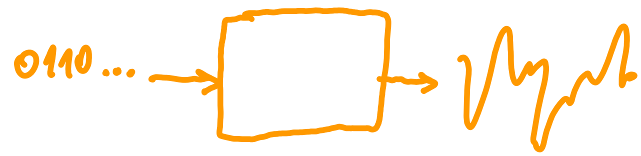
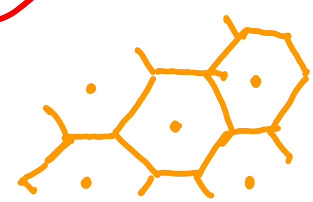
# Multi-level Coded Modulation




# and

# Lattice Construction D



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Joint work with Uri Erez & Or Ordentlich

# Multi-level Coded Modulation and Lattice Construction D

Are lattice codes better than non-lattice codes?

(question raised while teaching a graduate class on digital communication...)

# Lattices are everywhere!

* picture editing
by Kessem Zamir

# Lattice: Definition

Lattice = discrete subgroup of Euclidean space

$$\underline{\Lambda} = \{ \underline{\underline{G}} \cdot \underline{i} \; : \; \underline{i} = \text{vector of integers } (0, \pm 1, \pm 2, \dots) \}$$

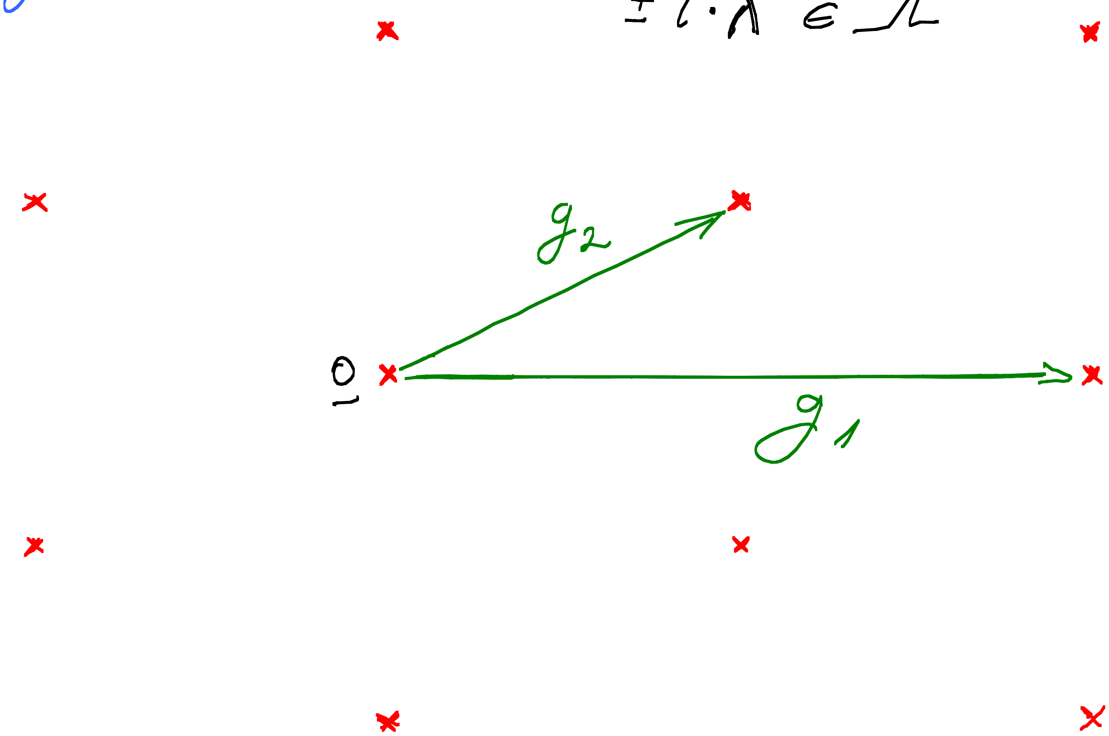
Lattice in $\mathbb{R}^n$

Generator Matrix $n \times n$

Closed under reflection & addition:

linearity: $\lambda_1, \lambda_2 \in \Lambda \Rightarrow \lambda_1 + \lambda_2 \in \Lambda$

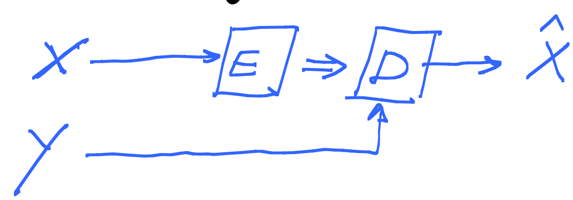
$\pm i \cdot \lambda \in \Lambda$



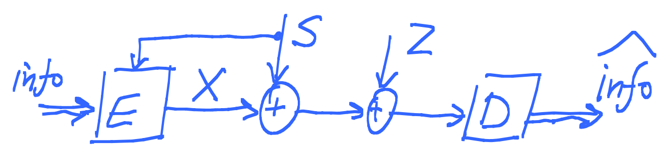
# Lattice codes in information theory

* dithered quantization → ECDQ achieves $\frac{1}{2}\log(S/D)$
* Voronoi modulation ("shaping") → achieves $\frac{1}{2}\log(1+SNR)$ as $n \to \infty$
* side information settings ("lattice binning")

Lattice Wyner-Ziv coding

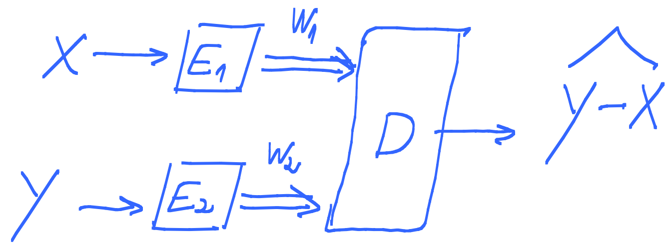



Lattice "dirty paper" coding

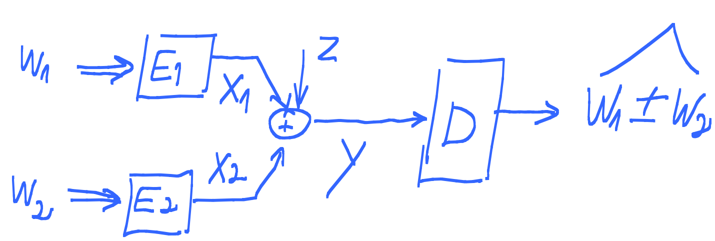


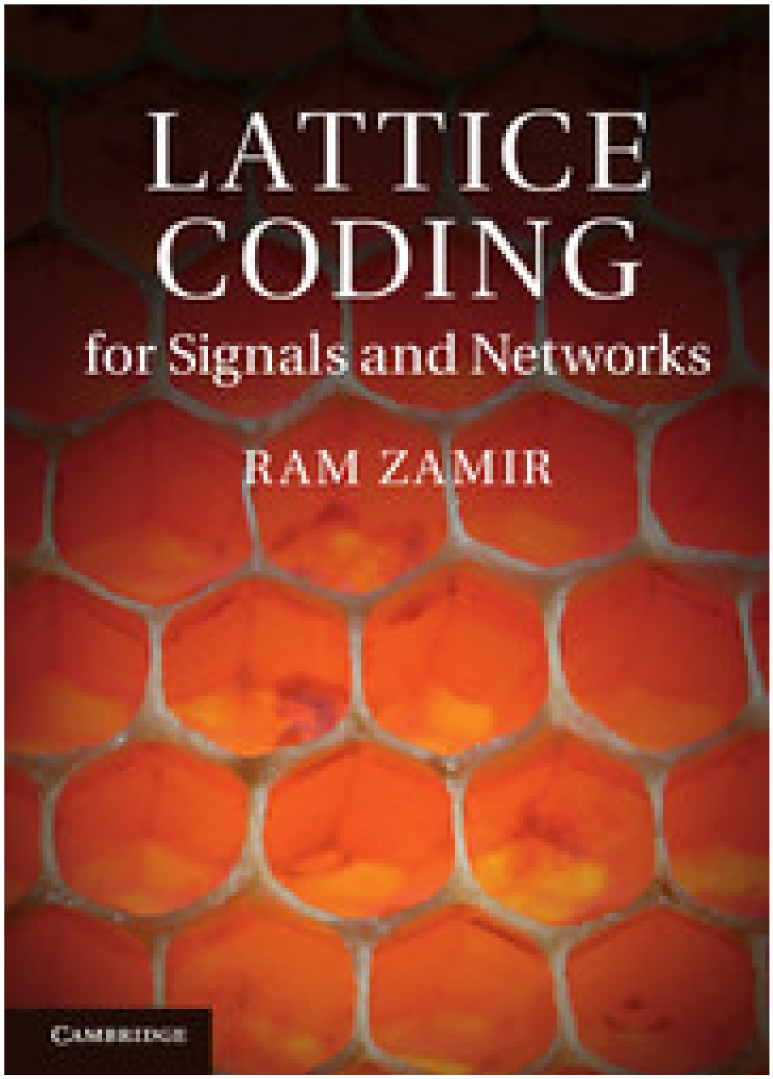

* structure beats random (distributed settings)

Lattice Korner-Marton (distributed computation):




Lattice network coding (distributed relaying):





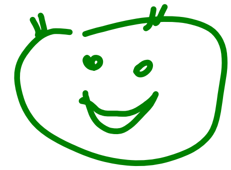

"black box"

$1 < n < \infty$

existence result

$n \to \infty$

# Goodness as $n \to \infty$



Random lattice ensemble + AEP

Minkowski-Hlawka-Siegel

Construction A with random linear codes [Loeliger 1997]

$\Rightarrow$ Sphere packing: $\rho_{pack}(\Lambda_n) \to 1/2$

Sphere covering: $\rho_{cov}(\Lambda_n) \to 1$

Normalized second moment: $G(\Lambda_n) \to \frac{1}{2\pi e}$

Normalized volume-to-noise ratio: $\mu(\Lambda_n, p_e) \to 2\pi e$, $0 < p_e < 1$

as lattice dimension $n \to \infty$.

# Back from infinity... 

Algebraic constructions in finite dimensions
$n < \infty$

Constructions A, B, C, D, E

See [Conway & Sloane Book 1988].

# Back from infinity... 

## Algebraic constructions in finite dimensions

$n < \infty$

Constructions $\underline{\underline{A}}, B, \underline{\underline{C}}, \underline{\underline{D}}, E$

non lattice (C) ? $\lessgtr$ lattice (D)

# Classification of "almost"-lattice codes

(infinite constellations)

Lattice $\Lambda$

$\downarrow$

Geometrically Uniform

$\downarrow$

Equi-Distance Spectrum

$\downarrow$

Equi-Minimum distance

(& Equi-kissing number)

$\vdots$

Random, $n \to \infty$

# Classification of "almost"-lattice codes

(infinite constellations in AWGN channel)

Lattice $\Lambda$ $\Rightarrow$ $Pe_{max} = \overline{Pe}$

$\downarrow$

Geometrically Uniform $\Rightarrow$ – // –

$\downarrow$

Equi-Distance Spectrum $\Rightarrow$ Union (& Gallager's) Bound - identical for all codewords

$\downarrow$

Equi-Minimum distance (& Equi-kissing number) $\Rightarrow$ Union Bound Estimate – // – (exponential UBE)

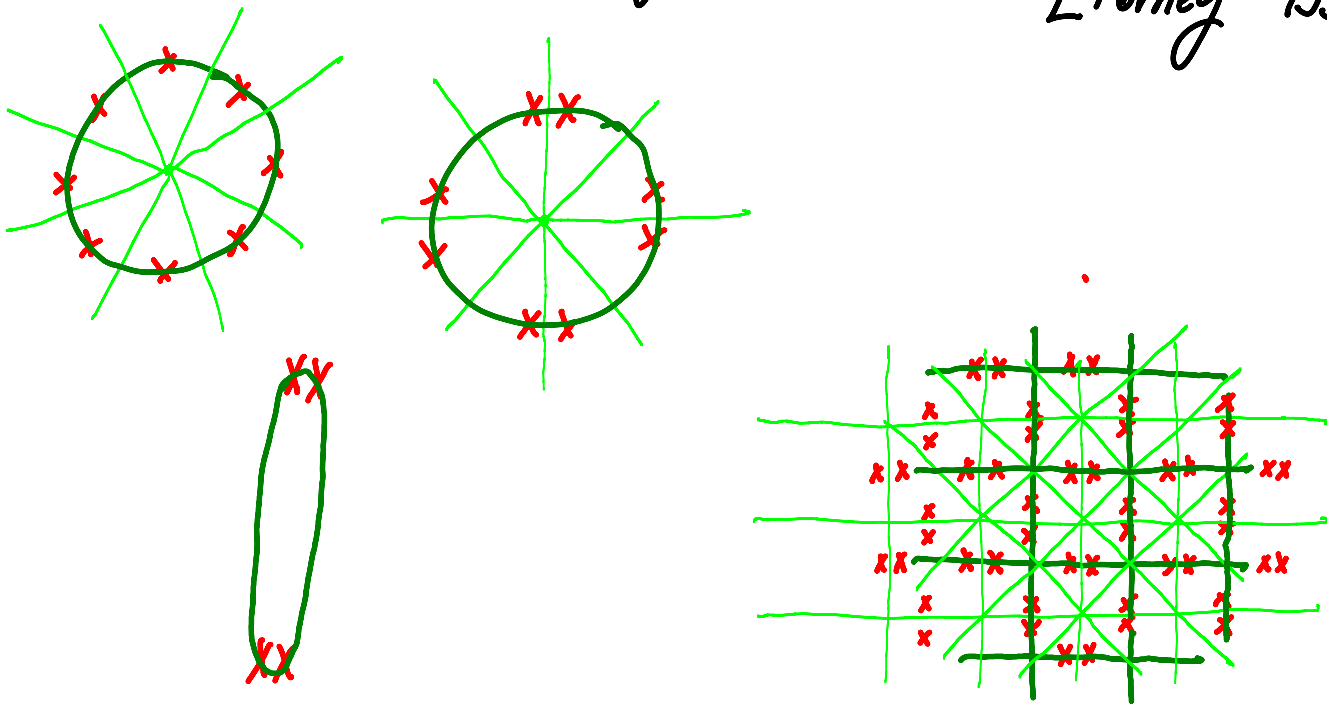
$\vdots$

Random, $n \to \infty$ $\Rightarrow$ $C$, $E(R)$

(most codewords are "good" as $n \to \infty$)

# Reminder : Geometrically Uniform Constellation

[Forney 1991]



**Definition:** $\Gamma$ is GU if for any two codewords $c, c' \in \Gamma$, there exists a distance-preserving transformation $T$ (translation, reflection, rotation) such that

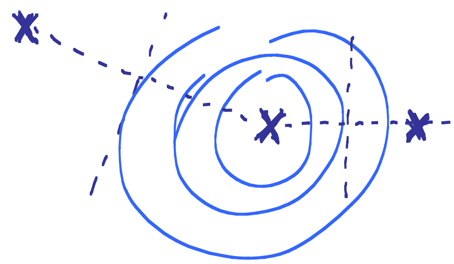
$$c' = T(c) \quad \text{and} \quad T(\Gamma) = \Gamma.$$

$\Rightarrow$ The world seen by any codeword is the same, up to rotation and reflection.

$\Rightarrow$ Same Voronoi cells (Euclidean distance)
Same $p_e(c)$ (under AWGN).

# Reminder : Union Bound

Channel: $Y = c + Z$ , $c \in \Gamma$ , $Z \sim AWGN(\sigma)$



$$P_e(c) \leq \sum_{d \geq d_{min}(c)} N(c,d) \cdot Q\left(\frac{d/2}{\sigma}\right)$$

where $N(c,d)$ = number of codewords in $\Gamma$ at distance $d$ from $c \in \Gamma$,

and $d_{min}(c) = \min \{ d : N(c,d) > 0 \}$.

UBE: $P_e(c) \simeq N_{min}(c) \cdot Q\left(\frac{d_{min}(c)/2}{\sigma}\right)$

where $N_{min}(c)$ = "kissing number" $= N(c, d = d_{min}(c))$

(For a lattice = number of spheres at radius $\frac{d_{min}}{2}$ touching sphere @ origin.)

## Equi-Distance Spectrum Definition:

$$N(c,d) = N(d) \quad \forall c \in \Gamma$$

$\Rightarrow$ UB is identical for all codewords.
(as well as the Gallager bounds)

## Equi-minimum distance definition:

$$d_{min}(c) = d_{min} = \text{constant} \quad \forall c \in \Gamma$$

$\Rightarrow$ (Since $Q(x) \sim e^{-x^2/2}$)

UBE is exponentially (in SNR) identical for all codewords.

# Construction A

Let $\mathcal{C}$ be an $(n, M, d)$ binary code:

$\mathcal{C} = \{\underline{c}_i\}_{i=1}^{M}$ , $\underline{c}_i \in \{0,1\}^n$ , $d$ = minimum Hamming distance.

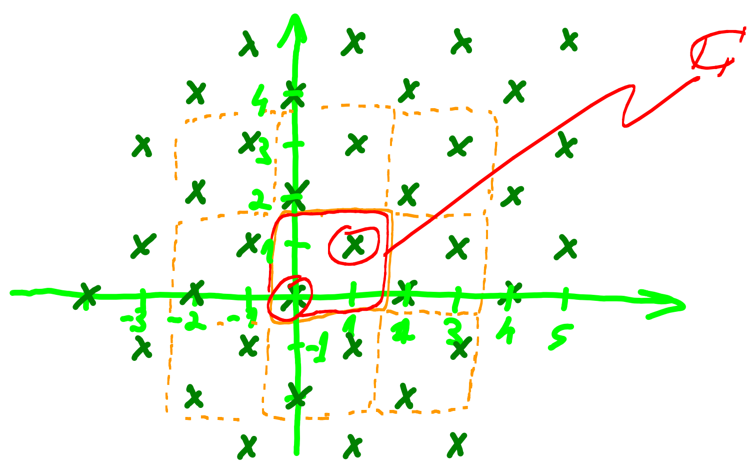
Construction A lifts $\mathcal{C}$ to $\mathbb{R}^n$ periodically:

Def. I

$$\Gamma_{\mathcal{C}} = \{\underline{x} \in \mathbb{Z}^n : \ \underline{x} \bmod 2 \in \mathcal{C}\}$$

integer vectors

modulo 2 per each component




**Equivalent definitions:**

1) $$\Gamma_{\mathcal{C}} = \mathcal{C} + 2 \cdot \mathbb{Z}^n$$ Def. II

2) Let $z = (LSB(z), MSB_1(z), MSB_2(z), \dots)$ = binary expansion of $z$

$$\Gamma_{\mathcal{C}} = \{\underline{x} \in \mathbb{Z}^n : \ LSB(\underline{x}) \in \mathcal{C}\}$$ Def. III

# Construction A : Properties

1) $d^E_{\min}(\Gamma) \triangleq$ min Euclidean distance $\triangleq \min\limits_{\substack{x,y\in\Gamma \\ x\neq y}} \|x-y\|$

$$= \min\{2, \sqrt{d}\}$$

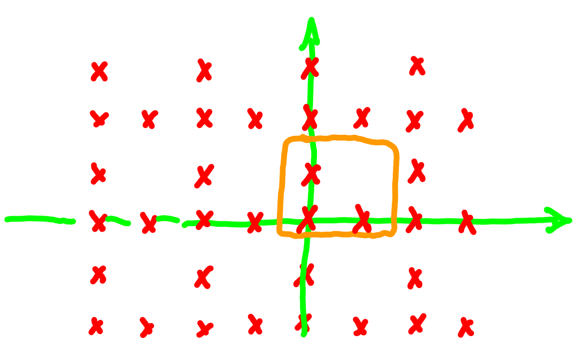
min Euclidean dist in coset $\underline{c} + 2\mathbb{Z}^n$ for $\underline{c} \in \mathcal{C}$

$d_H(\underline{c}_1, \underline{c}_2) = d$

$\Rightarrow \|\underline{c}_1 - \underline{c}_2\| = \sqrt{d}$ (Pythagoras)

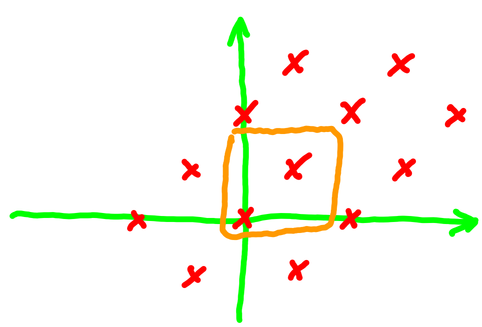
2) If $\mathcal{C}$ is a linear $(n,k,d)$ code $(M = 2^k)$

$\Rightarrow$ $\Gamma_{\mathcal{C}} = \Lambda_{\mathcal{C}}$ is a modulo-2 lattice.



non lattice

$\mathcal{C} = \{(00), (01), (10)\}$



lattice

$\mathcal{C} = \{(00), (11)\}$

# Construction A : Properties

1) $d^E_{min}(\Gamma) \triangleq$ min Euclidean distance $\triangleq \min\limits_{\substack{x,y\in\Gamma \\ x\neq y}} \|x-y\|$

$$= \min\left\{2\,,\ \sqrt{d}\right\}$$

min Euclidean dist in coset $\underline{c}+2\mathbb{Z}^n$ for $\underline{c}\in\mathcal{C}$

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2) If $\mathcal{C}$ is a linear $(n,k,d)$ code $(M=2^k)$

$\Longrightarrow$ $\Gamma_{\mathcal{C}}=\Lambda_{\mathcal{C}}$ is a modulo-2 lattice.

Def. IV

$$\Longrightarrow \Gamma_{\mathcal{C}}=\left\{\underbrace{\underline{w}\cdot\underline{\underline{G}}}+2\underline{z}\ :\ \begin{array}{l}\underline{w}\in\{0,1\}^k\\ \underline{z}\in\mathbb{Z}^n\end{array}\right\}$$

mod-2 or real multiplication (due to $2\mathbb{Z}^n$ term)

where $\underline{\underline{G}}=\begin{bmatrix}-\ \underline{g}_1\ -\\ \vdots\\ -\ \underline{g}_k\ -\end{bmatrix}=(k\times n)$ generator matrix for $\mathcal{C}$.

# Construction A : Properties

1) $d^{E}_{\min}(\Gamma) \triangleq$ min Euclidean distance $\triangleq \min\limits_{\substack{x,y\in\Gamma \\ x\neq y}} \|x-y\|$

$$= \min\{2, \sqrt{d}\}$$

min Euclidean dist in coset $\underline{c} + 2\mathbb{Z}^n$ for $\underline{c} \in \mathcal{C}$

$d_H(\underline{c}_1, \underline{c}_2) = d$

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2) If $\mathcal{C}$ is a linear $(n,k,d)$ code $(M = 2^k)$

$\Longrightarrow$ $\Gamma_{\mathcal{C}} = \Lambda_{\mathcal{C}}$ is a modulo-2 lattice.

Example 1: Construction A of Gosset lattice E8 via extended (8, 4, 4) Hamming code.

Example 2: Trellis-Coded Modulation ($\mathcal{C}$ = convolutional code) [Ungerboeck 1982]

3) Extension to p-ary (linear) codes ("mod-p lattices")

# Construction C

* "Multi-level coded modulation"
* Natural extension (?) of construction A to $L$ levels
* Bound on minimum distance $2 \to 2^{L-1}$
* Super-position of $L$ binary codes : $\mathcal{C}_1, \dots, \mathcal{C}_L$

$$\Gamma = \mathcal{C}_1 + 2\cdot\mathcal{C}_2 + 4\cdot\mathcal{C}_3 + \dots + 2^{L-1}\cdot\mathcal{C}_L + 2^L\cdot\mathbb{Z}^n$$

* Equivalent definitions:

binary expansion

$$\left\{ \underline{x} \in \mathbb{Z}^n : \ LSB(\underline{x}) \in \mathcal{C}_1 ,\ MSB_1(\underline{x}) \in \mathcal{C}_2 ,\ \dots ,\ MSB_{L-1}(\underline{x}) \in \mathcal{C}_{L-1} \right\}$$

$\Longleftrightarrow$

recursive law

$$\left\{ \underline{x} \in \mathbb{Z}^n : \begin{array}{ll} \hat{c}_1 \triangleq \underline{x} \bmod 2 & \in \mathcal{C}_1 \\ \hat{c}_2 \triangleq \frac{1}{2}\cdot(\underline{x} - \hat{c}_1) \bmod 2 & \in \mathcal{C}_2 \\ \hat{c}_3 \triangleq \frac{1}{4}(\underline{x} - \hat{c}_1 - 2\hat{c}_2) \bmod 2 & \in \mathcal{C}_3 \\ \vdots & \\ \hat{c}_L \triangleq \frac{1}{2^{L-1}}(\underline{x} - \hat{c}_1 - 2\hat{c}_2 - 4\hat{c}_3 - \dots) \bmod 2 & \in \mathcal{C}_L \end{array} \right\}$$

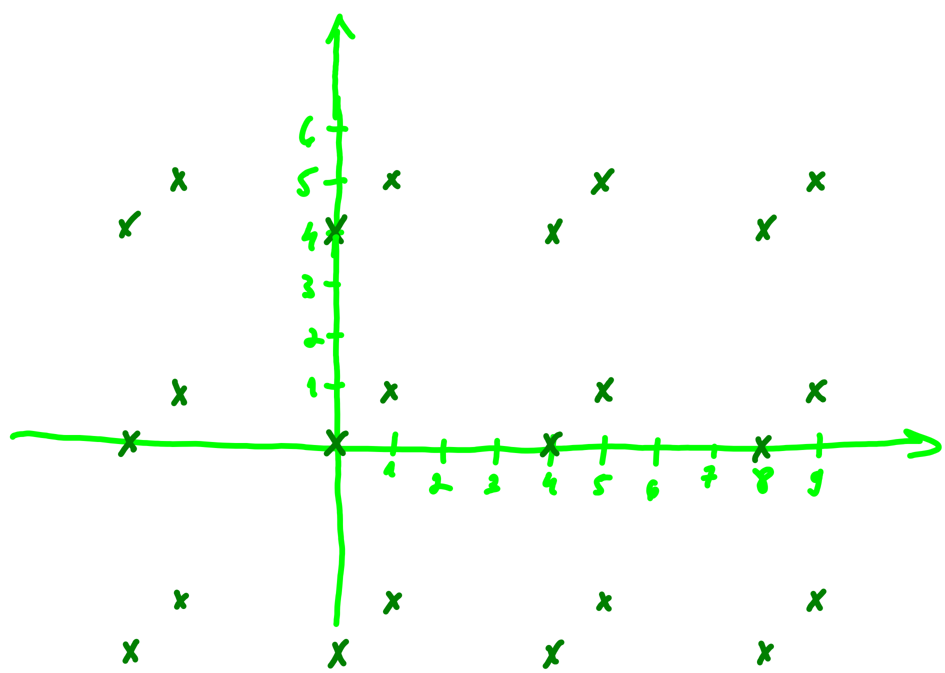
# Construction C : general context

* Single level $(L=1)$ $\Rightarrow$ Construction A
* Multiple levels $(L>1)$ $\Rightarrow$ not necessarily a lattice even if all component codes are linear !

$C_1 = \{(00), (11)\}$

$C_2 = \{(00)\}$

$\Gamma = C_1 + 2C_2 + 4\cdot\mathbb{Z}^2$




* Multi-level coset codes [Forney - Trott - Chang 2000] :

Special case where

$$\Lambda_1/\Lambda_2/\dots/\Lambda_L = \mathbb{Z}/2\mathbb{Z}/\dots/2^{L-1}\mathbb{Z}$$

# Construction C : basic properties

1) unique decomposition for $\underline{x} \in \Gamma$:

$$\underline{x} = \underline{c}_1 + 2\underline{c}_2 + \dots + 2^{L-1}\underline{c}_L + \underline{z} \quad , \quad c_j \in \mathcal{C}_j \quad , \quad \underline{z} \in \mathbb{Z}^n$$

2) Distance improvement (monotonic with $L$):

$$\|\underline{x} - \underline{y}\| = \|\underbrace{c_{x1} - c_{y1}}_{\in\{-1,0,+1\}} + 2\cdot\underbrace{(c_{x2} - c_{y2})}_{\in\{-2,0,+2\}} + \underbrace{4(\underline{z}_x - \underline{z}_y)}_{\in\{0,\pm4,\pm8,\dots\}}\| \geq \|c_{x1} - c_{y1}\|$$

2 levels

$\Rightarrow$ upper levels retain distance (if equal) or increase distance (if different) but never decrease it !

3) Equi-minimum distance property $\forall\, \underline{x} \in \Gamma$:

$$d_{min}^{E}(\underline{x}) = \min\left\{\sqrt{d_1}, \dots, 2^{L-1}\cdot\sqrt{d_L}, 2^L\right\} \quad , \quad d_i = d_{min}^{Hamming}(\mathcal{C}_i)$$

achieved when all levels but one are equal.

4) Balanced Hamming distances:

$$d_i = \frac{d_1}{4^{i-1}} \quad \Rightarrow \quad d_{min}^{E}(\Gamma) = \min\left\{\sqrt{d_1}, 2^L\right\} \overset{d_1 = 4^L}{=} 2^L$$

$\Rightarrow$ maximum point density per minimum distance.

# Construction G: more properties

## 5) Point density:

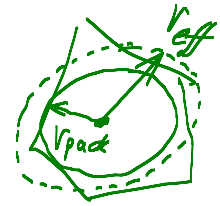
$M = M_1 \cdot M_2 \cdot \cdots M_L$ codewords per a $[2^L \times \ldots \times 2^L]$ cube.

For linear component codes, $M_i = 2^{k_i} = 2^{n r_i}$, $i = 1 \ldots L$

$\Rightarrow$ $\bar{V}$ = average cell volume $= \frac{2^{nL}}{M} = 2^{n \cdot \sum_{i=1}^{L} 1 - r_i}$

## 6) Packing efficiency $\rho_{pack} \triangleq \frac{r_{pack}}{r_{eff}}$:

$r_{pack} = d_{min}^{\epsilon}/2 \qquad r_{eff} = \sqrt[n]{V / V_n}$




Assume balanced linear codes $(L \to \infty)$, satisfying the Varshamov-Gilbert bound $\Rightarrow$ $r_i \underset{n \gg 1}{\simeq} 1 - H_B(d_i/n)$

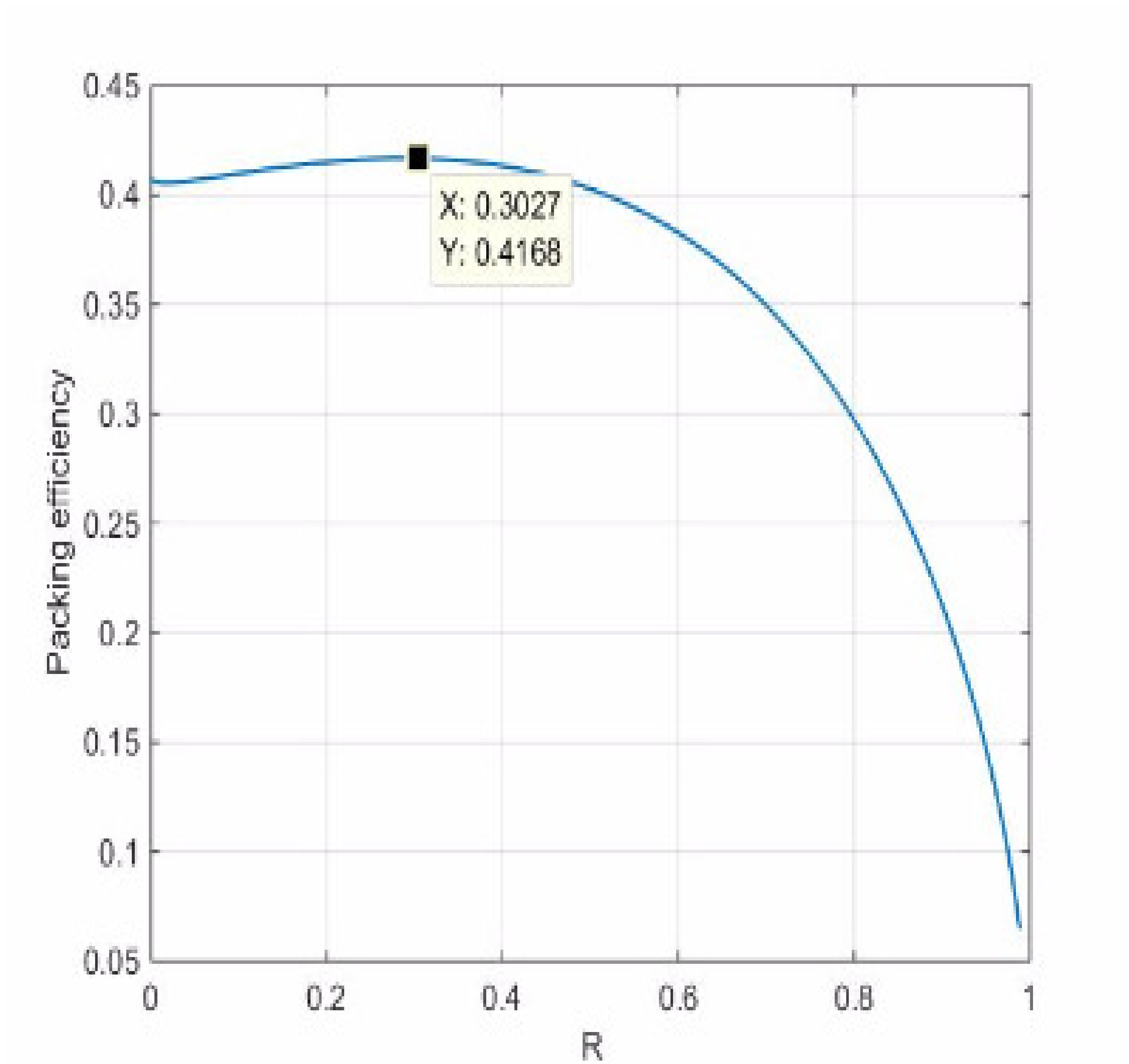
$$\Rightarrow \quad \rho_{pack}(\Gamma) \simeq \frac{1}{2} \cdot \sqrt{2\pi e \cdot \alpha \cdot 2^{-2 \sum_{i=1}^{\infty} H_B(\alpha/4^{i-1})}} \cong 0.42$$

$n \gg 1$
$L \to \infty$
$d_1 = \alpha \cdot n$
$r_i \simeq 1 - H_B\left(\frac{\alpha}{4^{i-1}}\right)$

$\alpha = 0.11$
$(r_1 = 1/2)$

$\Rightarrow$ Close to maximum known packing efficiency for large $n$ (Minkowski bound) !

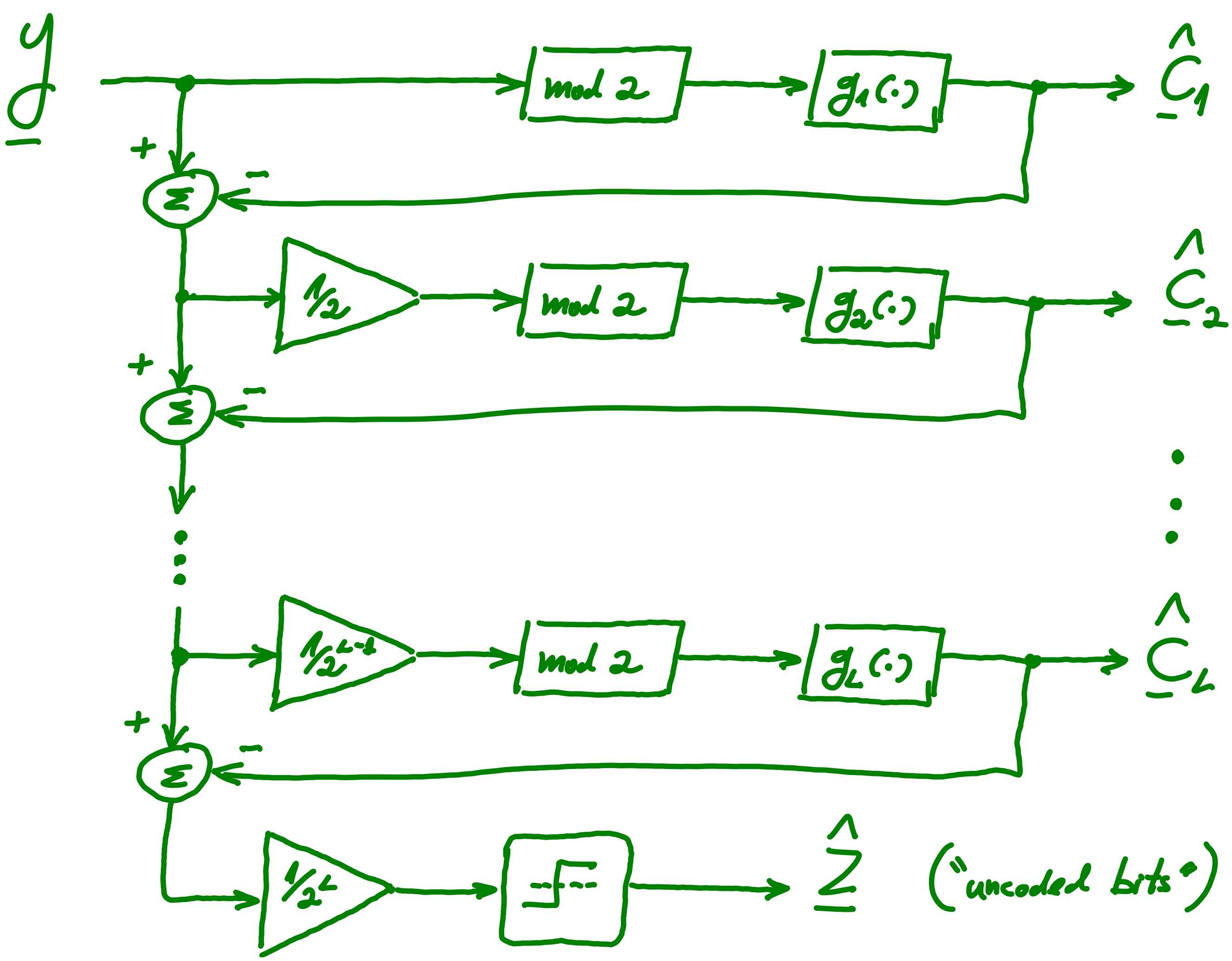
$\rho_{pack}(r_c^*)$ as a function of $r_1 = 1 - H_B(\alpha)$



# Multi-Stage Decoding

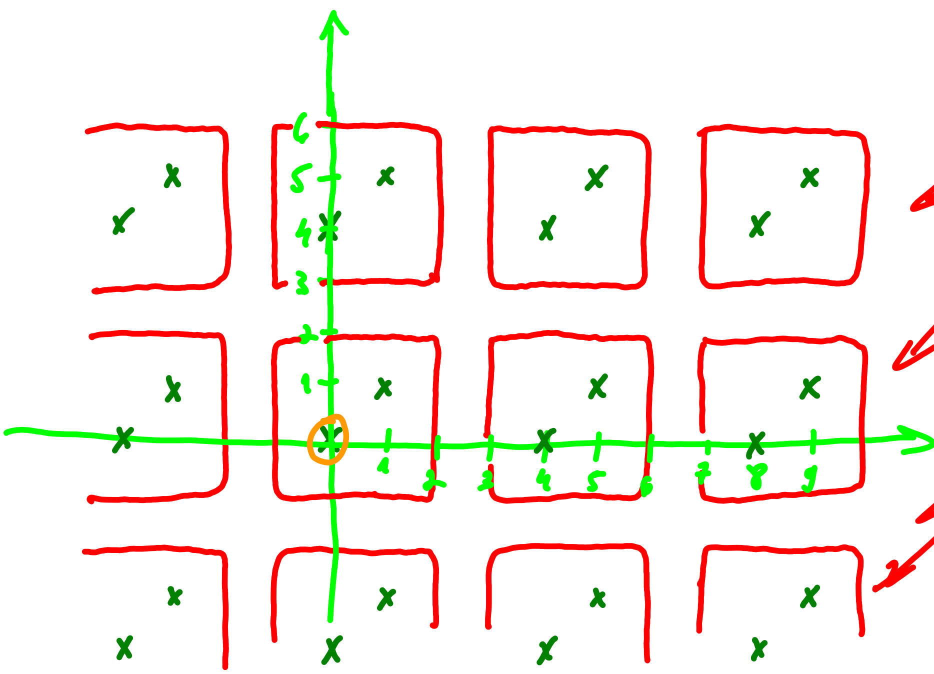
$$\underline{Y} = \underline{x} + \underline{N}, \quad \underline{x} \in \Gamma$$

Let $g_i(\cdot)$ = "soft-decision" decoder for $\underline{c} \in C_i$ in a modulo-2 channel: $\tilde{\underline{y}} = [\underline{c} + \underline{N}/2^{i-1}] \bmod 2$.



# Multi-Stage Decoding

$\underline{Y} = \underline{X} + \underline{N}$ , $\underline{X} \in \Gamma$



regions where

$\underline{\hat{X}}_{NN} = \underline{\hat{X}}_{MSD}$ ,

given $\underline{X} = \underline{0}$.

Not optimal, nevertheless...

* Forney - Trott - Chung 2000 :

Construction G with MSD can achieve the high-SNR AWGN channel capacity as $n \to \infty$.

( $L$ is tuned to SNR level )

# When Construction C is a lattice?

* Binary addition: $a, b \in \{0,1\} \Rightarrow a+b = (a+b) \bmod 2 + \underbrace{2\, a \cdot b}_{\text{"carry"}}$

* Linearity of construction A:
$\underline{x}, \underline{y} \in \Lambda_{\mathcal{C}} \triangleq \mathcal{C} + 2\mathbb{Z}^n$, where $\mathcal{C}$ is a linear code.

$\Rightarrow \underline{x}+\underline{y} = (c_x + 2z_x) + (c_y + 2z_y) = \underbrace{(c_x + c_y) \bmod 2}_{\in \mathcal{C}} + \underbrace{2 \cdot c_x \odot c_y + 2(z_x + z_y)}_{\in 2\mathbb{Z}^n} \in \Lambda_{\mathcal{C}}$

* Const. C is not always linear (even if component codes are):
$\underline{x}, \underline{y} \in \Gamma \triangleq \mathcal{C}_1 + 2\mathcal{C}_2 + 4\mathbb{Z}^n$ (2 levels)

$\Rightarrow \underline{x} + \underline{y} = (c_{x1} + 2c_{x2} + 4z_x) + (c_{y1} + 2c_{y2} + 4z_y)$

$$= \underbrace{(c_{x1} + c_{y1}) \bmod 2}_{\in \mathcal{C}_1} + \underbrace{2\, c_x \odot c_y}_{\text{"problem"}} + \underbrace{2(c_{x2} + c_{y2}) \bmod 2}_{\in 2\mathcal{C}_2} + \underbrace{4\, c_{x2} \odot c_{y2} + 4(z_x + z_y)}_{\in 4\mathbb{Z}^n}$$

$\therefore\ \underline{x} + \underline{y} \notin \Gamma$, unless $\mathcal{C}_2$ is closed under products in $\mathcal{C}_1$:
$c_x \odot c_y \in \mathcal{C}_2 \quad \forall c_x, c_y \in \mathcal{C}_1$ [Kositwattanarerk-Oggier 14].

* Example: If $\mathcal{C}_1 \subset \ldots \subset \mathcal{C}_L$ is a chain of Reed-Muller codes, then $\Gamma = \Lambda_{BW}$ is the Barnes-Wall lattice.

So... what can we say when construction C' is not a lattice? ...

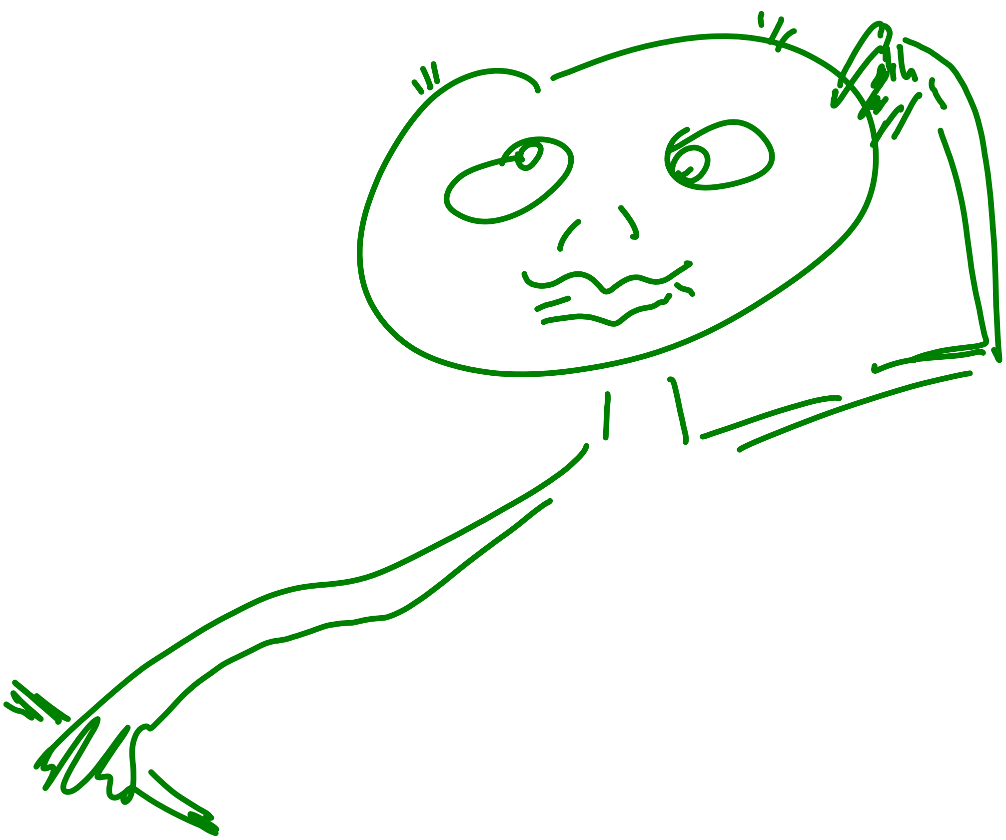
$\Lambda$ ✗

GU ?

EDS ?

EMD ✓

EKN ?



# Construction D

* Multi-level lattice construction
* Natural extension (?) of construction A (Def. IV)
* Similar to (non-lattice) construction C
  (same $d_{min}$, allows MSD)
* Based on a chain of nested linear binary codes:

  $\mathcal{C}_1 \subset \dots \subset \mathcal{C}_L$, where $\mathcal{C}_j = (n, k_j, d_j)$ code, $k_1 \leq \dots \leq k_L$
* Super-position of basis vectors (rather than of the codes)
* Let $\underline{g}_1 \dots \underline{g}_n$ be a basis for $\{0,1\}^n$, such that the $k_j \times n$ matrix $\underline{\underline{G}}_j = \begin{bmatrix} - \underline{g}_1 - \\ \vdots \\ - \underline{g}_{k_j} - \end{bmatrix}$ is a generator matrix for $\mathcal{C}_j$, $j = 1 \dots L$.

$$\Lambda_D = \Bigg\{ \underbrace{\sum_{j=1}^{L} \overbrace{2^{j-1} \cdot \underline{w}_j \cdot \underline{\underline{G}}_j}^{\text{real (not modulo 2) multiplication}}}_{\text{code nesting} \Rightarrow \text{closed under mod-}2^L \text{ addition} \Rightarrow \Lambda_D \text{ is a lattice}} + 2^L \cdot \underline{z} : \underset{j=1\dots L}{\underline{w}_j \in \{0,1\}^{k_j}}, \ \underline{z} \in \mathbb{Z}^n \Bigg\}$$

# Construction D versus C

* Similar form: replication by $2^L \cdot \mathbb{Z}^n$ of

$$\left[ \sum_{j=1}^{L} 2^{j-1} \left( \underline{w}_j \cdot \underline{\underline{G}}_j \bmod * \right) \right] \bmod 2^L$$

mod 2 for $\Gamma$
no modulo (real multiplication) for $\Lambda_D$

* Similar features:
  * upper levels can only increase distance

$\Longrightarrow$ $d^E_{min}(\Lambda_D) = \min\{\sqrt{d_1}, \ldots, 2^{L-1}\sqrt{d_L}, 2^L\} = d^E_{min}(\Gamma)$

  * Unique decomposition: $\underline{x} = \underline{w}_1 \underline{\underline{G}}_1 + \ldots + 2^{L-1} \underline{w}_L \cdot \underline{\underline{G}}_L + 2^L \cdot \underline{z}$

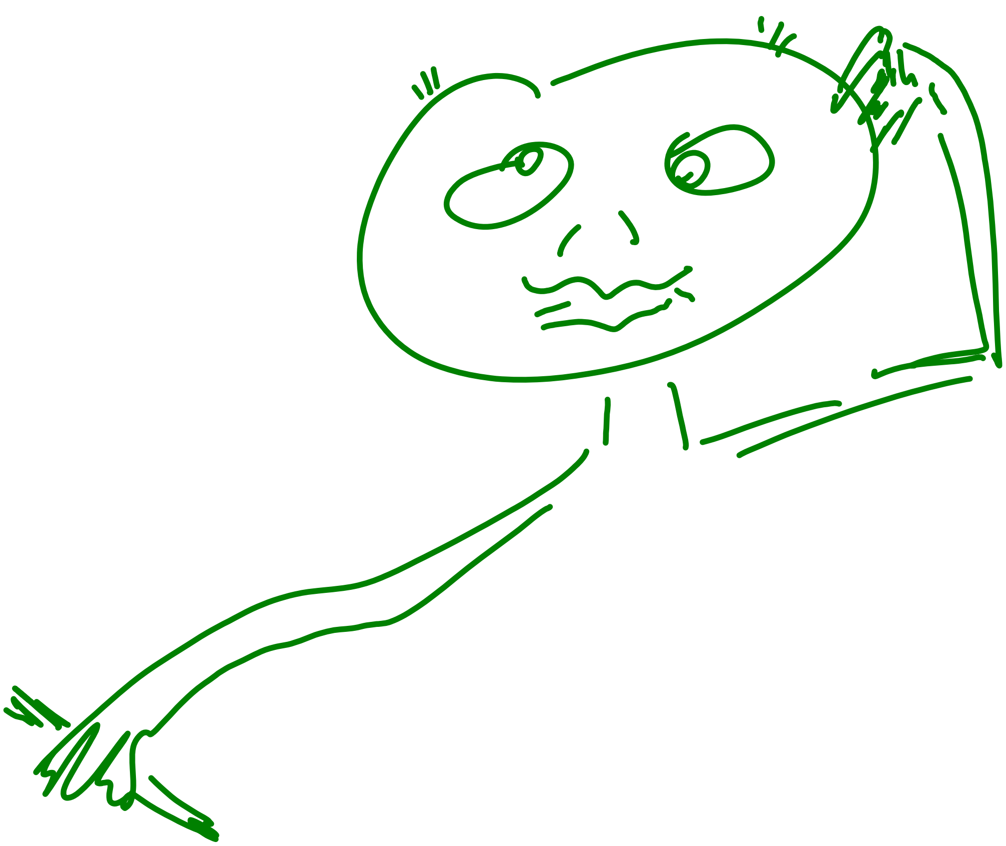
$\Longrightarrow$ $V(\Lambda_D)$ = cell volume = $2^{n \cdot \sum_{j=1}^{L} 1 - r_j} = V(\Gamma)$

  * achieves AWGN capacity w MSD, as $n \to \infty$.

* $\Lambda_D$ is always a lattice
* Given $C_1 \ldots C_L$, $\Lambda_D$ may depend on the selected basis $\underline{\underline{G}}_1 \ldots \underline{\underline{G}}_L$
* If code chain is closed under products, then $\Lambda_D$ is unique $= \Gamma$ [Kositwattanarerk-Oggier 14].

So... how close to a lattice can construction C get? ...

- ✗
- GU ?
- EDS ?
- EMD ✓
- EKN ?



So... how close to a lattice can construction C get ?...

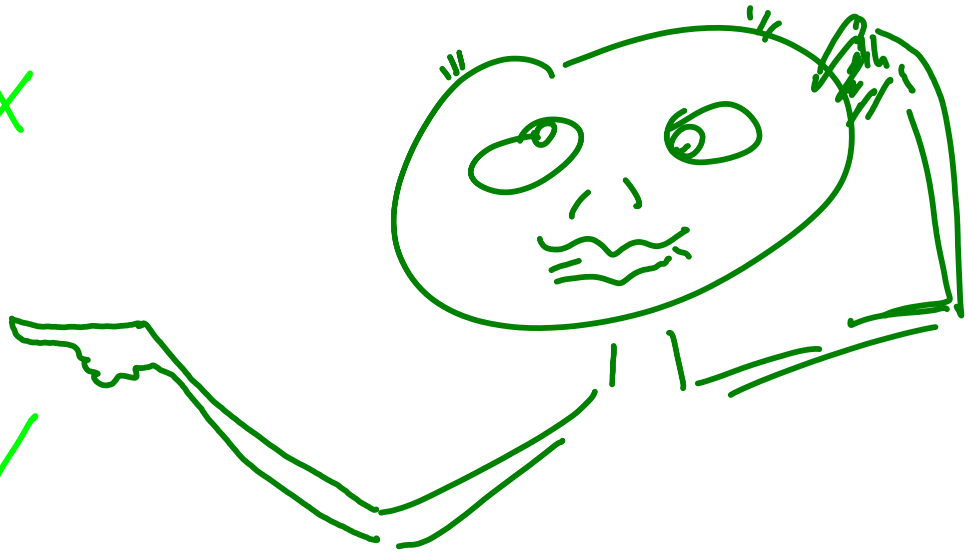
✗

GU ?

EDS

EMD ✓

EKN ?



# Construction $C$ is EDS for $L=2$ linear levels

(EDS = equi distance spectrum)

Def (equal vectors up to sign of some components):

$$(a_1 \dots a_n) \overset{\shortparallel}{=} (b_1 \dots b_n) \quad \text{if} \quad |a_i| = |b_i| \quad i=1\dots n.$$

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**Lemma:** Consider $\Gamma = C_1 + 2C_2 + 4\mathbb{Z}^n$, where $C_1, C_2$ - linear. Then, for any $\underline{x}, \underline{y}, \underline{x}' \in \Gamma$, there exist $\underline{y}' \in \Gamma$ s.t. the error vectors $\underline{y}'-\underline{x}'$ and $\underline{y}-\underline{x}$ are equal up to signs: $\underline{y}'-\underline{x}' \overset{\shortparallel}{=} \underline{y}-\underline{x}$. Furthermore, for $\underline{x}, \underline{x}' \in \Gamma$ and $\underline{e}$, the number of $\underline{y}'$s for which $\underline{y}'-\underline{x}' \overset{\shortparallel}{=} \underline{e}$ is equal to the number of $\underline{y}$s " " $\underline{y}-\underline{x} \overset{\shortparallel}{=} \underline{e}$.

**Corollary:** The distance spectrum is identical for all codewords in $\Gamma$: $N(c,d) = N(d) \quad \forall c \in \Gamma$.

**Proof:** First, choose component linear codes of $\underline{y}$ such that:

$$\underline{C}_{y'1} \oplus \underline{C}_{x'_1} = \underline{C}_{y1} \oplus \underline{C}_{x1}, \qquad \underline{C}_{y'_2} \oplus \underline{C}_{x'_2} = \underline{C}_{y2} \oplus \underline{C}_{x_2}$$

Note that: $\Delta C \in \{-1, 0, +1\} + \{-2, 0, +2\} \in \{-3, -2, -1, 0, +1, +2, +3\}$

$$\Delta 4Z \in \{\dots -8, -4, 0, +4, +8, \dots\}$$

$\Longrightarrow$ Can choose $z'$ such that $|\Delta C' + 4\Delta Z'| = |\Delta C + 4\Delta Z|$. ∎

Summary: what we know 

what we think 

and what we don't 

## Is it true also that for $L=2$ levels...

* Construction C is GU ?
* All possible const. D lattices have the same spectrum ?
* Spectra of const. C and D are equal ?
  (for the same component codes)

## Is it true that for $L \geq 3$ levels...

* Const. C is not GU ? not EDS ?
  (unless it is a lattice - [K-O-14] condition)
* Average spectrum of const. C $\not\equiv$ const. D ?
  (for the same component codes)
* If not, is const. C at least EKN ?

## Finally, is const. D better than C
## (for the same component codes) ? in what sense ?

The End