# NLSE Soliton Spectral Efficiency for Gaussian Input

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## Abstract

- Imminent "**Capacity Crunch**" necessitates new look at channel nonlinearities in Fiber Optic Communications.
- The channel nonlinearities can be adequately described by the **non-linear Schroedinger equation (NLSE)**, which is integrable.
- **In this work** we take advantage of exact results when the input signal distribution is Gaussian to derive an **analytic lower bound for the spectral efficiency** of solitonic transmission in the low noise limit, using scattering data parameters which can be evaluated numerically.

## Channel Model

- Single Fiber Channel NLSE:

$$i\frac{\partial u}{\partial z}+\frac{\partial^2 u}{\partial t^2}+2|u|^2 u=\varepsilon\sum_{k=1}^{M} f_k(t)\delta(z-kL)$$

  - Normalized units:

$$z\to\frac{z}{L_s}\qquad t\to\frac{t}{t_s}\qquad u\to\frac{u}{\sqrt{P t_s}}$$

$$t_s=\sqrt{\frac{\beta_2}{\gamma P}}\qquad L_s=\frac{2}{P t_s}\qquad \varepsilon^2=\frac{\sigma^2}{P t_s}=\frac{\alpha h\nu K}{P t_s}$$

  - Bandwidth: (-B, B) $\quad L_d^{-1}=\beta_2 B^2$
  - Noise Model: M amplifiers, L apart, with white noise:

$$E\left[f_k(t)\bar{f}_{k'}(t')\right]=\delta(t-t')$$

## Inverse Scattering Transform

- Exploit integrability of NLSE:
  - Given input signal $u(t)$ calculate scattering data of associated linear Zakharov-Shabat equation

$$\begin{bmatrix} i\partial_t & u \\ -\bar{u} & -i\partial_t \end{bmatrix}\begin{pmatrix}\Psi_1\\ \Psi_2\end{pmatrix}=\lambda\begin{pmatrix}\Psi_1\\ \Psi_2\end{pmatrix}$$

  - 4 degrees of freedom per soliton

$$\eta_n,\ \xi_n,\ \log b_n=\eta_n X_n+i\phi_n$$

  - Jost functions $\Psi_1(t),\ \Psi_2(t)$ are an important byproduct of the calculation
- **Basic Benefit:** In the absence of noise, scattering data have no (or trivial) spatial propagation

## Effect of Noise

- Noise shifts of the scattering data:

$$\delta\lambda_n=\varepsilon\sum_k\frac{\int\left(f_k\Psi_{1n}^{\;2}+\bar{f}_k\Psi_{2n}^{\;2}\right)}{\int\Psi_{1n}\Psi_{2n}}$$

$$\delta\log b_n=-i2\lambda_n L\sum_k\delta\lambda_{nk}(M-k)$$

$$+i\frac{\varepsilon\, b_n}{a'(\lambda_n)}\sum_k\int\left(f_k\partial\Psi_{1n}\Psi_{1n}-\bar{f}_k\partial\Psi_{2n}\Psi_{2n}\right)$$

$$\partial\Psi_{in}=\left\lfloor\partial_\lambda\Psi_i-b_n^{-1}\partial_\lambda\phi_i\right\rfloor_{\lambda=\lambda_n}\pm\log b_n\Psi_i$$

- In scattering data, first term corresponds to Gordon-Haus effect: Random shift in velocity
- Second term: "intrinsic" shift

## Soliton Information Theory

- Input Distribution: From signal to scattering data

$$u(t)\to\{\eta,\xi,X,\phi\}$$
$$P(u)\to P(\eta,\xi,X,\phi)$$

- Gaussian Noise Model:

$$f(t)\to\{\delta\eta,\delta\xi,\delta X,\delta\phi\}$$

  - For low noise, effectively AWGN channel (but with **colored noise**)
- From Sequences to Sets:
  - Solitons are indistiguishable (Gordon-Haus effect mixes any sequence)
  - Thus, for X iid:

$$H_{set}(X)=\sum_{n=1}^{N}H(x_n)-\log N!$$

  - Noise Entropy: Use of Permanent

$$H_{set}(Y\,|\,X)=-\sum_P P(Y\,|\,X_P)\log\left(\sum_{P'}P(Y\,|\,X_{P'})\right)$$

  - Low noise limit: Only one permutation probable. Thus:

$$H_{set}(Y\,|\,X)\approx N\left(\log\sqrt{2\pi e}+\frac{1}{2N}\log\det\Sigma\right)$$

  - **Σ** 4x4 block matrix of NxN correlations

## Gaussian Input

- Assume Gaussian input distribution

$$P(\{u(t)\})\propto e^{-\frac{1}{2D}\int|u(t)|^2 dt}$$

  - Correlations:

$$E\left[u(t)\bar{u}(t')\right]=2D\,\delta_B(t-t')$$

  - From dimensional analysis: $D=\dfrac{1}{4Bt_s}$
- Why?
  - Exact distribution:

$$P(\eta,\xi)=\frac{2}{D}\frac{\frac{\eta}{D}\coth\left(\frac{\eta}{D}\right)-1}{\sinh^2\left(\frac{\eta}{D}\right)}$$

  - Conserved during propagation
  - $\Psi_1(t),\ \Psi_2(t)$ localized (positive Lyapunov exponent), thus scattering data (nearly) independent and (a.s.) solitonic.
  - Uniform distribution in X and φ
- D only remaining scale in ZS system: Hence

$$t\to tD^{-1}\qquad u\to Du\qquad \lambda\to\lambda D$$

- Number of solitons

$$E\left[\int|u|^2 dt\right]=4DBT=4NE[\eta]=4ND$$

  - Hence $\quad N=BT$

## Spectral Efficiency

- 2BT complex degrees of freedom
- N = BT solitons with 4 real degrees of freedom

$$SE\geq\frac{N}{2BT}\begin{pmatrix}H(\eta)+\mathrm{H}(\xi)+\mathrm{H}(\mathrm{X})+\mathrm{H}(\phi)\\ -\log\left(\frac{N}{e}\right)\\ -\log\sqrt{2\pi e}-\frac{1}{2N}\log\det\Sigma\end{pmatrix}$$

  - After calculating input distributions and reintroducing units

$$SE\geq\log\left(\frac{Pe}{B\sigma^2\mathrm{M}}\right)-\frac{1}{4N}E\left[\log\det S\right]$$

  - Symmetric S matrix:

$$\mathbf{S}=\begin{bmatrix} R_{\eta\eta} & R_{\eta\xi} & F_1+q^2F_2 & D_1+q^2D_2\\ \dots & R_{\xi\xi} & G_1+q^2G_2 & A_1+q^2A_2\\ \dots & \dots & C_1+q^2C_2+q^4C_3 & B_1+q^2B_2+q^4B_3\\ \dots & \dots & \dots & P_1+q^2P_2+q^4P_3\end{bmatrix}$$

  - Submatrices are O(1) correlation matrices

$$q=\frac{1}{4}\sqrt{\frac{L_{tot}}{\beta_2}\frac{\gamma P}{B}}=\frac{\sqrt{L_{tot}L_d}}{2L_{nl}}$$

  - When $q\ll 1$ nonlinearity weak and Shannon limit correct

$$SE\approx\log(SNR)$$

  - When $q\gg 1$ nonlinearity is strong (hence Gordon-Haus effect dominates)

$$SE\approx\log\left(\frac{\beta_2\mathrm{B}}{\mathrm{M}L_{tot}\sigma^2\gamma^2P}\right)+O(1)$$

## Conclusions

- Analytic expression of soliton spectral efficiency in terms of fundamental parameters of fiber.
- Expressed in terms of correlations functions, written as integrals over Jost functions that can be averaged over realizations.
- Expected to self-average in large system-size limit.
- Crossover between linear AWGN and non-linear regime determined by scalar parameter q.
- Methodology can also be applied to defocusing NLSE and multimode integrable systems.