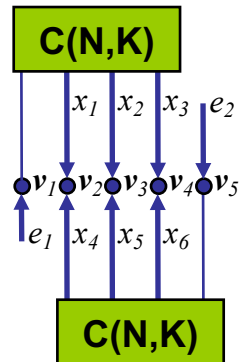
- Unlike for binary LDPC codes, the sign of messages generated by the CNs in GLDPC codes does not depend only on them being satisfied or not.
- Practical decoders: Max-Log CNs, messages quantized and saturated (e.g. to ± 1 by function $\text{sat}(\cdot)$)

$$E_j = \text{sat} \left(\max_{\{c \in C: c_j = +1\}} \sum_{k \neq j} \frac{c_k L_k}{2} - \max_{\{c \in C: c_j = -1\}} \sum_{k \neq j} \frac{c_k L_k}{2} \right)$$

- We can write a linear relation (apart from saturation) between input and output messages of the CNs. Assuming $c = +1$ transmitted and a competing codeword selected by the max operator that differ in the smallest possible number of bits (minimum Hamming distance d_H)

$$E_j = \text{sat}(L_1 + L_2 \dots + L_{d_H} - L_j), \quad j = 1 \dots d_H$$

State-space model example



For instance, the message x_4 at the k -th iteration, reads

$$x_4^{(k)} = \text{sat} \left(\lambda_3 + x_2^{(k-1)} + \lambda_4 + x_3^{(k-1)} + \lambda_5 + e_2^{(k-1)} \right)$$

i.e. in matrix form

$$\mathbf{x}^{(k)} = \text{sat} \left(\mathbf{A}\mathbf{x}^{(k-1)} + \mathbf{R}\mathbf{e} + \mathbf{C}\boldsymbol{\lambda} \right)$$

with

$$\mathbf{A} = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad \mathbf{R} = \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \\ 0 & 1 \end{bmatrix}, \quad \mathbf{C} = \begin{bmatrix} 1 & 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 1 \end{bmatrix}$$

This system, that locally describes iterative decoding, allows misleading equilibria, i.e. pairs $(\mathbf{x}, \boldsymbol{\lambda})$, (for instance, $(-\mathbf{1}, -\mathbf{1})$) such that $\mathbf{x}$ is stable along the iterations and produces wrong decisions: an *absorbing set*.

State-space model analysis

- We look for *sufficient conditions* for the system

$$\mathbf{x}^{(k)} = \text{sat} \left(\mathbf{A}\mathbf{x}^{(k-1)} + \mathbf{R}\mathbf{e} + \mathbf{C}\boldsymbol{\lambda} \right)$$

to converge to an *equilibrium* corresponding to correct decisions.

- We assume that in the rest of the graph messages converge towards correct decisions.
- Start the analysis of the iterations when the messages received by the VNs $\in D$, from external CNs, are already saturated to their maximal value $\mathbf{e} = +\mathbf{1}$.
- By this choice we decouple internal from external messages evolution.
- We should use an initial vector $\mathbf{x}^{(0)}$ which is the result of the message evolution up to that iteration, which is unknown.
- If no $\mathbf{x}^{(0)}$ results in a convergence failure, the AS cannot trap the decoder independently of this message evolution.

AS State-Space Model (2)

- With saturated external messages

$$\mathbf{Re} = (d_H - 1)\mathbf{1} - \mathbf{A}\mathbf{1}$$

letting $\boldsymbol{\mu} \triangleq \mathbf{C}\boldsymbol{\lambda}$ the local evolution of messages along the iterations is

$$\mathbf{x}^{(k+1)} = \text{sat}\left(\mathbf{A}(\mathbf{x}^{(k)} - \mathbf{1}) + (d_H - 1)\mathbf{1} + \boldsymbol{\mu} \right) \triangleq f(\mathbf{x}^{(k)}, \boldsymbol{\mu})$$

- Linear State-space model formally identical to the system derived in [TomTCOM17] for binary LDPC AS.
- Depending on the starting point $(\mathbf{x}^{(0)}, \boldsymbol{\mu})$ and on the structural elements $(\mathbf{A}, d_H)$ the vector $\mathbf{x}^{(k)}$ could describe a periodical or chaotic sequence or reach an equilibrium, i.e. a value $\mathbf{x} = f(\mathbf{x}, \boldsymbol{\mu})$.
- Of particular interest is the equilibrium $(\mathbf{x} = +\mathbf{1}, \boldsymbol{\mu})$. If $f(+\mathbf{1}, \boldsymbol{\mu}) = +\mathbf{1}$, the decoder makes correct decisions about the set of VNs D : the posterior LLRs of the VNs are $\boldsymbol{\lambda} + \mathbf{2} \geq \mathbf{1}$.

Absorbing Set threshold

- Sufficient condition on the channel messages μ , for the existence of the unique equilibrium $(+\mathbf{1}, \mu)$, based on the *aggregate threshold* τ_μ

$$\tau_\mu = \max_{(x, \mu)} \min(\mu) \\ \text{s.t. } -\mathbf{1} \leq x \leq +\mathbf{1}, \quad \exists j: x_j < 1, \quad x = f(x, \mu)$$

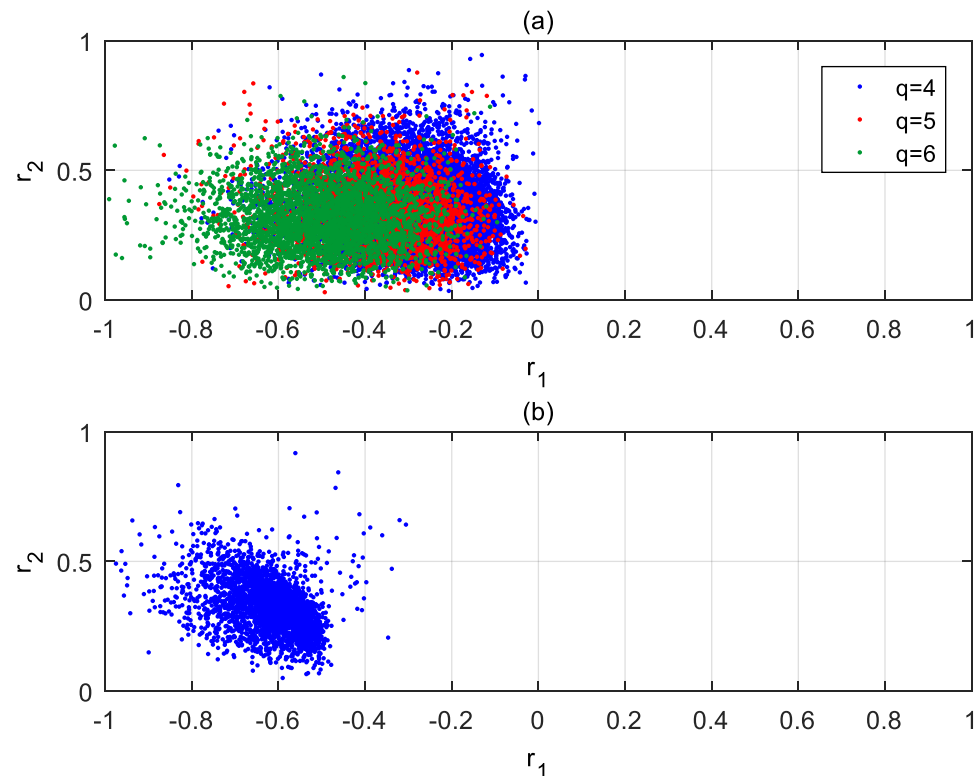
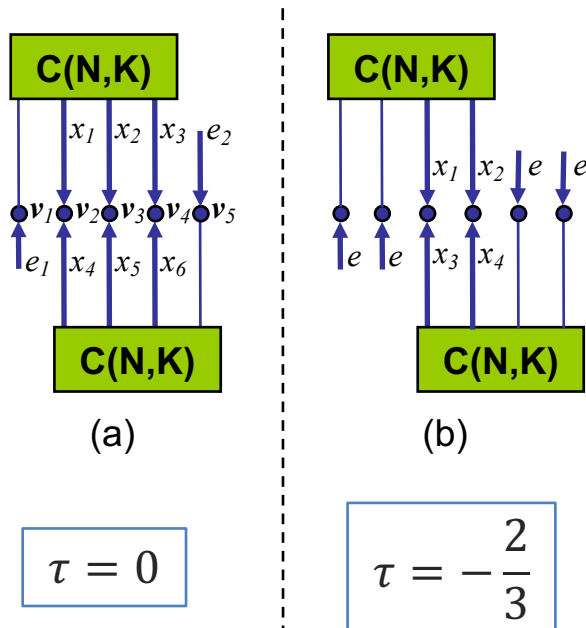
- If $\mu_i > \tau_\mu, \forall i$, the only equilibrium allowed is $(+\mathbf{1}, \mu)$ and no periodic or aperiodic trajectory can be described by the vector $x^{(k)}$, i.e. correct decoding of D is guaranteed [TomTCOM17].
- The corresponding **threshold** for the single LLR can be taken as

$$\tau \triangleq \frac{\tau_\mu}{(d_H-1)} \quad \longrightarrow \quad \text{If } \lambda_i > \tau, \forall v_i \in D \Rightarrow \mu_i > \tau_\mu, \forall i$$

- When $\tau < 0$, the condition $\lambda_i > \tau, \forall v_i$ can be guaranteed by setting different saturation levels λ_{max} and E_{max} for the channel and extrinsic messages: if $\lambda_{max}/E_{max} < |\tau|$ the AS is *deactivated*.

Absorbing Set threshold: Examples

- Importance Sampling simulation with $q_I = 4$ bits for the channel messages ($\lambda_{max} = 7$) and $q = 4, 5$ or 6 bits for the extrinsic messages ($E_{max} = 7, 15, 31$).



Absorbing Set definition

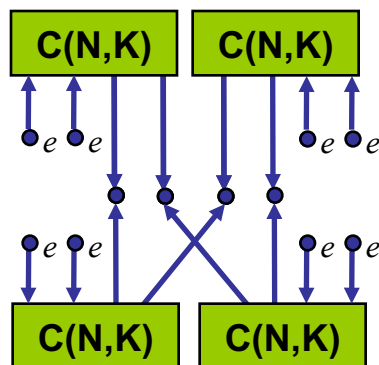
The Tanner graph nodes that must be considered to define the linear model that locally describes iterative decoding, are the union D of all VNs corresponding to (low) weight codewords of the component codes, and the set $\mathcal{E}$ of all their neighboring CNs with at least two connections in D_2 (subset of D with both neighboring CNs in $\mathcal{E}$).

Definition III.1. *In the Tanner graph of a binary GLDPC code, with VN-degree $d_v = 2$, an Absorbing Set is a subgraph with a subset $\mathcal{E}$ of the CNs, and a subset $\mathcal{D} = \mathcal{D}_1 \cup \mathcal{D}_2$ of the VNs, where $\mathcal{D}_i$ are the VNs in $\mathcal{D}$ with i neighboring CNs in $\mathcal{E}$, if*

- 1) $\mathcal{D}$ is the union of low Hamming weight codewords for each CN $c \in \mathcal{E}$.
- 2) $\mathcal{E}$ is the subset of the neighboring CNs of $\mathcal{D}$, that are connected at least twice to $\mathcal{D}_2$.

AS in girth-constrained GLDPC codes

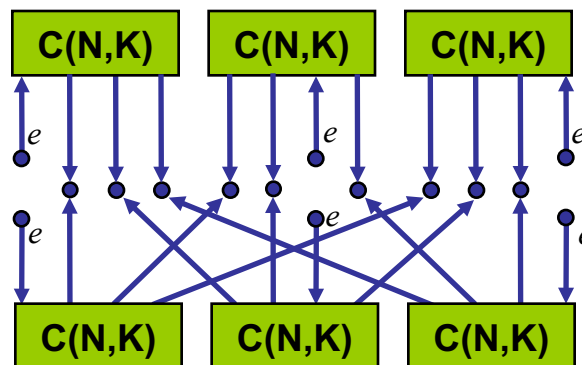
- If π is designed with girth > 4 , AS of the types (a) and (b) cannot exist in the Tanner graph of the GLDPC code.
- Due to the structure of the GLDPC graph, the girth is at least 8 and the smallest AS that can be found according to our definition are



(c)

$$(a, b) = (12, 4)$$

$$\tau = -\frac{2}{3}$$



(d)

$$(a, b) = (15, 6)$$

$$\tau = 0$$

AS search in GLDPC codes

- The AS (5,2) and (15,6) of type (a) and (d) cannot be deactivated.
- GLDPC code with extended Hamming (64,57) component codes: from a single AS (15,6) we measured by IS a WER contribution of $3 \cdot 10^{-19}$, $6 \cdot 10^{-22}$ and $5 \cdot 10^{-26}$, with $q = 4, 5$ or 6 bits for the extrinsic messages respectively ($E_s/N_0 = 2.5$ dB).
- The total WER due to these AS depends on their multiplicity, that needs to be estimated.
- AS search quite difficult: joint inspection of the biadjacency matrix of the GLDPC code and of the component codebook C .
- The problem is simplified by eH codes, for which any triplet of ones can be turned into a codeword with a 4th one in a specific unique position.
- Exploiting this property, exhaustive search of these AS in the Tanner graph of an unconstrained code C_1 and of a girth-constrained code C_2 .

AS search in GLDPC codes (2)

- $\mathcal{C}_1$: $N_v = 16384$ bits, eH(128,120) component codes, $R = 7/8$, random permutation matrix π .
- The total amount of AS (5,2) found is 2705 (the expectation under random permutation returns 2667), responsible for an error floor at WER $8 \cdot 10^{-5}$, $7 \cdot 10^{-6}$ and $5 \cdot 10^{-8}$ for Max-Log decoders with $q = 4, 5$ and 6 , respectively, at $E_s/N_0 = 3.6$ dB, after 20 iterations.
- $\mathcal{C}_2$: $N_v = 32768$ bits, eH(64,57) component codes, $R = 25/32$, girth-8 biadjacency matrix [LivTCOM08].
- The total amount of AS (15,6) found is approximately $2.3 \cdot 10^6$ (also checked against a probabilistic argument that returned an expected $3 \cdot 10^6$), responsible for an error floor at WER $7 \cdot 10^{-13}$, 10^{-15} and $2 \cdot 10^{-19}$ for Max-Log decoders with $q = 4, 5$ and 6 , respectively (at $E_s/N_0 = 2.5$ dB, after 20 iterations).

Conclusions

- We propose a definition for combinatorial substructures of the Tanner graph of binary VN-degree 2, GLDPC codes, that can trap practical Max-Log decoders over AWGN channels, i.e., *Absorbing Sets* of GLDPC codes.
- For these structures we can derive a quasi-linear model that reveals a threshold behavior similar to ASs in binary LDPC codes.
- Design constraints on the adjacency matrix of the code can avoid the smallest structures, but larger ASs able to trap the iterative decoders, of size smaller than the minimum Hamming distance of the code, do exist.
- In case of extended Hamming component codes we enumerated by exhaustive search the most critical ASs and we checked our results against combinatorial arguments.
- More details in [FerIZS2018], pdf paper online available at <https://www.research-collection.ethz.ch/handle/20.500.11850/245076>.

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