# A Study of Capacity and Spectral Efficiency of Fiber Channels

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# 1) Main Message

An upper bound on the spectral efficiency of a standard optical fiber model

$$\eta \leq \log(1+SNR) \quad \text{[bits/sec/Hz]}$$

- this is the first upper bound on a "full" model
- the bound is tight at low SNR;
- the bound may be extremely loose at high SNR;
  but it's better than nothing

# 2) Information Theory Basics

- Capacity C of a channel $P_{Y|X}(.)$ is the maximum I(**X**;**Y**) under constraints put on **X**
- Example: real-alphabet additive white Gaussian noise (AWGN) channel

$$Y = X + Z$$

with Var[Z]=N and an input power constraint $E[X^2] \leq P$ has

$$I(X;Y) \leq C = \frac{1}{2}\log\left(1+\frac{P}{N}\right)$$

- Complex alphabet AWGN channels: C = log(1+P/N)
- N is usually taken as $N_0W$ where $N_0$ is the (one-sided) noise PSD and W is the bandwidth
- Spectral efficiency is $\eta$ =C if one uses sinc-pulses of bandwidth W

# Maximum Entropy

- Maximum Entropy: consider $\mathbf{R}_{\underline{X}}=E[\underline{X}\,\underline{X}^{\dagger}]$ where $\underline{X}$ has length L. Then

$$h(\underline{X}) \le \log\left[(\pi e)^{L} \det \mathbf{R}_{\underline{X}}\right]$$

  with equality if and only if $\underline{X}$ is Gaussian and circularly symmetric
- For a complex square matrix **M** we have

$$h(\mathbf{M}\,\underline{X}) = h(\underline{X}) + 2\log\left|\det(\mathbf{M})\right|$$

  In particular, if **M** is unitary then $h(\mathbf{M}\,\underline{X}) = h(\underline{X})$

# Entropy Power Inequality

- Entropy Power:

$$V(\underline{X}) = e^{h(\underline{X})/L} / (\pi e)$$

- Entropy Power Inequality: for independent $\underline{X}$ and $\underline{Y}$ we have

$$\boxed{V(\underline{X}+\underline{Y}) \geq V(\underline{X}) + V(\underline{Y})}$$

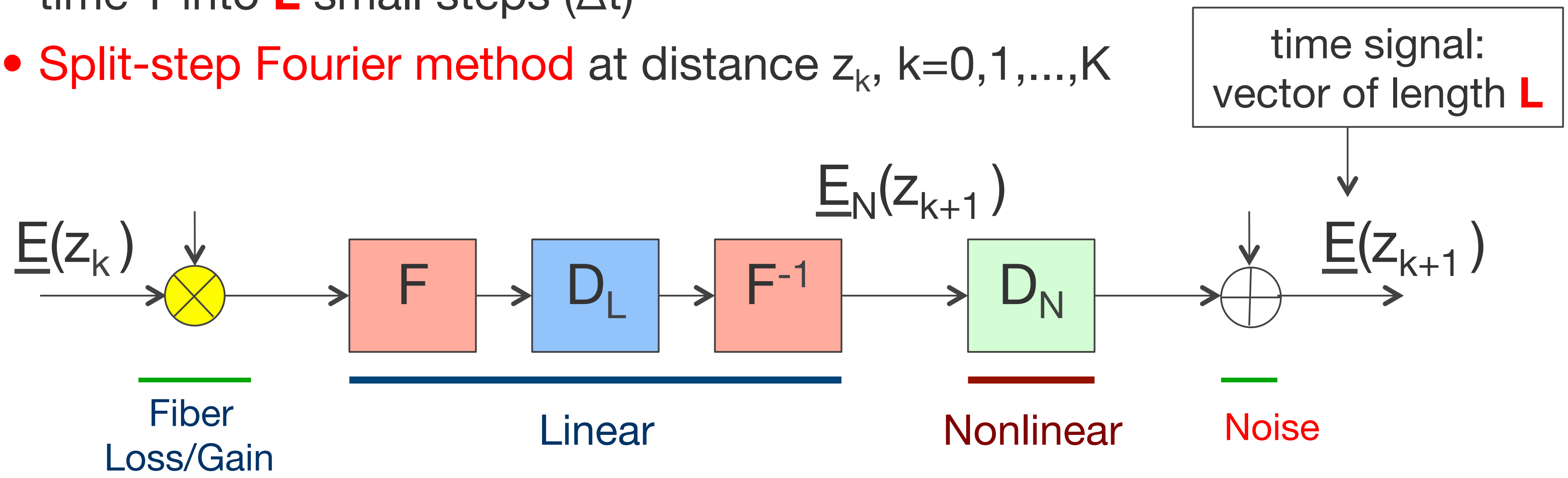
- Conditional version: for conditionally independent $\underline{X}$ and $\underline{Y}$ we have

$$V(\underline{X}|\underline{U}) = e^{h(\underline{X}|\underline{U})/L} / (\pi e)$$

$$V(\underline{X}+\underline{Y}|\underline{U}) \geq V(\underline{X}|\underline{U}) + V(\underline{Y}|\underline{U})$$

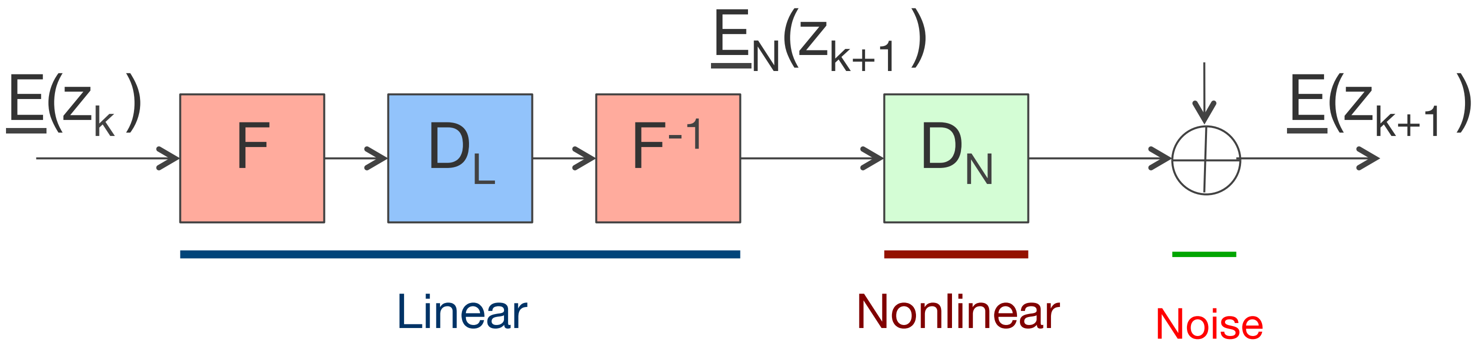
# 3) Fiber Channel(s)

- To simulate, split the fiber length z* into K small steps (Δz) and the time T into **L** small steps (Δt)
- Split-step Fourier method at distance $z_k$, k=0,1,...,K




- Ideal Raman amplification: removes the loss but adds noise
- F = Fourier transform
- $D_L$ = diagonal matrix with fixed entries of unit amplitude (all-pass filter)
- $D_N$ = diagonal matrix with unit amplitude entries; the $(\ell,\ell)$-entry phase shift is proportional to the magnitude-squared of the $\ell^{th}$ entry of $\underline{E}_N(z_{k+1})$

# 4) An Upper Bound



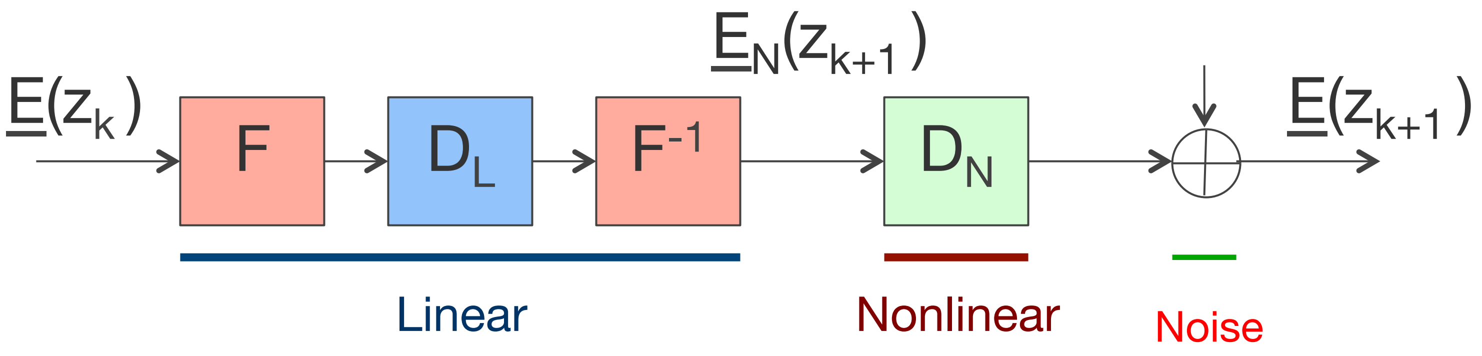

## Main Observations

- The linear step conserves energy and entropy
- The non-linear step also conserves energy and entropy

$$h\left(|a|e^{j\arg(a)+jf(|a|)}\right) = h\big(|a|, \arg(a)+f(|a|)\big) + \mathrm{E}\big[\log|a|\big]$$

$$= \underbrace{h\big(|a|\big) + h\big(\arg(a)+f(|a|)\,\big|\,|a|\big)}_{h(|a|,\arg(a))} + \mathrm{E}\big[\log|a|\big] = h(a)$$

# Energy Recursion



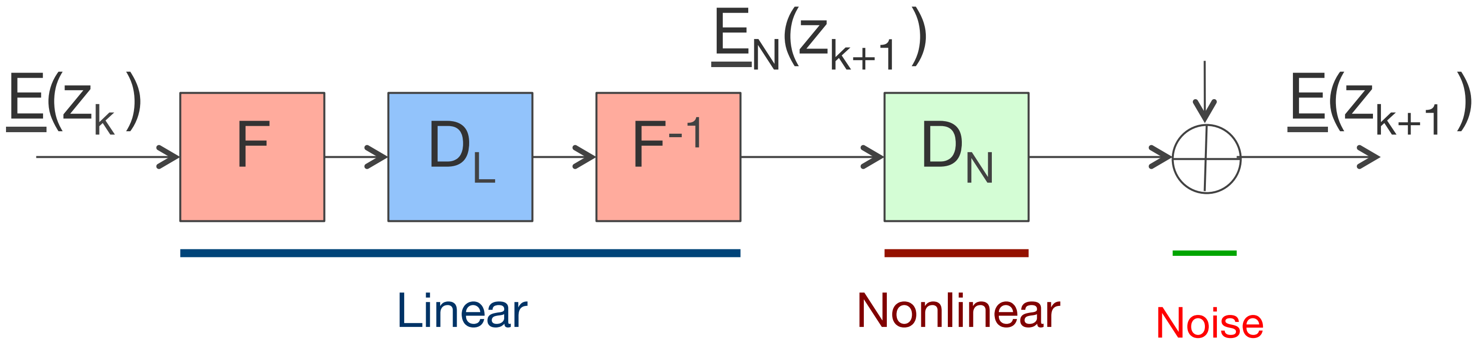

- Energy after K steps: $\text{Energy}_{\text{Launch}} + KN$ . We thus have:

$$h\left(\underline{E}(z_K)\right) \le \log\left[(\pi e)^L \det\left(\mathbf{R}\left(\underline{E}(z_K)\right)\right)\right] \quad \ldots \text{ maximum entropy}$$

$$\le \sum_{i=1}^{L} \log\left[\pi e\, R_{i,i}\left(\underline{E}(z_K)\right)\right] \quad \ldots \text{ Hadamard's inequality}$$

$$\le L \cdot \log\left[\pi e\left(Energy_{\text{Launch}} + KN\right)/L\right] \quad \ldots \text{ Jensen's inequality}$$

# Entropy Recursion



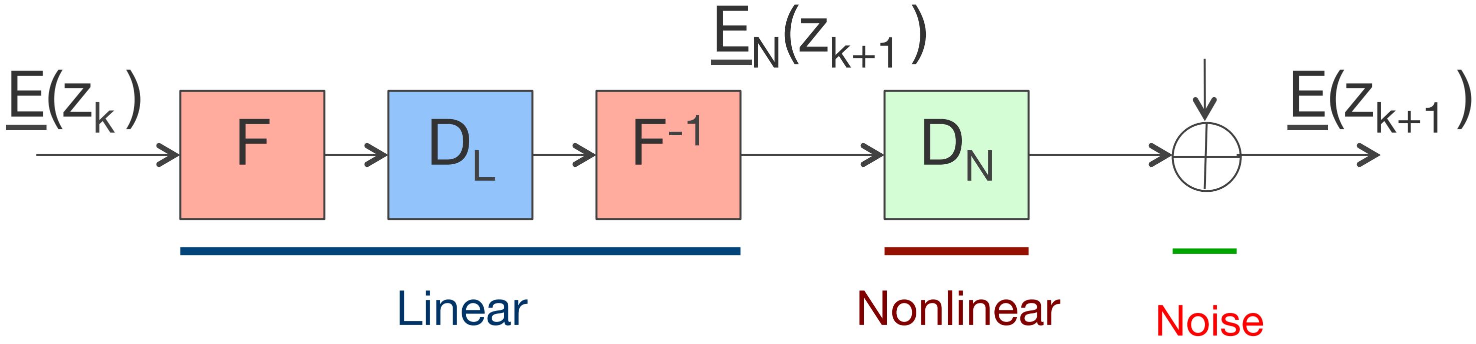

- Entropy recursion:

$$V\left(\underline{E}(z_{k+1})\middle|\underline{E}(z_0)\right) \geq V\left(\underline{E}(z_k)\middle|\underline{E}(z_0)\right) + N/L$$

- We thus have:

$$V\left(\underline{E}(z_K)\middle|\underline{E}(z_0)\right) \geq KN/L$$

$$\text{or} \quad h\left(\underline{E}(z_K)\middle|\underline{E}(z_0)\right) \geq L\log(\pi e\, KN/L)$$




So for every step we have:

- Signal energy grows by the noise variance: can upper bound h( $\underline{E}(z_K)$ )
- Entropy power grows by at least the noise variance: can lower bound h( $\underline{E}(z_K)$ | $\underline{E}(z_0)$ )
- Result*:

$$I\left(\underline{E}(z_0);\underline{E}(z_K)\right) = h\left(\underline{E}(z_K)\right) - h\left(\underline{E}(z_K)\middle|\underline{E}(z_0)\right)$$

$$\leq L \cdot \log\left(1 + SNR\right)$$

*SNR = receiver signal-to-noise ratio

$$\Rightarrow \frac{1}{L} I\left(\underline{E}(z_0);\underline{E}(z_K)\right) \leq \log\left(1+SNR\right)$$

- Let B = 1/Δt be the "bandwidth" of the simulation
- So L = T/Δt = TB is the time-bandwidth product
- The spectral efficiency is thus bounded by

$$\eta \leq \log\left(1+SNR\right) \quad \text{[bits/sec/Hz]}$$

# Discussion

$$\eta \leq \log(1 + SNR) \quad \text{[bits/sec/Hz]}$$

Q1: Why normalize by the simulation bandwidth B?
The “real” bandwidth W can be smaller.

A1: B can be chosen (this is even desirable) as the smallest bandwidth for which simulations give accurate results

Q2: What about capacity?

A2: Any real fiber has a maximal bandwidth $B_{max}$.
A capacity upper bound follows by multiplying $B_{max}$ by log(1+SNR)

# Discussion

$$\eta \leq \log(1 + SNR) \quad \text{[bits/sec/Hz]}$$

Q3: What about MIMO fiber?
A3: If energy is preserved by the linear and non-linear steps, and the noise is AWGN then the above bound remains valid per mode

Q4: What about frequency-dependent (or mode-dependent) loss?

A4: Open research!

Q5: What about lower bounds?

A5: Apply entropy recursion to V( $E(z_k)$ ) and energy recursion to h( $E(z_k)$ | $E(z_0)$ ). Issues (looks solvable): bandwidth expansion bounds

# Conclusions

1) Nonlinear cascade models are fun to study ... many other applications
2) Spectral efficiency of SMF with linear polarization is ≤ log(1+SNR)
3) Many extensions are possible:
   - lumped amplification, 3rd-order dispersion, delayed Kerr effect
   - uniform loss, linear filters (for capacity results)
   - MIMO fiber (MMF or MCF) if the linear and non-linear steps conserve energy and entropy, and the noise is Gaussian and white
4) More difficult:
   - better bounds and understanding at high SNR
   - frequency-dependent loss, dispersion, non-linearity
5) Network information theory for fiber should be developed