Maximising Capacity of a Nonlinear Optical Fibre Channel (using solitons and nonlinear Fourier transforms)

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What is the problem?

- ❑ What is the most appropriate way to suppress nonlinearity-induced distortion?
- ❑ What is the capacity of the nonlinear optical channel?
- ❑ What is theoretically maximum number of bit/s/Hz that can be reliably transmitted through fibre when encoding information on the amplitude of solitons?

Master model of nonlinear optical fibre channel

Single-mode optical fibre model: scalar nonlinear Schrödinger equation (NLSE)

$$i\frac{\partial E(z,t)}{\partial z} + \frac{\beta_2}{2}\frac{\partial^2 E(z,t)}{\partial t^2} + \gamma\,|E(z,t)|^2 E(z,t) = N(z,t)$$

Chromatic dispersion

$$\beta_2 \triangleq -\frac{1}{v_g}\frac{dv_g}{d\omega} < 0$$

$$D \triangleq -\frac{2\pi c}{\lambda_0^2}\beta_2 > 0$$

Kerr nonlinearity

$$\gamma \triangleq \frac{\omega_0\, n_2}{c\, A_{eff}}$$

$$n_2 \triangleq \frac{3}{8n_0}\Re\{\chi^{(3)}_{1111}\}$$

Amplifier spontaneous emission noise

Mean value

$$\mathbb{E}[N(z,t)] = 0$$

Autocorrelation

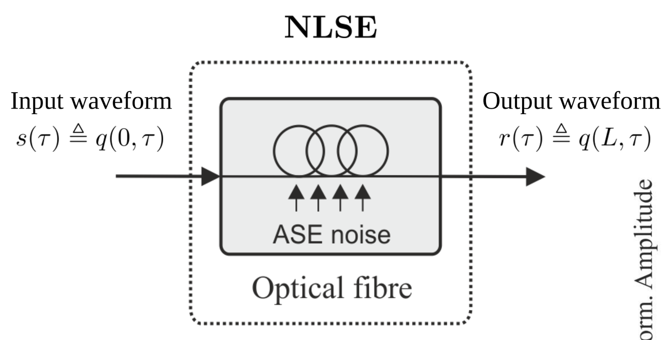
$$\mathbb{E}[N(z,t)\,N^*(z',t')] = 2\sigma_0^2\,\delta(z-z')\,\delta(t-t')$$

Noise power spectral density

$\sigma_0^2 \triangleq \alpha K_T \cdot \hbar\omega_0 \rightarrow$ *Raman amplification*

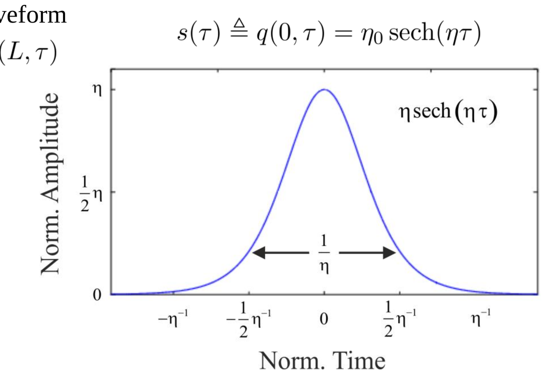
$K_T \approx 1 \rightarrow$ *characterises Raman pump beam providing the gain*

Waveform channel



❑ Initial condition: Fundamental bright soliton

$$s(\tau) \triangleq q(0,\tau) = \eta_0\,\mathrm{sech}(\eta\tau)$$



❑ Unperturbed soliton solution

$$r(\tau) \triangleq q(L,\tau) \approx \eta\,\mathrm{sech}(\eta\tau)\exp\left(i\frac{\eta^2 L}{2}\right)$$

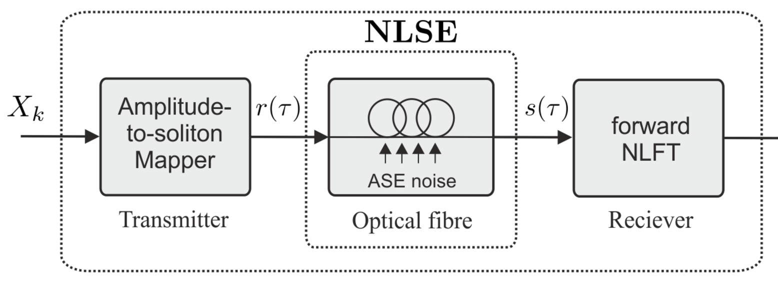
❑ Soliton (dimensionless) units

$$q \triangleq \frac{Q}{\sqrt{P_n}} \qquad \tau \triangleq \frac{t}{T_S}$$

$P_n \triangleq \frac{|\beta_2|}{\gamma T_S^2} \rightarrow$ *normalisation power*

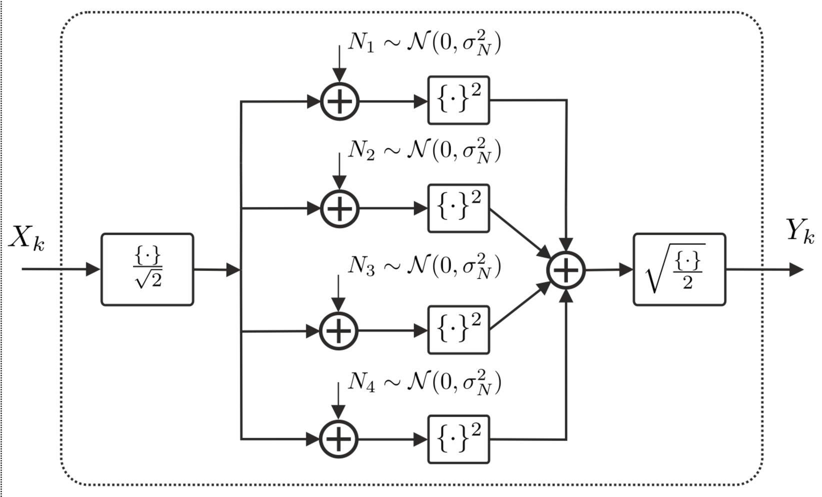
Information is encoded on solitons amplitudes η

Discrete-time channel



Channel law

$p_{Y|X}(y|x)$



Input signal

$$s(\tau) = \sum_{k=1}^{\infty} X_k\,\mathrm{sech}[X_k(\tau-k)],\;\; X_k \triangleq \sqrt{\eta_k}$$

Energy of k th soliton waveform

$$E(\eta_k) \triangleq \int_{-(k-1/2)}^{(k+1/2)} |s(\tau)|^2 d\tau \approx 2\eta_k$$

Probability density function

(noncentral $\chi^2_{p=4}(\lambda)$ - distribution) [1]

$$p_{Y|X}(y|x) = \frac{1}{\sigma_N^2}\sqrt{\frac{y}{x}}\exp\left(-\frac{x+y}{\sigma_N^2}\right)I_1\left(\frac{2\sqrt{xy}}{\sigma_N^2}\right)$$

Channel model (*input – output relationship*)

$$Y_k = \frac{1}{2}\sum_{i=1}^{4}\left(\sqrt{\frac{X_k}{2}} + N_i\right)^2$$

$N_i \rightarrow$ *independent Gaussian random variables with* $\sigma_N^2 \triangleq \mathrm{Var}[N_i]$

Forward NLFT [2]

✓ **NLFT of the signal associated with NLSE arises via the spectral analysis of the Zakharov-Shabat operator L**

Zakharov-Shabat scattering problem

$$\hat{L}\phi = \zeta\phi, \quad \hat{L} \triangleq i\begin{pmatrix} \frac{\partial}{\partial\tau} & r(\tau) \\ -r^*(\tau) & -\frac{\partial}{\partial\tau}\end{pmatrix}$$

Jost solution

$$\Phi(\zeta,\tau) = \lim_{\tau\to-\infty}\phi(\zeta,\tau) = \begin{pmatrix}1\\0\end{pmatrix}\exp(-i\zeta\tau)$$

$\phi = (\phi_1\;\phi_2)^T \rightarrow$ auxiliary scattering functions

$\zeta \rightarrow$ scattering parameter (complex soliton eigenvalues) or real *nonlinear frequencies*

Continuous nonlinear spectral data

$$\begin{cases} a(\zeta) = \lim_{\tau\to\infty}\phi_1(\zeta,\tau)\exp(i\zeta\tau) \\ b(\zeta) = \lim_{\tau\to\infty}\phi_2(\zeta,\tau)\exp(-i\zeta\tau)\end{cases}$$

Nonlinear spectrum

$a(\zeta_n) = 0 \;\rightarrow \zeta_n = \frac{1}{2}(\xi_n + i\eta_n) \;\rightarrow$ soliton velocities and **amplitudes**

$g(\zeta_n) \triangleq \frac{b(\zeta_n)}{a'(\zeta_n)} \;\rightarrow$ determines the phase and centre position of the soliton

Information theory in a nutshell

$h_Y \triangleq -\int p_Y(y)\log[p_Y(y)]\,dy \;\rightarrow$ **differential Shannon entropy**

$p_Y(y) \triangleq \int p_{Y|X}(y|x)p_X(x)\,dx \;\rightarrow$ **output distribution**

$h_{Y|X} \triangleq -\iint p_{XY}(x,y)\log[p_{Y|X}(y|x)]\,dxdy \;\rightarrow$ **differential conditional entropy**

$I_{XY} \triangleq h_Y - h_{Y|X} \;\rightarrow$ **mutual information**

$C \triangleq \max_{p_X(x):\,\mathbb{E}[|X|^2]\le\mathcal{P}_0} I_{XY} \ge I_{XY}\Big|_{p_X(x)=\tilde{p}_X(x)} \;\rightarrow$ **channel capacity**

Lower Bound on the per Soliton Capacity [3]

Trial input: Rayleigh distribution $\longrightarrow$ $\tilde{p}_X(x) = \frac{2x}{\sigma_S^2}\exp\left(-\frac{x^2}{\sigma_S^2}\right)$

Lemma 1. *For the channel given by non-central chi-squared distribution with four degrees of freedom and Rayleigh input distribution, the output entropy has the following form*

$$h_Y = \log\sqrt{\sigma_S^2} - \log\sqrt{1+\rho^{-1}} - \rho^{-1}\log\sqrt{1+\rho} + \rho + \psi(\rho^{-1}) - \frac{3}{2}\psi(1) - \log 2 + 1.$$

Lemma 2. *For the channel given by non-central chi-squared distribution with four degrees of freedom and Rayleigh input distribution, the conditional entropy has the following form*

$$h_{Y|X} = \log\sqrt{\sigma_S^2} + 2(1+\rho) - (1+\rho^{-1})\log(1+\rho) - \rho^{-1}\sqrt{1+\rho^{-1}}\,F(\rho) - \frac{\psi(1)}{2} - \log 2.$$

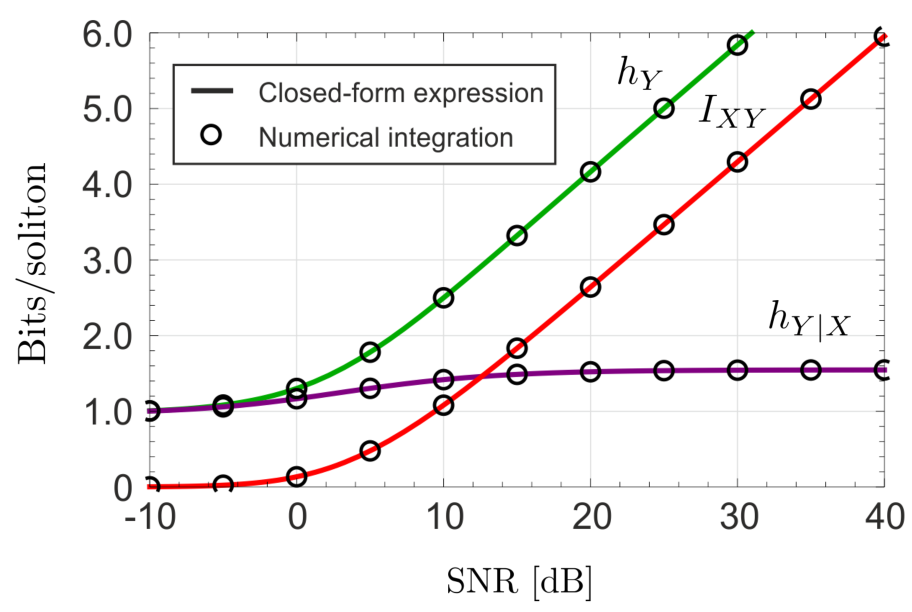
Theorem 1. *For the channel given by non-central chi-squared distribution with four degrees of freedom and Rayleigh input distribution, the mutual information (MI) has the following form*

$$I_{XY} = \log\left(\rho\sqrt{1+\rho^{-1}}\right) + \rho^{-1}\log\left(\sqrt{1+\rho}\right) - \rho + \rho^{-1}\sqrt{1+\rho^{-1}}\,F(\rho) + \psi(\rho^{-1}) - \psi(1) - 1,$$

where $F(\rho) \triangleq \int_0^\infty \xi\, K_1\left(\sqrt{1+\rho^{-1}}\,\xi\right) I_1(\xi)\log\left[I_1(\xi)\right]d\xi,$ *and* $\psi(\rho) \rightarrow$ *digamma function.*

Theorem 2. *The MI I_{XY} satisfies*

$$\lim_{\rho\to\infty}\frac{\frac{1}{2}\log\rho}{I_{XY}} = 1.$$



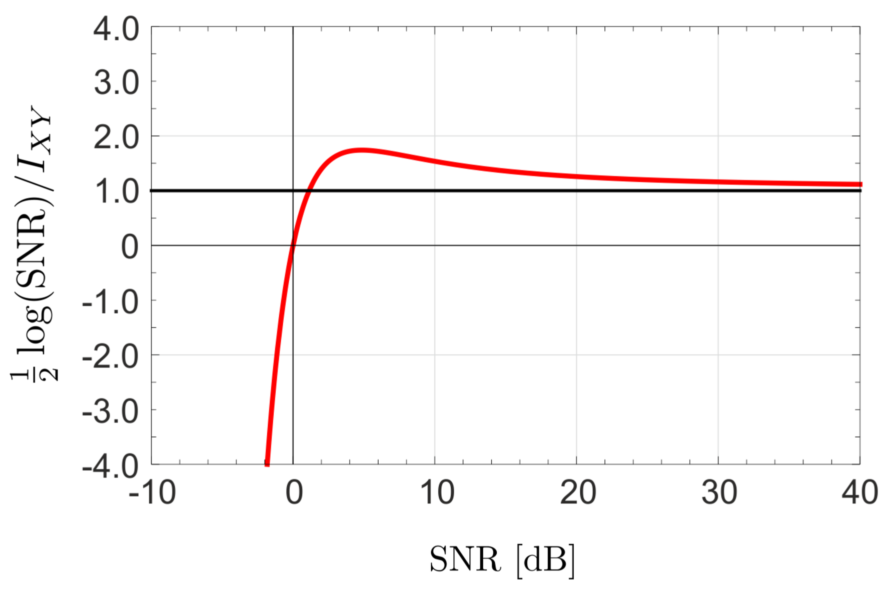
Signal-to-noise ratio (SNR):

$$\mathrm{SNR} \triangleq \frac{\mathbb{E}[E(\eta_0)]}{\sigma_N^2 T_S} = 2\kappa\cdot\rho$$

$\rho \triangleq \frac{\sigma_S^2}{\sigma_N^2} \;\rightarrow$ ratio between variances

$R_S \triangleq \frac{1}{T_S} \;\rightarrow$ symbol rate

$\kappa \;\rightarrow$ ratio between available bandwidth (BW) and symbol rate R_S



Main conclusion:

Lower bound on the *per soliton* capacity for the channel based on the individual amplitudes of well separated soliton pulses displays an unbounded (logarithmic) growth similarly to the linear Gaussian channel, i.e.,

$$C \ge I_{XY} = \frac{1}{2}\log(\mathrm{SNR}) + \mathcal{O}(1)$$

Next steps

- ❑ Take into account the BW used for soliton transmission, giving practically more relevant channel capacity in [bit/s/Hz].
- ❑ Numerically (split-step Fourier method) and/or experimentally demonstrate the analytical results.
- ❑ Use other degrees of freedom of optical solitons (phase, frequency, position, etc.).
- ❑ Generalise this to vector solitons (Manakov equation)
- ❑ Investigate the accuracy of Gaussian noise models in Nyquist WDM transmission

References

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