Design of Coarsely-Quantized Message Passing Decoders

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Binary Message Passing (BMP) for LDPC Codes

- Gallager, “Low density parity check codes,” IRE Trans. IT, 1962
- Kou, Lin, Fossorier, LDPC codes based on finite geometries, IEEE Trans. IT, 2001
- Zhang, Fossorier, Modified bit-flipping decoding, IEEE Comm. Lett., 2004
- Miladinovic, Fossorier, Improved bit-flipping decoding, IEEE Trans. IT, 2005
- Jiang, Zhao, Shi, Chen, Improved bit flipping decoding, IEEE Comm. Lett., 2005
- Ardakani, Kschischang, Properties of binary message-passing, IEEE Trans. IT, 2005
- Sankaranarayanan et al., Failures of the Gallager B decoder, ISIT 2006
- Reliability-based majority-logic decoding for LDPC codes, IEEE Trans. Comm., 2009
- Planjery, Declercq, Danjean, Vasic, Finite alphabet iterative decoders, 2013-
- Many other papers

Here Review and Expand on:

- Lechner, Pedersen, Kramer, Analysis and design of binary message passing decoders, IEEE Trans. Comm., 2012 (and ISIT 2007)

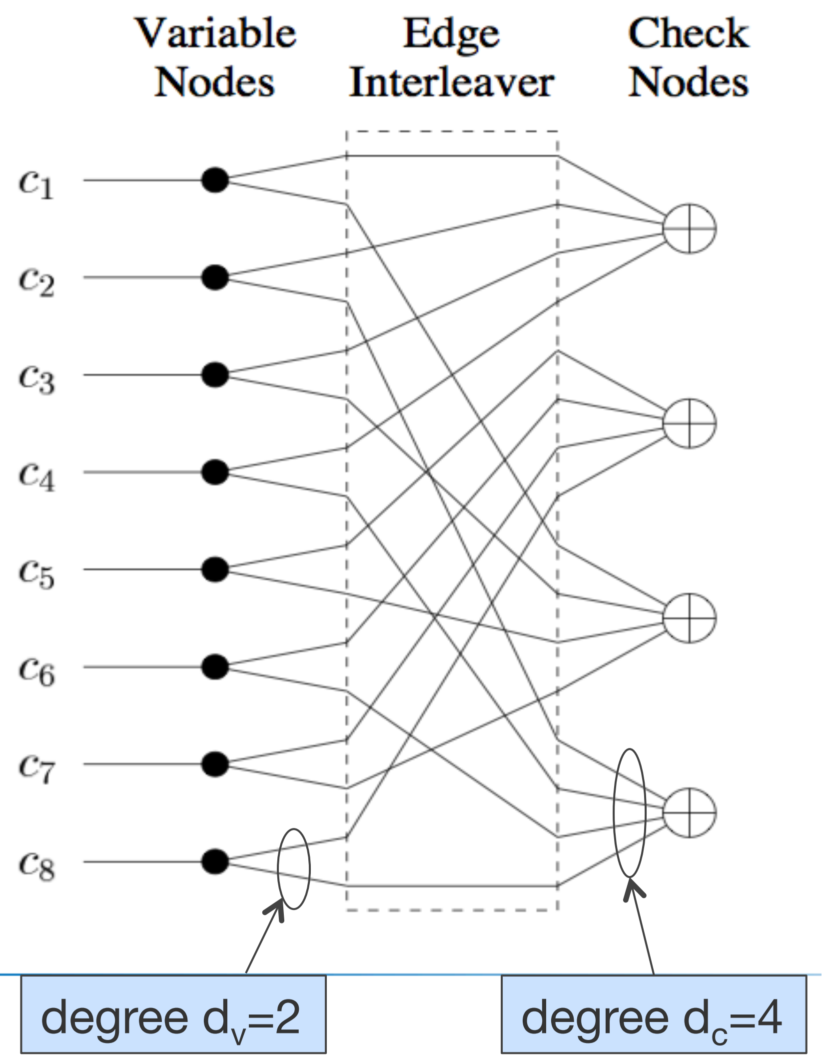
I. Low-Density Parity-Check (LDPC) Codes

- A binary linear block code is the set of binary (row) vectors, or codewords, $\underline{c}$, satisfying, e.g.,

$$\underline{c} \underbrace{\begin{bmatrix} 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}}_{\text{matrix } H^T} = \underline{0}$$

where H is a (n-k) x n parity-check-matrix.
Rate is R=1-rank(H)/n (example: R=5/8).

Tanner Graph Representation of Parity-Check Constraints



- Code is low-density if each row and column of H^T has “few” 1’s
- Irregular LDPC code: variable number of 1’s in every column/row
- Decoding: use message passing on the graph
- Messages may be cond. probabilities

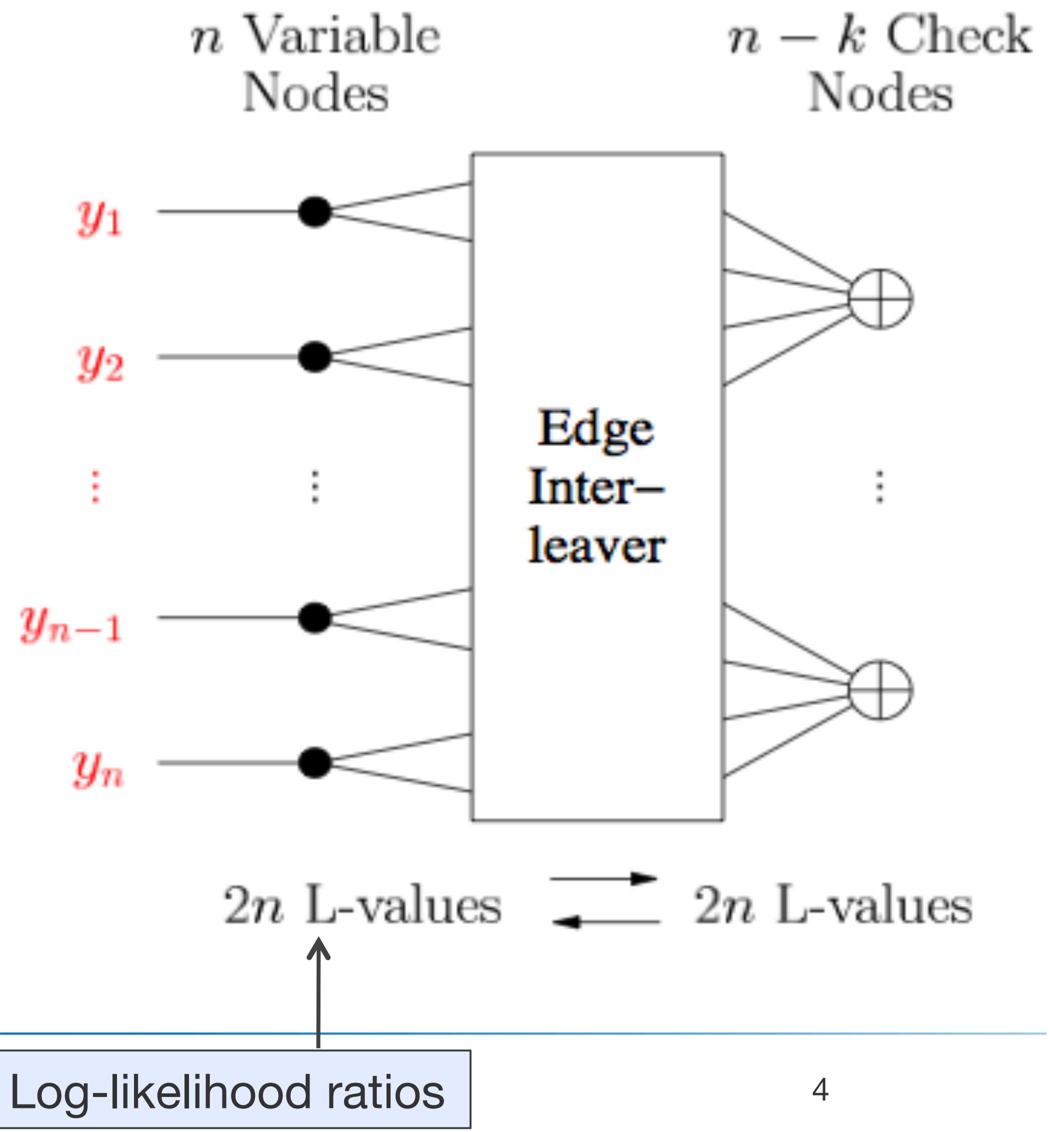
$$\Pr\left(c_1 = 0 \middle| \underline{y}\right)$$

or log-likelihood ratios (L-values)

$$L_1 = \log \frac{\Pr\left(c_1 = 0 \middle| \underline{y}\right)}{\Pr\left(c_1 = 1 \middle| \underline{y}\right)}$$

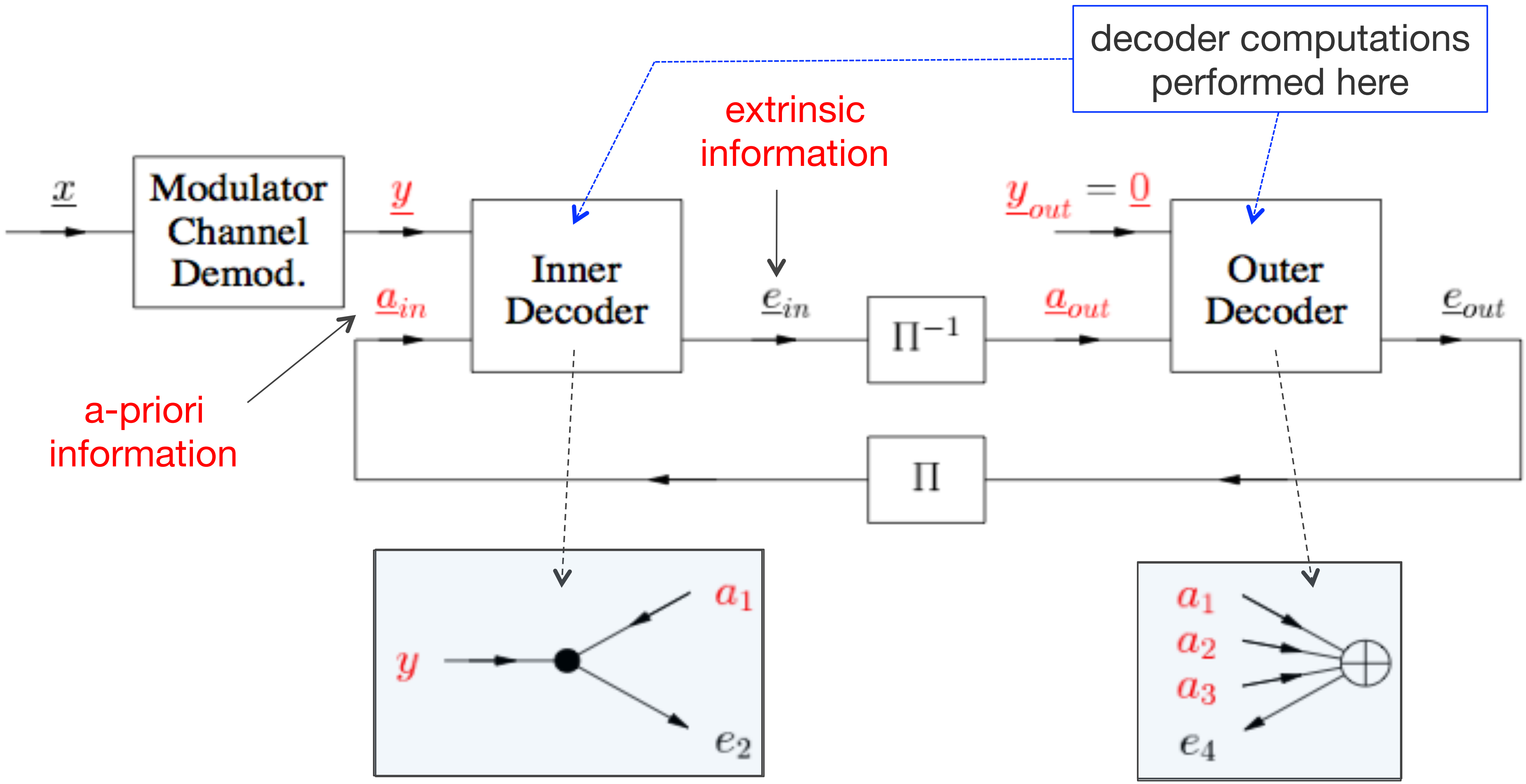
or, in practice, quantized L-values

Message Passing



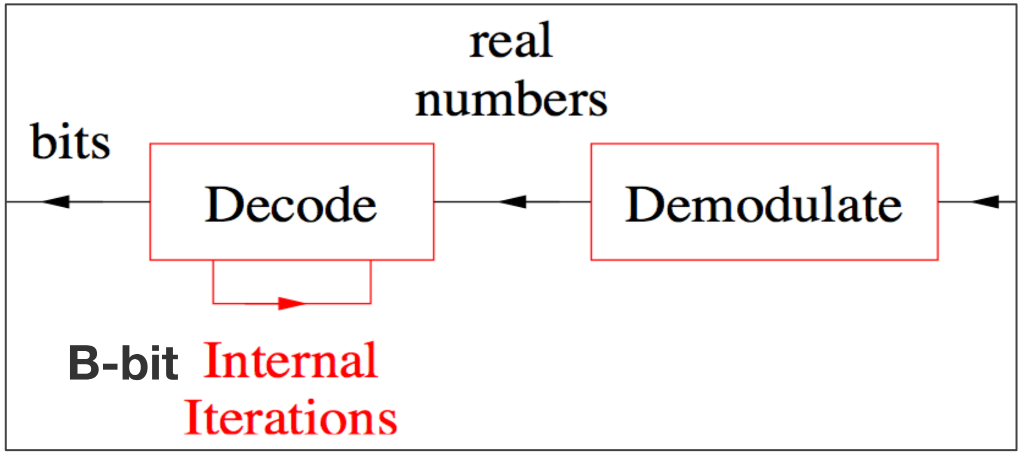
II. Iterative Decoding

LDPC code decoder iterations (turbo processing):

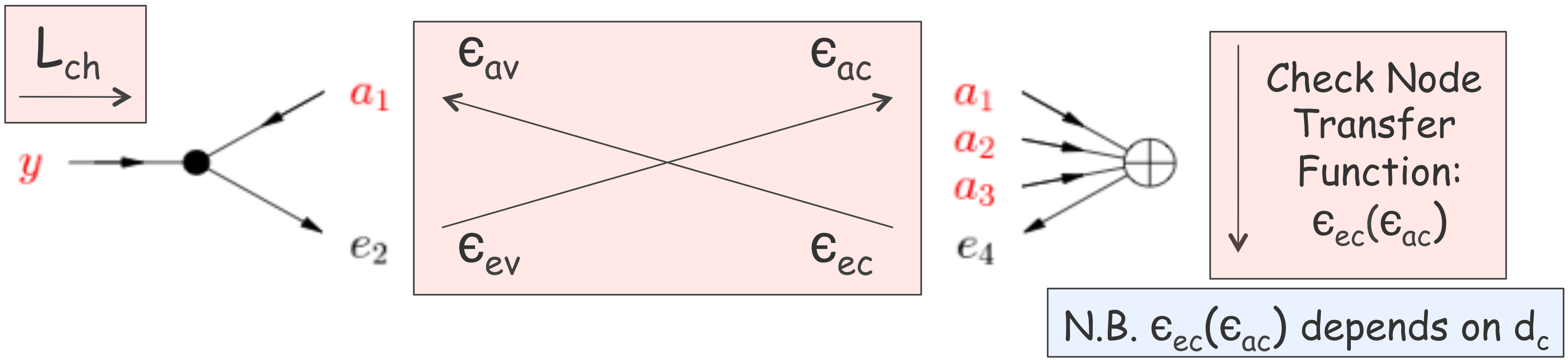


III. Demodulation and Decoding

- L-values are real but must be quantized, see figure below
    1) Demodulator: can put out soft decisions (>$\log_2(M)$ bits/symbol) or hard decisions (=$\log_2(M)$ bits/symbol)
    2) Decoder iterations: B-bit message passing:
       binary message passing (BMP, B=1)
       ternary message passing (TMP, B≈2)
- Motivation: high-speed devices (>100 Gb/s) need simplifications



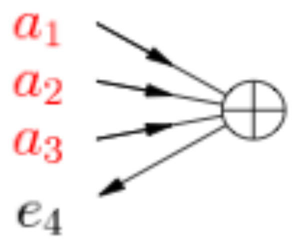
- BMP/TMP: natural approaches are as follows:
  1) Every edge bit represents a hard decision on an extrinsic L-value
  2) Variable nodes convert apriori bits to L-values, add L-values, and make binary (hard) or ternary decisions on output L-values
  3) Check nodes perform (extrinsic) XORs for binary message passing, and (extrinsic) XORs and erasures for ternary message passing
- Analysis: use distribution evolution (DE) to track extrinsic probabilities. BMP: track error probabilities; TMP: also track erasures



BMP Distribution Evolution

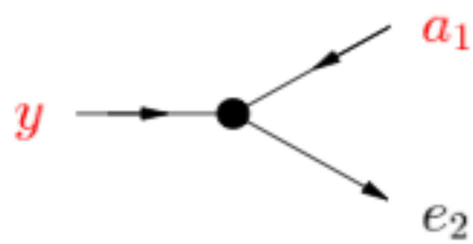
- Check node (degree d_c) and binary messages:

$$\epsilon_{ec} = f_c(\epsilon_{ac}; d_c) = \frac{1-(1-2\epsilon_{ac})^{d_c-1}}{2}$$



- Variable node (degree d_v): suppose $x_j=\pm1$ (BPSK)

$$\varepsilon_{ev} = \sum_{j=1}^{d_v} \frac{1}{d_v} \Pr\left[\operatorname{sgn}\left(L_{ev,j}\right) \neq x_j\right]$$



$$L_{ev,j} = L_{ch} + \sum_{i=1;i\neq j}^{d_v} L_{av,i} \qquad j = 1, 2, \ldots, d_v$$

$$L_{av,i} = a_i \log\frac{1-\varepsilon_{av}}{\varepsilon_{av}}, \ a_i = \pm 1, \text{ but what is } \varepsilon_{av}\ ?$$

L_{ch} depends on channel quantization

Two design issues

Issue 1: Variable Node Processing

- Processing depends on ϵ_{av} which
  - Varies from iteration to iteration
  - Is unknown, unless the codes have infinite length in which case ϵ_{av} can be computed from EXIT chart (see below)
- Two other approaches:
  - Optimize "choice" of ϵ_{av} offline by numerical simulation
  - Estimate ϵ_{av} online based on the number of unsatisfied checks
- 1^{st} approach is complex, but likely very good. This variant was used to design certain deployed LDPC codes
- 2^{nd} approach is used here

Issue 2: Channel Outputs

- Consider an AWGN channel, $x_j=\pm 1$, noise variance σ_n^2
- Let $D_{ch}=|L_{ch}|$ … called the reliability of the L-value
- For soft decisions:

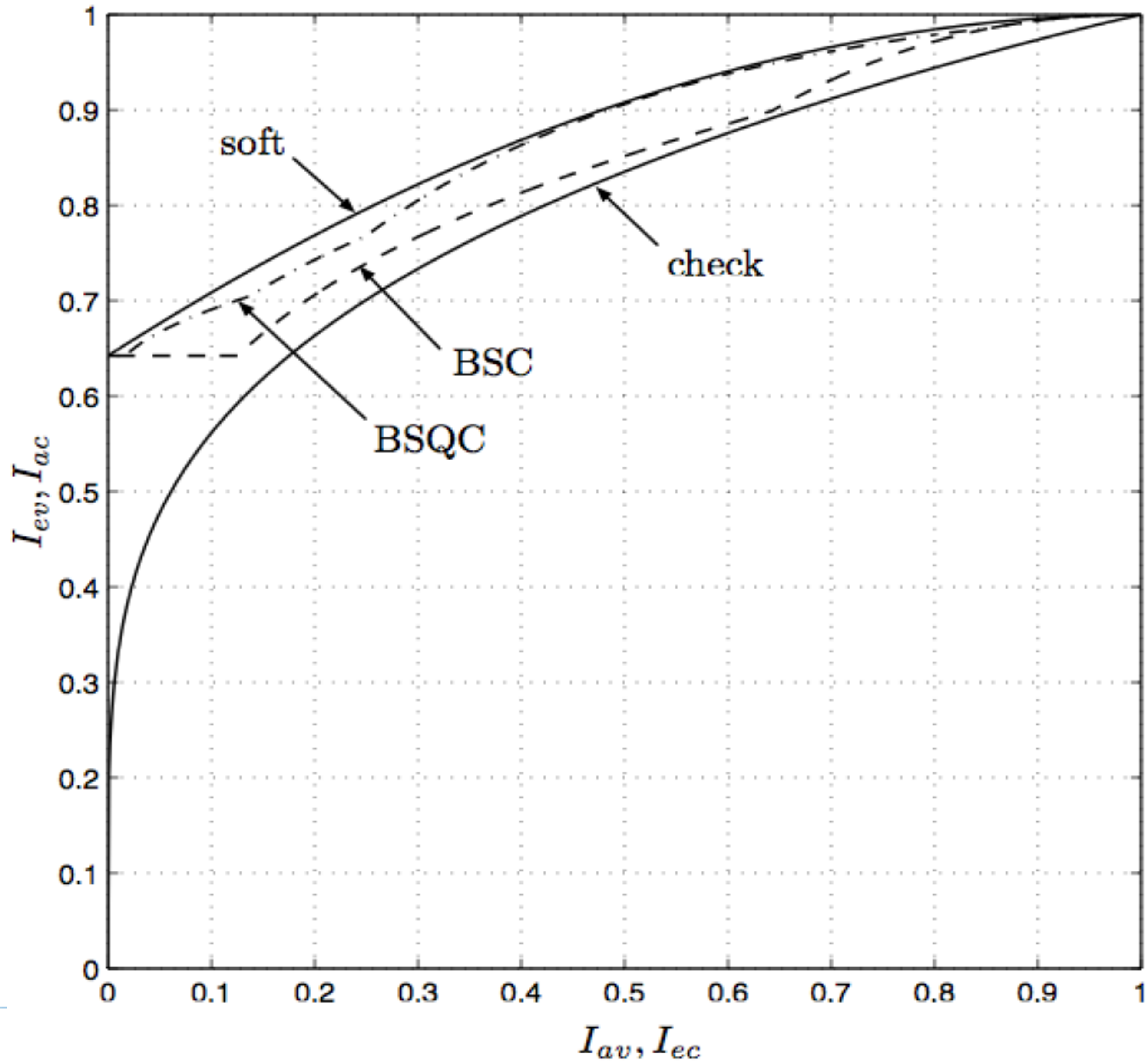
$$D_{ch} = \frac{2}{\sigma_n^2}$$

- For hard decisions get a binary symmetric channel (BSC) with crossover probability ϵ_{ch} $(0 \le \epsilon_{ch} \le 0.5)$

$$D_{ch} = \log \frac{1-\varepsilon_{ch}}{\varepsilon_{ch}}, \text{ where } \varepsilon_{ch} = Q\left(1/\sigma_n\right)$$

- For b-bit quantization: use mixture of b hard-decision channel reliabilities, e.g., 2-bit quantization with a binary symmetric quaternary output (BSQC) channnel

Transfer Functions for (4,6)-Regular Code



Channel: σ_n=0.67
x-axis to y-axis:
variable nodes
y-axis to x-axis:
check nodes

$I_{ac}=1-h(\epsilon_{ac})$ where h(x) is the binary entropy function:
$h(x) = -x\log_2 x-(1-x)\log_2(1-x)$
Example: h(0.11)=0.5
Similar for I_{ec}, I_{av}, I_{ev}

Comments:

- BSC quantization same as Gallager B algorithm
- BSQC quantization thresholds at 0 and ±1.9

IV. Optimization: Irregular LDPC Codes

- Each node's ϵ_{ev} depends on d_v: write as $\epsilon_{ev}(\epsilon_{av},d_v)$. Now use different degrees to shape avg. variable node curve:

$$\varepsilon_{ev}\left(\varepsilon_{av}\right)=\sum_{i}\lambda_i\,\varepsilon_{ev}\left(\varepsilon_{av},i\right)$$

  with λ_i=fraction of edges connected to var. nodes of degree i
- Can similarly shape the check node function $\epsilon_{ec}(\epsilon_{ac})$
- Degree distribution $\{\lambda_i\}$ design: use EXIT chart
  - $\epsilon_{ev}(\epsilon_{av})$ curve should lie above $\epsilon_{ec}(\epsilon_{ac})$ curve for convergence (and $n=\infty$)
  - L-value messages: Matching EXIT curves maximizes rate.
- BMP: new issues vs. L-value messages
  - Stability (decoder convergence when ϵ_{av} or ϵ_{ac} are small)
  - Cycles related to "absorbing sets" cause decoder to get stuck
- Approach: build optimization & remedies into a linear program

Rate, Stability, Cycles

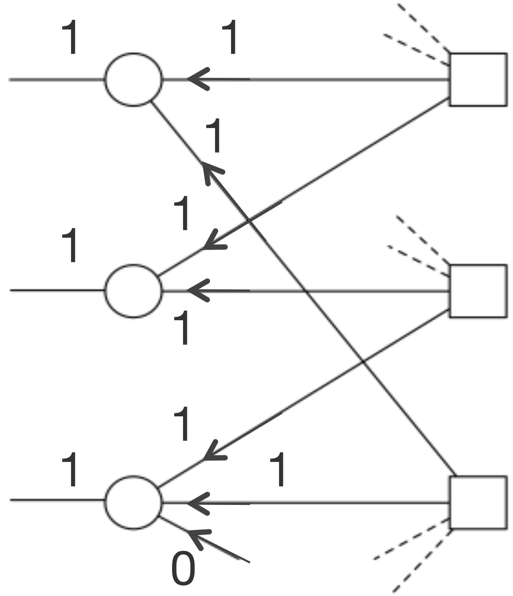
- Design Rate:

$$R = 1 - \frac{1/d_c}{\sum_i \lambda_i / i}$$

- Stability: satisfied for binary message passing and hard or soft channel messages if and only if (try $\lambda_2 = 1$)

$$\left(\lambda_2 + 2\varepsilon_{ch}\lambda_3\right)\left(d_c - 1\right) < 1$$

- Cycles:
  - Structure on right causes decoding failure if all channel messages in error, and if all other incoming messages correct
  - Obvious idea: avoid cycles of degree 2 or 3 variable nodes



Cycles and Linear Program

- Result: a Tanner graph with no cycles having degree 2 and 3 variable nodes exists if and only if (try $\lambda_3 = 1$)

$$3\lambda_2 + 4\lambda_3 \le \frac{6}{d_c}\left(1 - \frac{1}{(1-R)N}\right) < \frac{6}{d_c}$$

- Linear Program: $\lambda = \{\lambda_i\}$ is variable node degree distribution

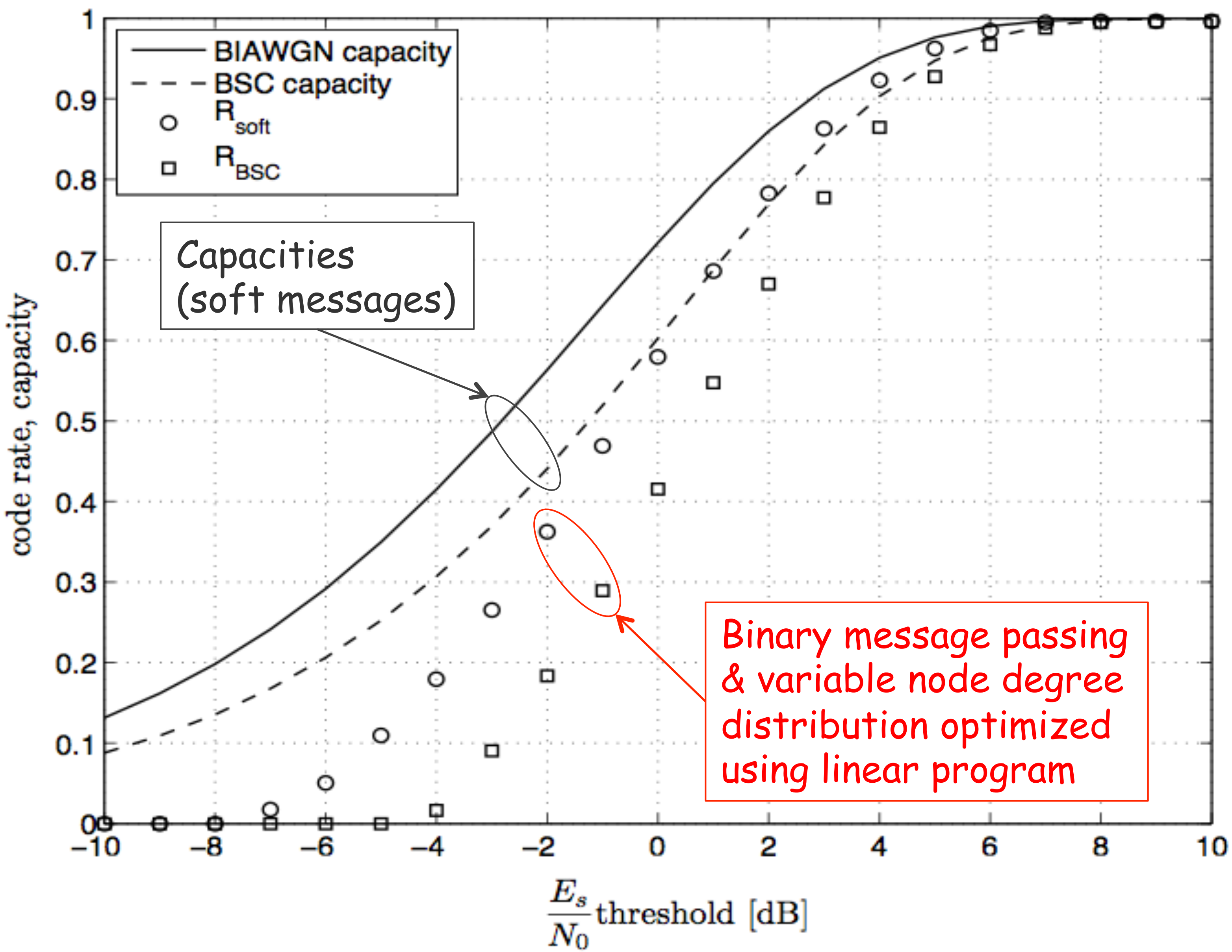
$$\lambda^* = \arg\max_{\lambda} R = \arg\max_{\lambda} \left(1 - \frac{1/d_c}{\sum_i \lambda_i / i}\right) = \arg\max_{\lambda} \sum_i \lambda_i / i$$

subject to [variable node EXIT curve above check node EXIT curve]

$$\sum_i \lambda_i = 1, \quad 0 \le \lambda_i \le 1$$

$$(\lambda_2 + 2\varepsilon_{ch}\lambda_3)(d_c - 1) < 1, \quad 3\lambda_2 + 4\lambda_3 < \frac{6}{d_c}$$

BMP Thresholds

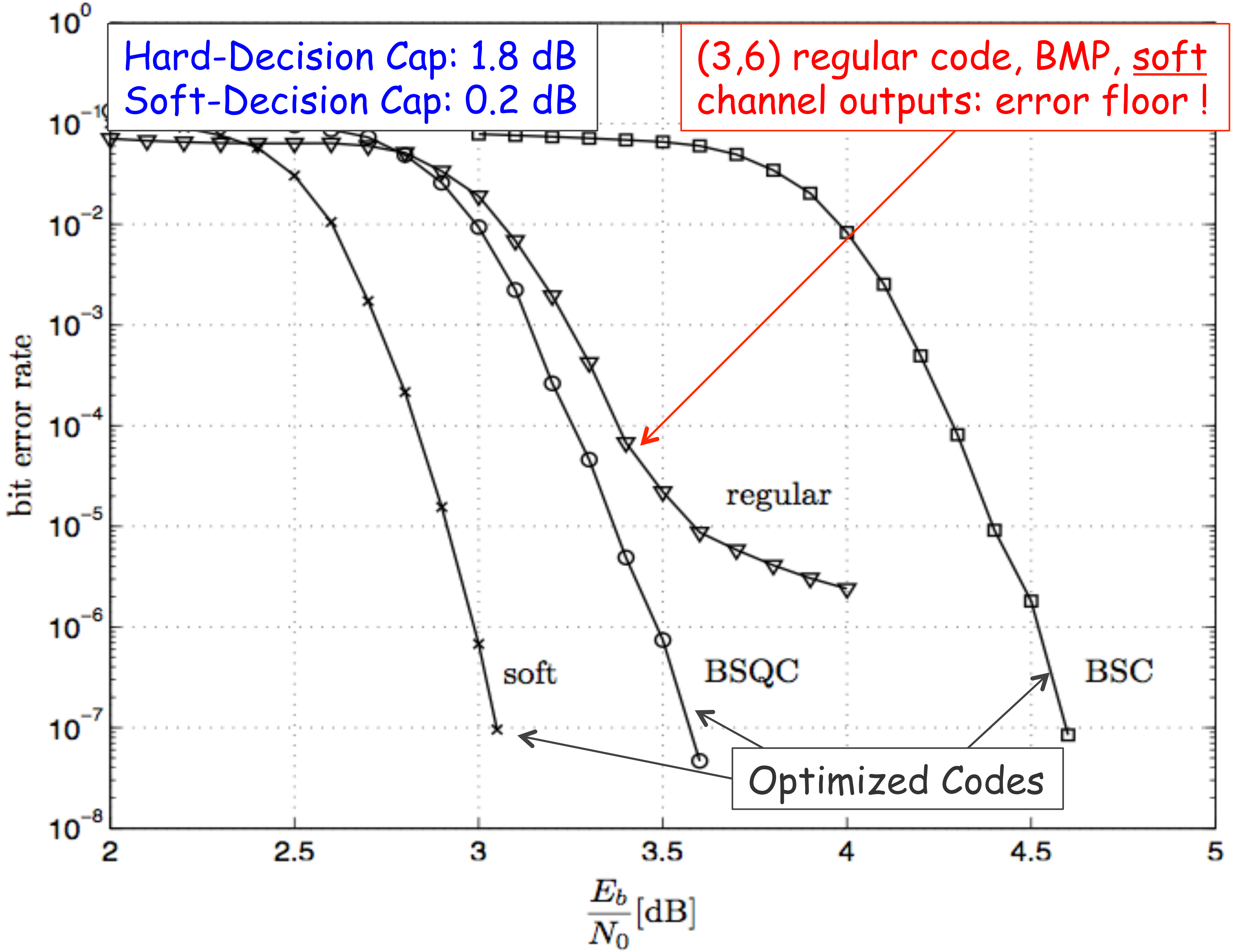


Comments:

- x-axis is E_s/N_0
- hard decision (BSC) capacity ≈2dB below AWGN capacity at low rate
- gap to capacity decreases as rate increases, for hard decisions and BMP

- Conclusion: high rate is good for BMP

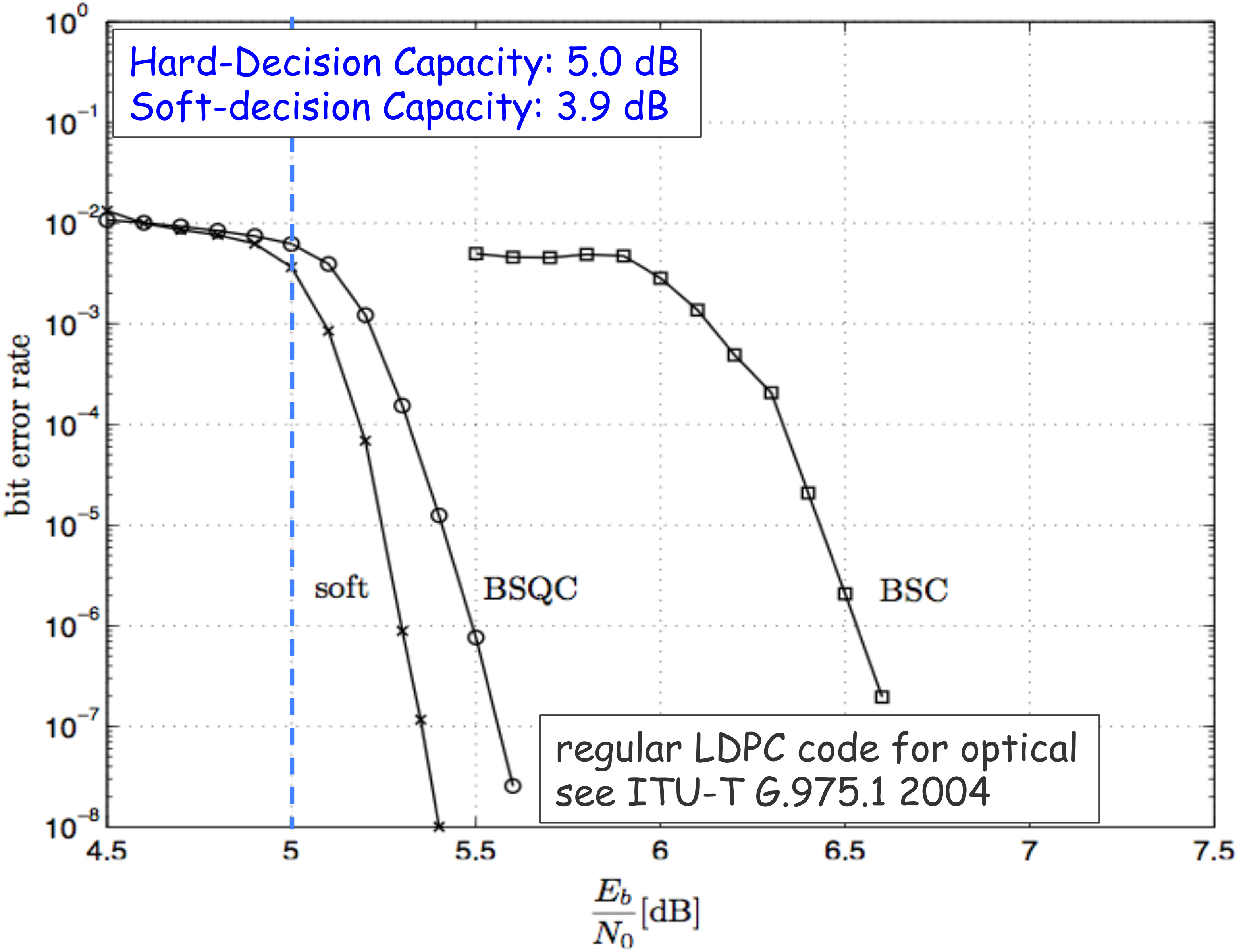
Performance: Rate 1/2, BMP



Comments:

- x-axis is E_b/N_0
- PEG interleavers automatically avoid undesirable cycles
- n = 10,000
- 2-bit quant. gains ≈1dB over Gallager B and loses ≈0.5dB as compared to soft outputs

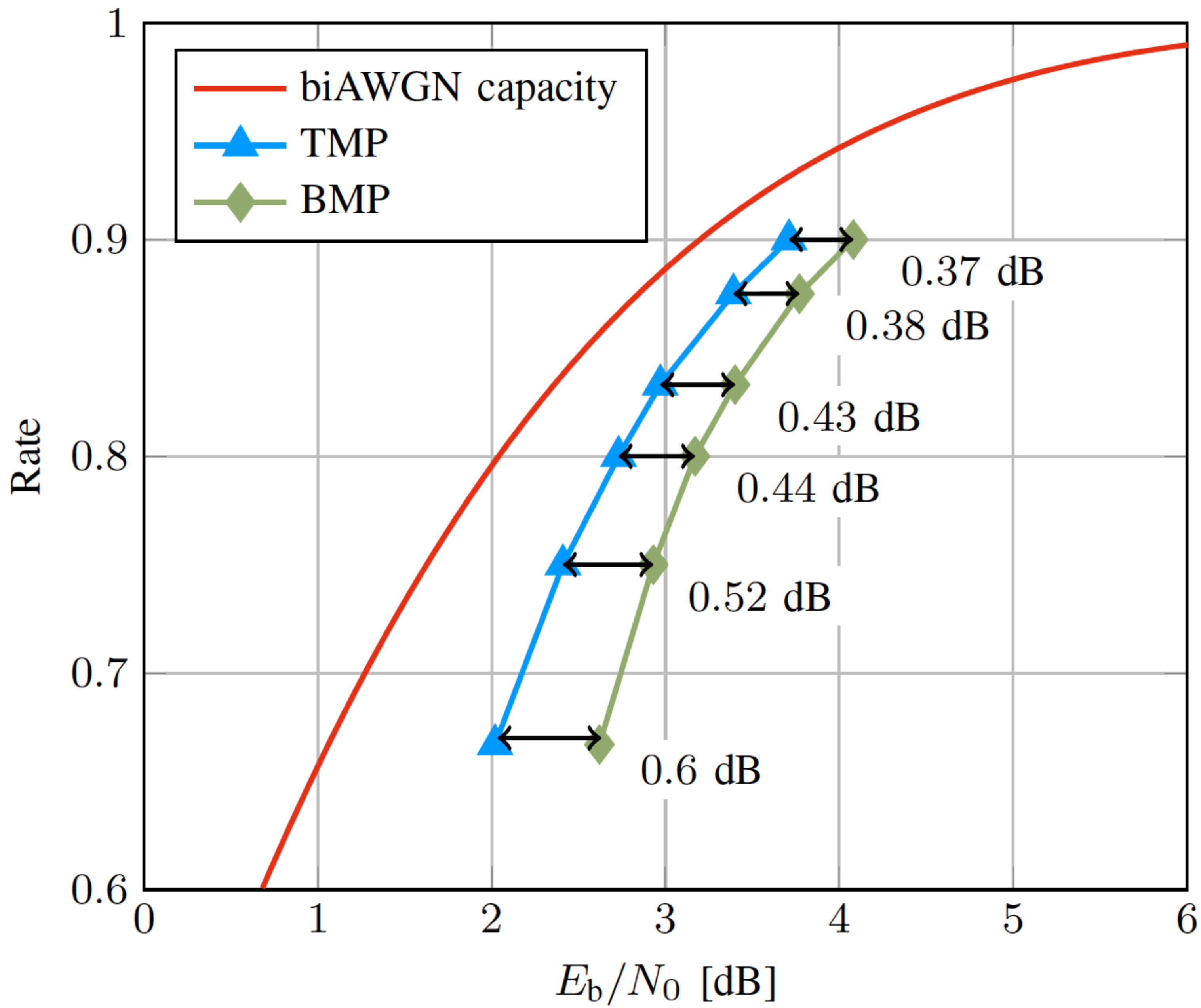
Performance: Rate 15/16, BMP



Comments:

- x-axis is E_b/N_0
- interleaver taken from standard
- 2-bit quant. gains ≈1dB over Gallager B and loses ≈0.2dB vs. soft outputs
- BMP is ≈1.5dB from L-value message capacity
- longer & irregular codes get closer

Performance: TMP



Comments:

- Figure taken from Emna Ben Yacoub's Master Thesis, Oct. 2018
- Curves show decoding thresholds with BMP and TMP for optimized protograph LDPC code ensembles

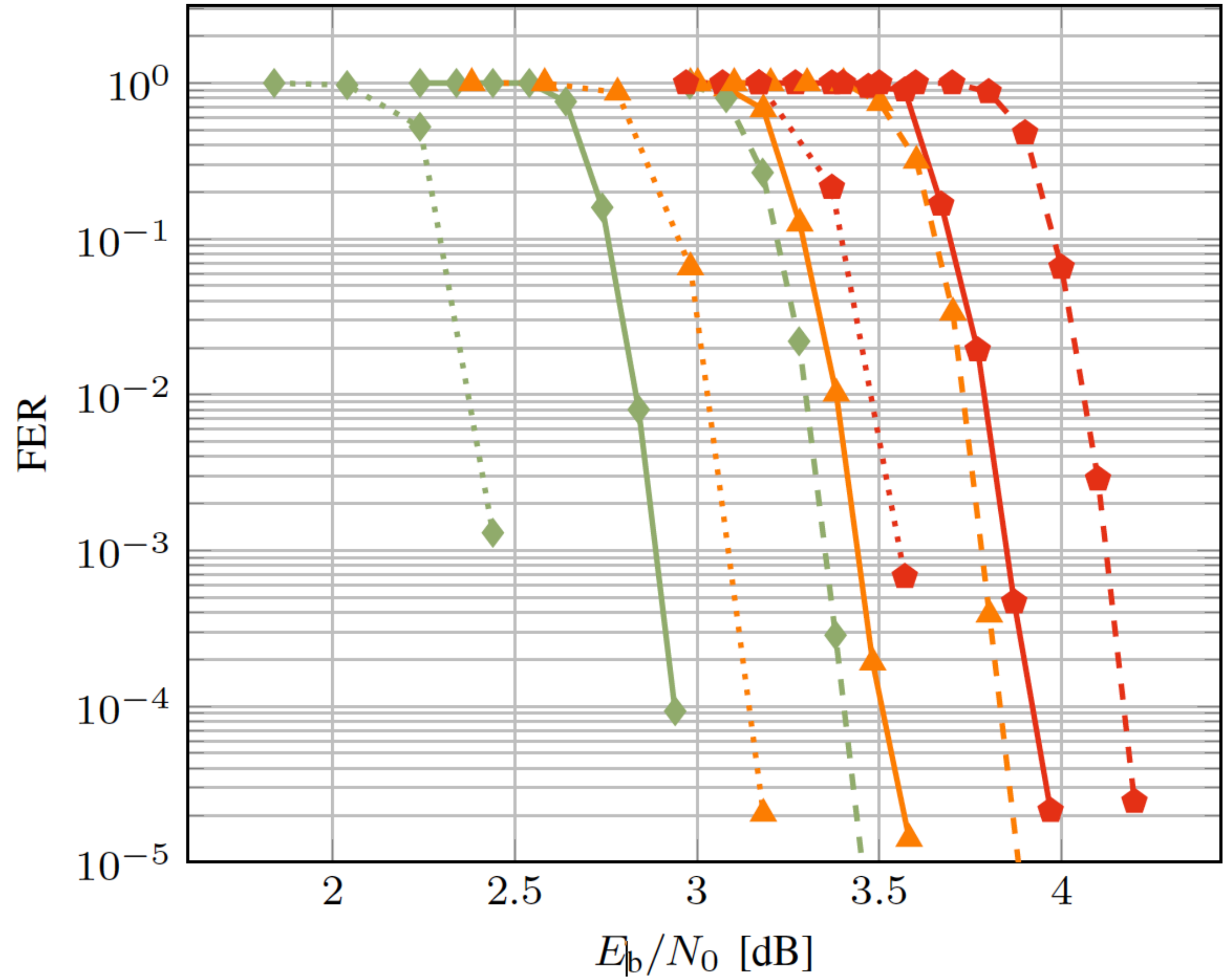


Fig. 2. FER versus E_b/N_0 for TMP and unquantized BP decoding for $R = 3/4$ (—), $R = 5/6$ (—) and $R = 7/8$ (—). We compare the TMP performance of optimized codes (—◆—, —▲—, —⬟—) to their AR4JA counterparts with unquantized BP (···◆···, ···▲···, ···⬟···) and TMP decoding (- -◆- -, - -▲- -, - -⬟- -).

Comments:

- Figure from E. Ben Yacoub et al.'s arxiv paper, Sep. 2018

- Curves show frame error rate (FER) of AR4JA and optimized codes

For More Details:

G. Lechner, T. Pedersen, and G. Kramer, “Analysis and design of binary message passing decoders,” IEEE Trans. Commun., 60(3), 601-607, 2012.
See also: http://arxiv.org/pdf/1004.4020v1

E. Ben Yacoub, “LDPC Decoding Algorithms Based on Ternary Message Passing,” Master’s Thesis, Technical University of Munich, Oct. 2018

E. Ben Yacoub, F. Steiner, B. Matuz, G. Liva, “Protograph-Based LDPC Code Design for Ternary Message Passing Decoding,” Sep. 2018
https://arxiv.org/abs/1809.10910v2

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