LUND UNIVERSITY

Threshold Saturation for Spatially Coupled Turbo-Like Codes over the Binary Erasure Channel

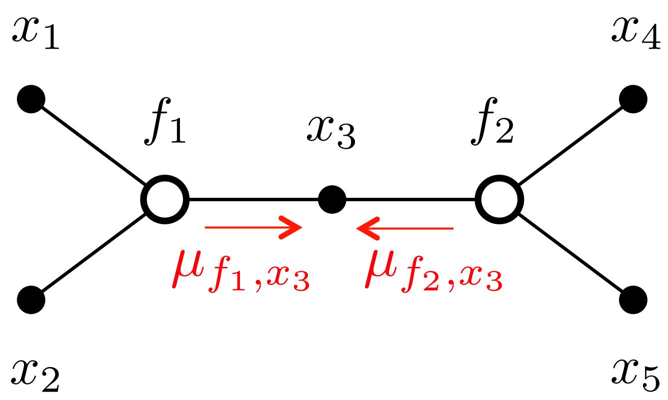
Saeedeh Moloudi†, Michael Lentmaier†, and Alexandre Graell i Amat‡

† Lund University, Lund, Sweden
‡ Chalmers University of Technology, Gothenburg, Sweden

Munich Workshop on Coding and Modulation
Munich, Germany, July 30, 2015

Background

- **Codes on graphs:**
  build large and powerful codes out of a set of smaller component codes



$$f(x_1, x_2, x_3, x_4, x_5) = f_1(x_1, x_2, x_3) \cdot f_2(x_3, x_4, x_5)$$

- **Message passing decoding:** (e.g. belief propagation)
  efficient parallel computation of marginals by local computations and exchange of messages within the factor graph

Question:
which influence does the structure of the component codes have and how simple or powerful should they be?

Convolutional Codes for Iterative Decoding

Parallel Concatenation:

\+ better BP decoding threshold
\- worse distance spectrum / error floor

\- worse MAP decoding threshold

Ensemble	Rate	ϵ_{BP}	ϵ_{MAP}
PCC	1/3	0.6428	0.6553
SCC	1/4	0.6896	0.7483

Serial Concatenation:

\- worse BP decoding threshold
\+ better distance spectrum / error floor

\+ better MAP decoding threshold

Ensemble	Rate	ϵ_{BP}	ϵ_{MAP}
PCC	1/3	0.6428	0.6553
SCC	1/3	0.6118	0.6615
PCC	1/2	0.4606	0.4689
SCC	1/2	0.4010	0.4973

Observation:

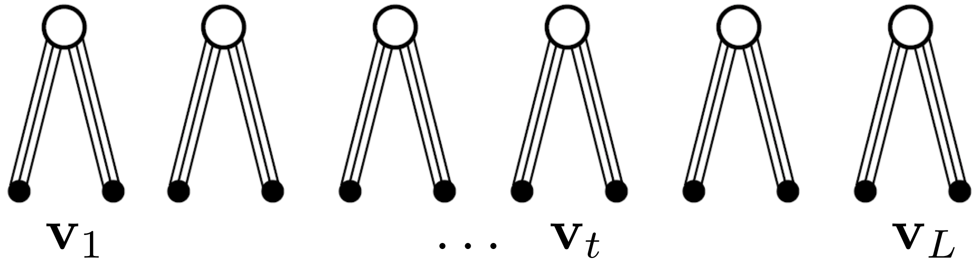
optimizing component codes for iterative decoding does not necessarily optimize the strength of the overall code

Spatial coupling:

can overcome this discrepancy due to threshold saturation

Spatially Coupled LDPC Codes

- Consider transmission of a sequence of codewords:

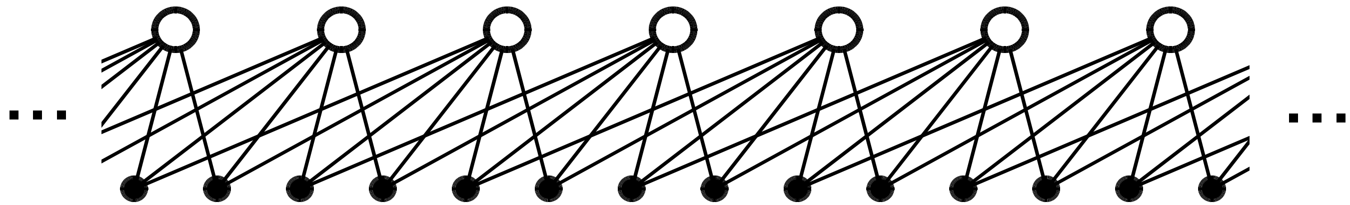


$$\mathbf{B} = [3, 3]$$

- Assuming a conventional LDPC code, each codeword satisfies:

$$\mathbf{v}_t \cdot \mathbf{H}^T = \mathbf{0} \quad \forall t = 1, \ldots, L$$

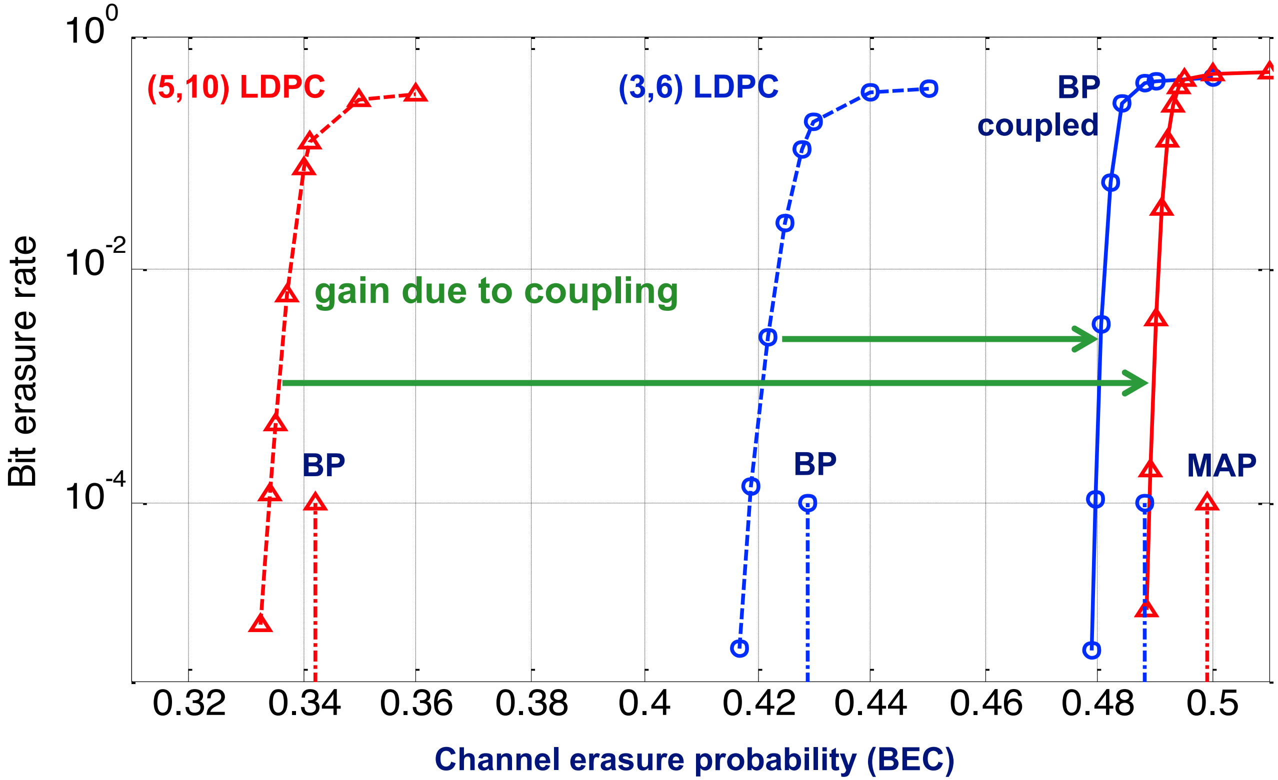
- **Spatial coupling:** codewords are interconnected with their neighbors



$$\mathbf{v}_t \cdot \mathbf{H}_0^T(t) + \mathbf{v}_{t-1} \cdot \mathbf{H}_1^T(t) + \cdots + \mathbf{v}_{t-m} \cdot \mathbf{H}_m^T(t) = \mathbf{0} \quad ,$$

Condition: $\mathbf{H}_0(t) + \mathbf{H}_1(t) + \cdots + \mathbf{H}_m(t) = \mathbf{H} \; \forall t$

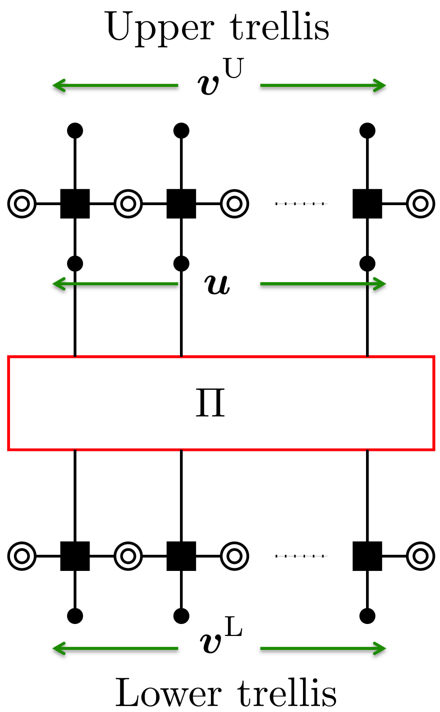
Threshold Saturation



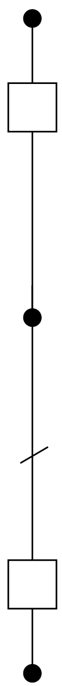
- Threshold saturation: as the coupling length L increases, the BP decoding threshold approaches the optimal MAP decoding threshold
- It follows that coupled regular LDPC codes achieve capacity as their node degrees tend to infinity

Parallel Concatenated Codes (PCCs)

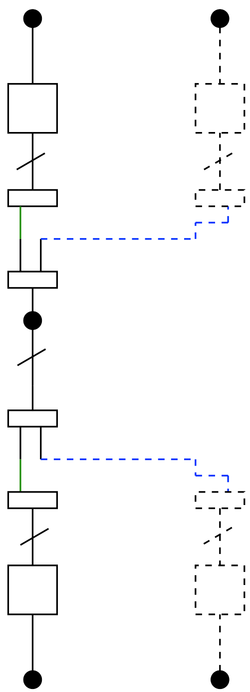
Factor graph:

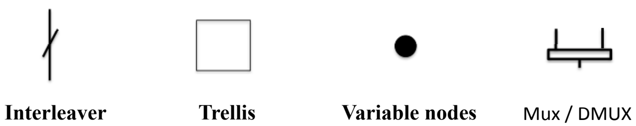


Compact form:



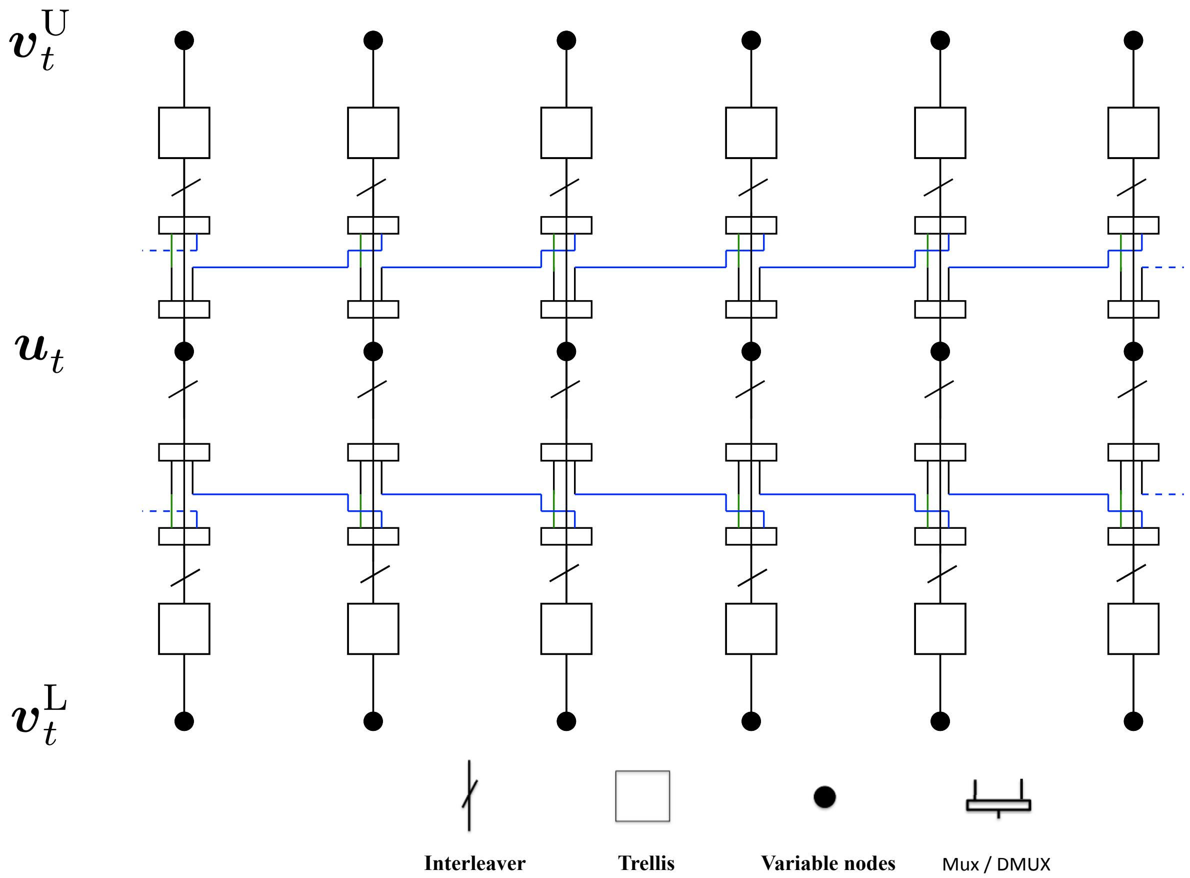
Coupling (edge spreading):



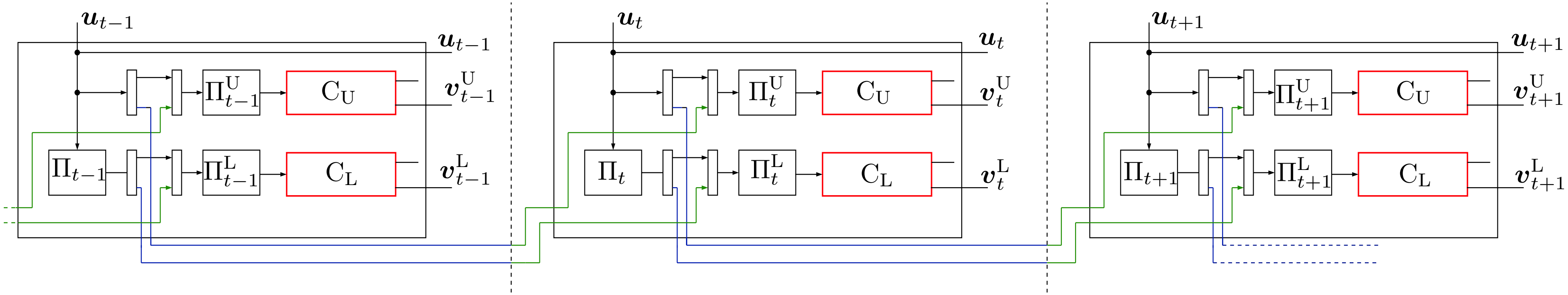


Construction of Spatially Coupled PCCs

1. Consider a sequence of L independent blocks
2. Interconnect (couple) neighboring blocks by spreading edges

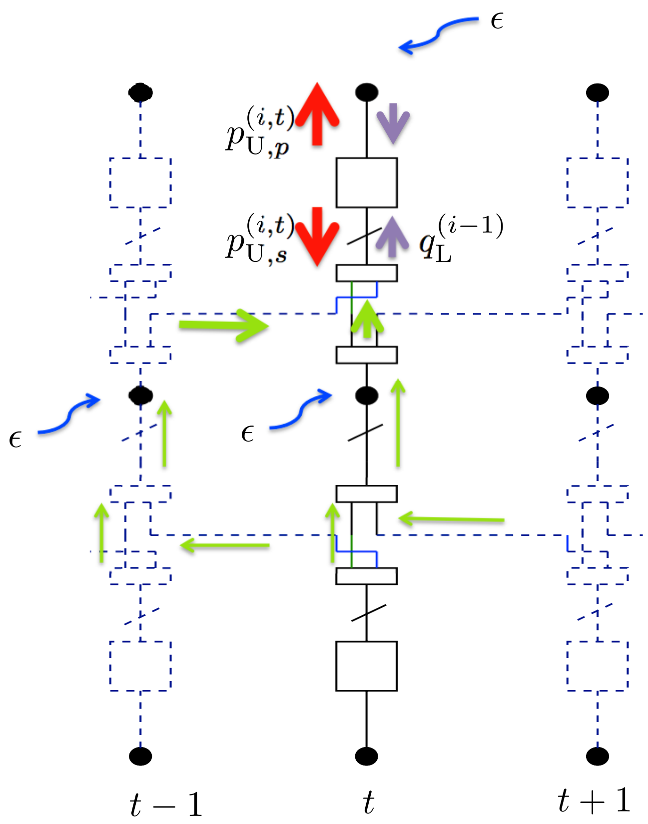


Spatially Coupled PCCs (SC-PCCs)



- Coupling memory $m=1$, can be generalized to $m>1$
- Termination:
  The information sequences at the end of the chain are chosen in such a way that the output sequence at time $t=L+1$ becomes $\mathbf{v}_{L+1}=0$
- Decoding works like for uncoupled PCCs, but the BCJR decoders exchange LLRs between different time slots

Density Evolution Analysis (BEC)



Constraint node update (trellis):

$$p_{\mathrm{U},s}^{(i,t)} = f_{\mathrm{U},s}\left(q_{\mathrm{L}}^{(i-1)}, \epsilon\right)$$

$$p_{\mathrm{U},p}^{(i,t)} = f_{\mathrm{U},p}\left(q_{\mathrm{L}}^{(i-1)}, \epsilon\right)$$

Variable node update:

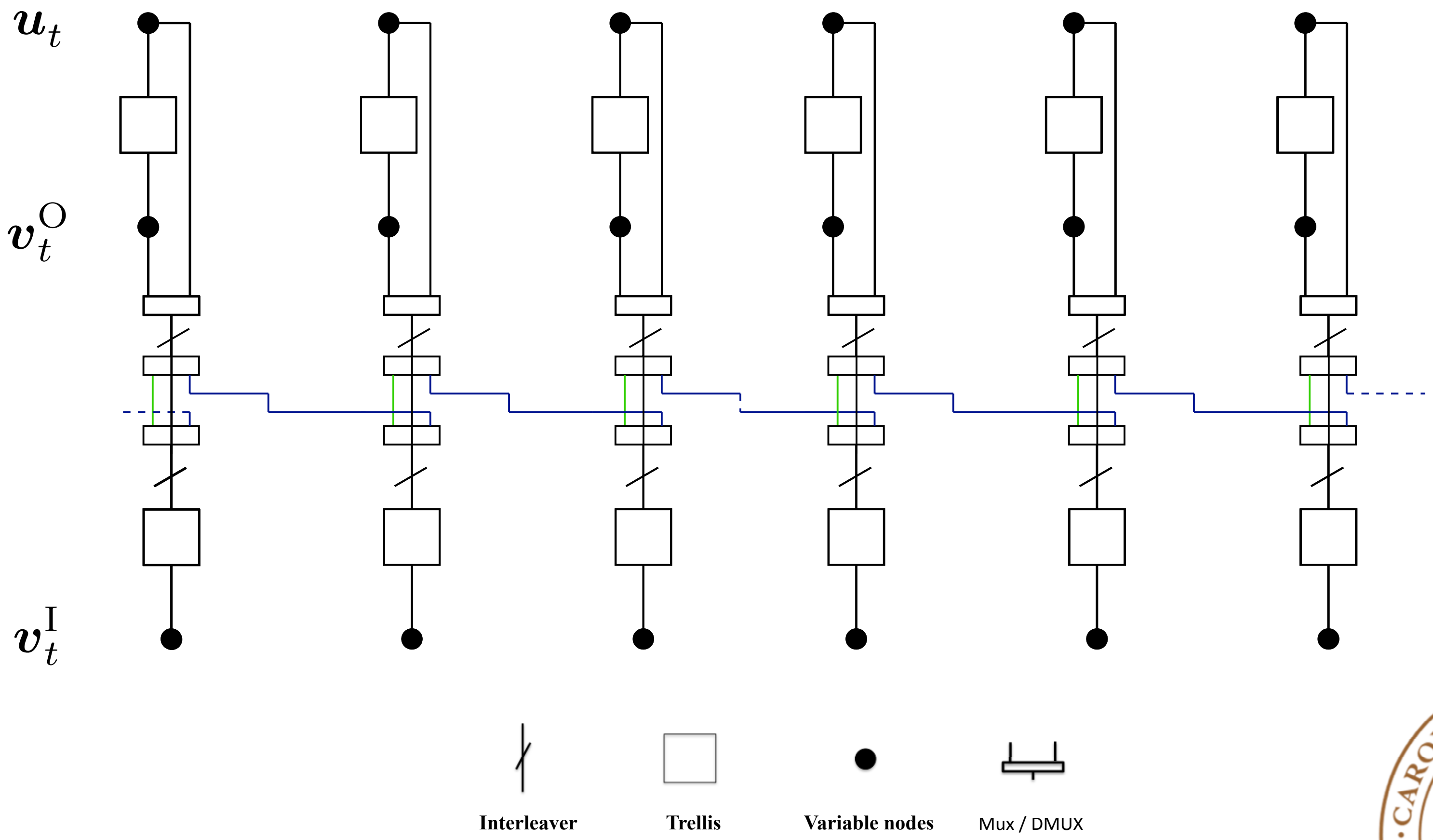
$$q_{\mathrm{L}}^{(i-1)} = \epsilon \cdot \frac{2p_{\mathrm{L},s}^{(i-1,t)} + p_{\mathrm{L},s}^{(i-1,t-1)} + p_{\mathrm{L},s}^{(i-1,t+1)}}{4}$$

For the BEC, it is possible to derive analytical expressions for the transfer functions [1].

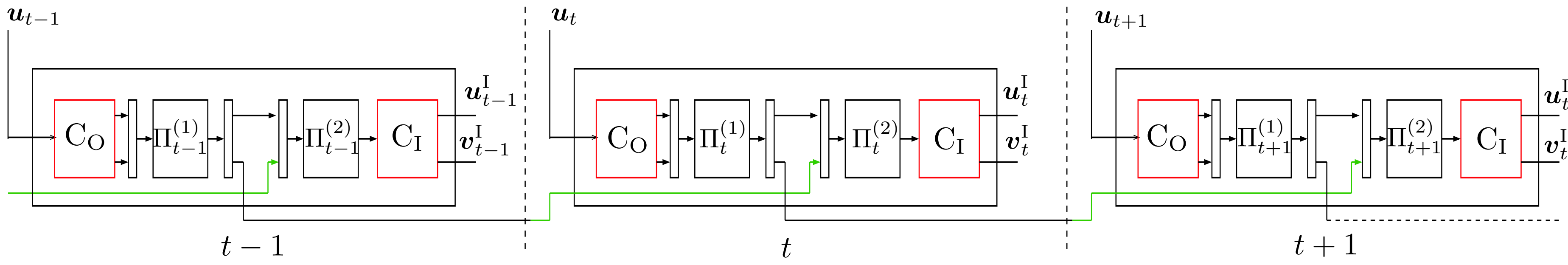
[1] B.M.Kurkoski, P.H.Siegel, and J.K.Wolf, "Exact probability of erasure and a decoding algorithm for convolutional codes on the binary erasure channel," in *Proc. IEEE Global Telecommunications Conference, 2003. GLOBECOM '03.*, Dec. 2003, vol. 3.

Spatially Coupled Serial Concatenated Codes

1. Consider a sequence of L independent blocks
2. Interconnect (couple) neighboring blocks by spreading edges



Spatially Coupled SCCs (SC-SCCs)



- Coupling memory $m=1$, can be generalized to $m>1$
- Termination:
  The information sequences at the end of the chain are chosen in such a way that the output sequence at time $t=L+1$ becomes $\mathbf{v}_{L+1}=0$
- Decoding works like for uncoupled PCCs, but the BCJR decoders exchange LLRs between different time slots

Numerical Results
(SC-TCs with 4-state components)

Thresholds for punctured SC-TCs with different coupling memories

Ensemble	Rate	ρ_2	ε_{BP}	ε_{MAP}	$\varepsilon^1_{\text{SC}}$	$\varepsilon^3_{\text{SC}}$	$\varepsilon^5_{\text{SC}}$	δ_{SH}
$\mathcal{C}_{\text{PCC}}/\mathcal{C}_{\text{SC-PCC}}$	$1/3$	1.0	0.6428	0.6553	0.6553	0.6553	0.6553	0.0113
$\mathcal{C}_{\text{SCC}}/\mathcal{C}_{\text{SC-SCC}}$	$1/3$	1.0	0.5405	0.6654	0.6437	0.6650	0.6654	0.0012
$\mathcal{C}_{\text{PCC}}/\mathcal{C}_{\text{SC-PCC}}$	$1/2$	0.5	0.4606	0.4689	0.4689	0.4689	0.4689	0.0311
$\mathcal{C}_{\text{SCC}}/\mathcal{C}_{\text{SC-SCC}}$	$1/2$	0.5	0.3594	0.4981	0.4708	0.4975	0.4981	0.0019
$\mathcal{C}_{\text{PCC}}/\mathcal{C}_{\text{SC-PCC}}$	$2/3$	0.25	0.2732	0.2772	0.2772	0.2772	0.2772	0.0561
$\mathcal{C}_{\text{SCC}}/\mathcal{C}_{\text{SC-SCC}}$	$2/3$	0.25	0.2038	0.3316	0.3303	0.3305	0.3315	0.0018
$\mathcal{C}_{\text{PCC}}/\mathcal{C}_{\text{SC-PCC}}$	$3/4$	0.166	0.1854	0.1876	0.1876	0.1876	0.1876	0.0624
$\mathcal{C}_{\text{SCC}}/\mathcal{C}_{\text{SC-SCC}}$	$3/4$	0.166	0.1337	0.2486	0.2155	0.2471	0.2486	0.0014
$\mathcal{C}_{\text{PCC}}/\mathcal{C}_{\text{SC-PCC}}$	$4/5$	0.125	0.1376	0.1391	0.1391	0.1391	0.1391	0.0609
$\mathcal{C}_{\text{SCC}}/\mathcal{C}_{\text{SC-SCC}}$	$4/5$	0.125	0.0942	0.1990	0.1644	0.1968	0.1989	0.0011

Observation:

threshold saturation occurs for large enough coupling memory m

[2] A. Graell i Amat, S. Moloudi, and M. Lentmaier,
“Spatially Coupled Turbo Codes: Principles and Finite Length Performance,” ISWCS 2014.

GLDPC Codes from Convolutional Codes

- **Braided block codes (BCCs)** [3]:
  a class of generalized low-density parity-check (GLDPC) convolutional codes that are closely related to product codes.
- BBCs with BCH component codes were recently considered for high speed optical communications [4].
- In this work we consider a counterpart of BBCs called braided convolutional codes (BCCs) [5].
- BCCs can be seen as a class of turbo-like GLDPC codes, i.e., a bridge between turbo codes and LDPC codes.

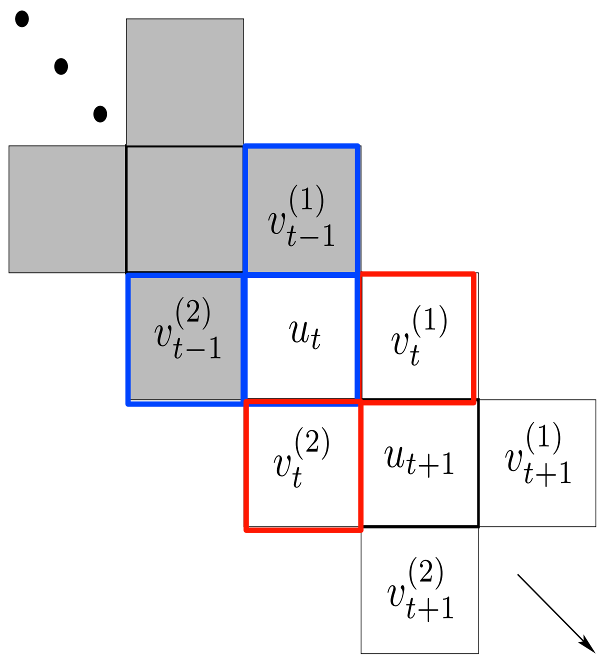
[3] A.J. Feltstrom, D. Truhachev, M. Lentmaier, and K.Sh. Zigangirov, “Braided block codes,” *IEEE Trans. Inf. Theory*, vol. 55, no. 6, pp. 2640–2658, June 2009.

[4] Y.Y. Jian, H.D. Pfister, K.R. Narayanan, R. Rao, and R. Mazahreh, “Iterative hard decision decoding of braided BCH codes for high-speed optical communication,” in Proc. IEEE Global Telecommunications Conference, 2013. GLOBECOM ’13., Dec. 2013.

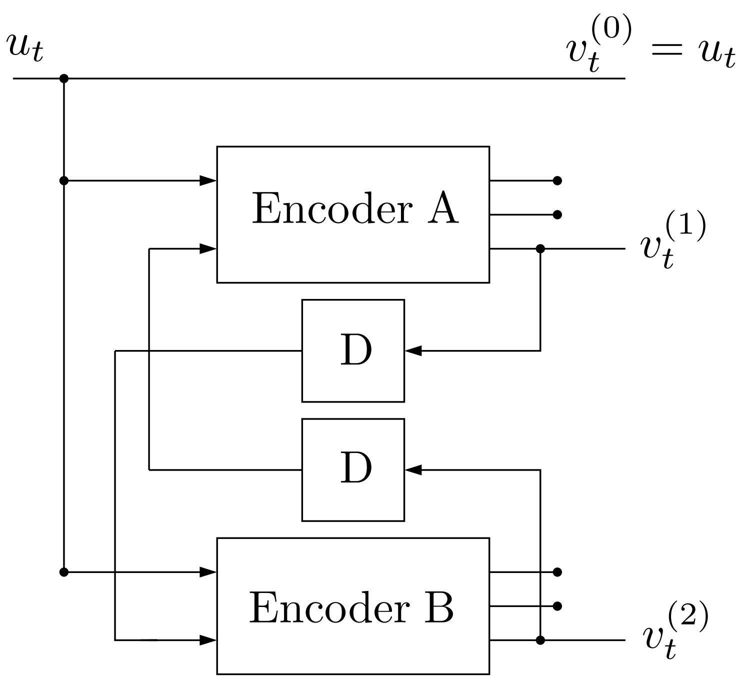
[5] W. Zhang, M. Lentmaier, K.Sh. Zigangirov, and D.J. Costello, Jr., “Braided convolutional codes: a new class of turbo-like codes,” IEEE Trans. Inf. Theory, vol. 56, no. 1, pp. 316–331, Jan. 2010.

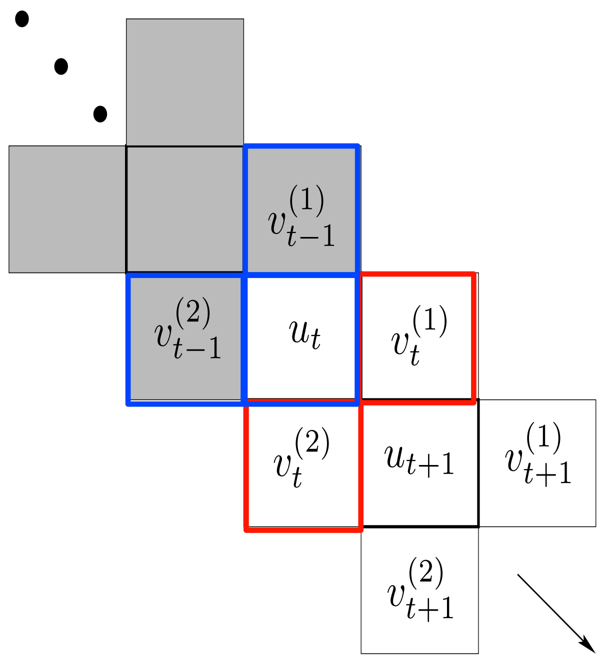
BCCs and product codes (Tightly BCC)

- Data is written in an infinite two-dimensional array with three diagonal ribbons.
- Rows and columns are coded by separate convolutional encoders with rate 2/3.
- Shaded squares contain the previous inputs and outputs, which are assumed to be known.

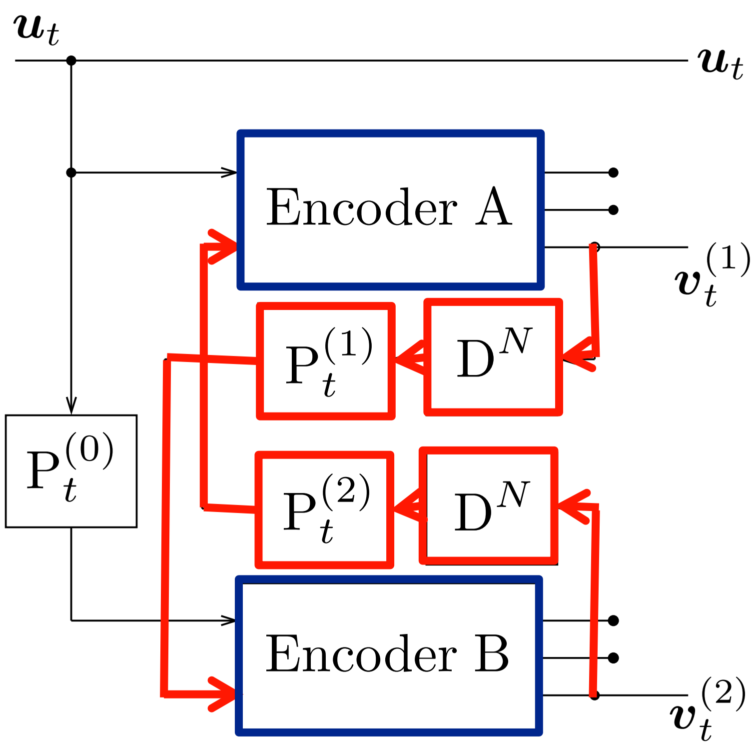


BCCs and product codes (Tightly BCC)





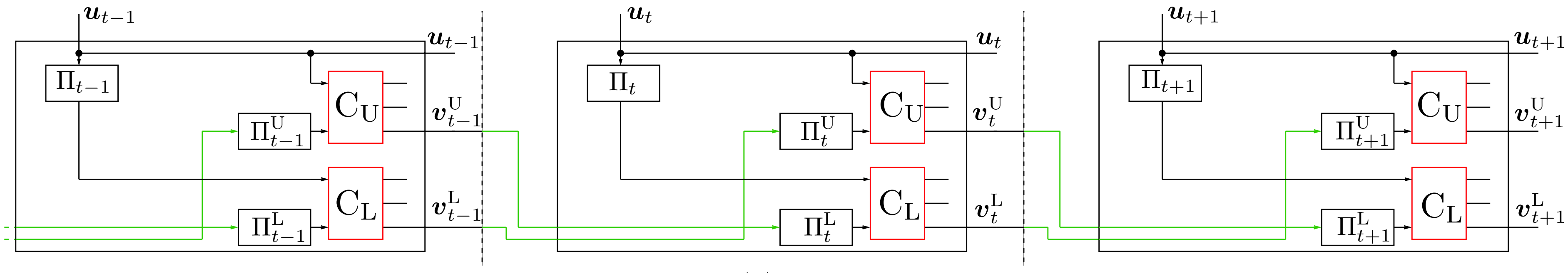
Sparsely braided convolutional codes (SBCC)



Usually data is transmitted in packets

➡ blockwise version of BCCs
block length: N

BCCs are inherently spatially coupled

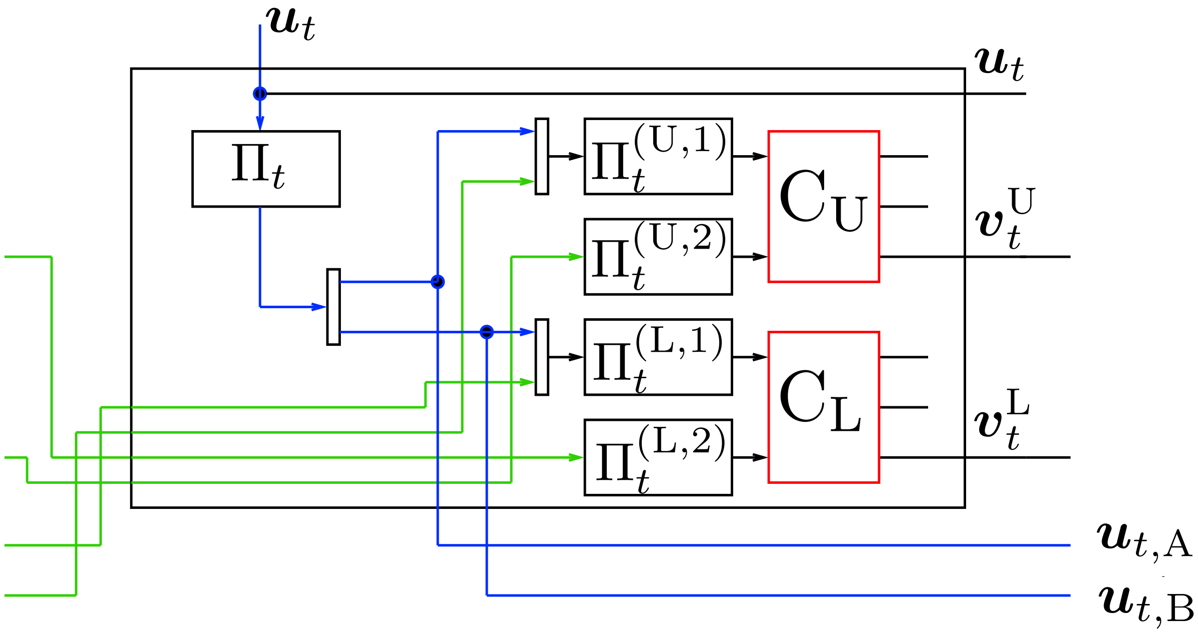


- Consider transmission of sequence of L blocks.
- At time t, encoders use the outputs of previous encoders
- BCCs are inherently spatially coupled

- Encoders are terminated
- The coded blocks at times $t < 1$ and $t > L$ are equal to zero.

- Uncoupled case: Consider circular structure.
  Can be defined by using tailbiting with $L = 1$.

Coupling of Information Symbols



Type II BCCs

Information sequence is divided randomly into $\boldsymbol{u}_{t,\mathrm{A}}$ and $\boldsymbol{u}_{t,\mathrm{B}}$

Upper encoder input: $(\boldsymbol{u}_{t,\mathrm{A}}, \boldsymbol{u}_{t-1,\mathrm{B}})$

Lower encoder input: $(\boldsymbol{u}_{t,\mathrm{B}}, \boldsymbol{u}_{t-1,\mathrm{A}})$

Thresholds of Type-I and Type-II BCCs

	ϵ_{BP}	ϵ_{MAP}	ϵ_{SC}			
			$m=1$	$m=3$	$m=5$	$m=7$
Type-I	0.55414	0.66539	0.66094	0.66447	0.66506	0.66524
Type-II	0.55414	0.66539	0.66534	0.66538	0.66539	0.66539

→ threshold saturation occurs for larger m [6].

Comparison with coupled turbo codes (parallel and serial) [7]:

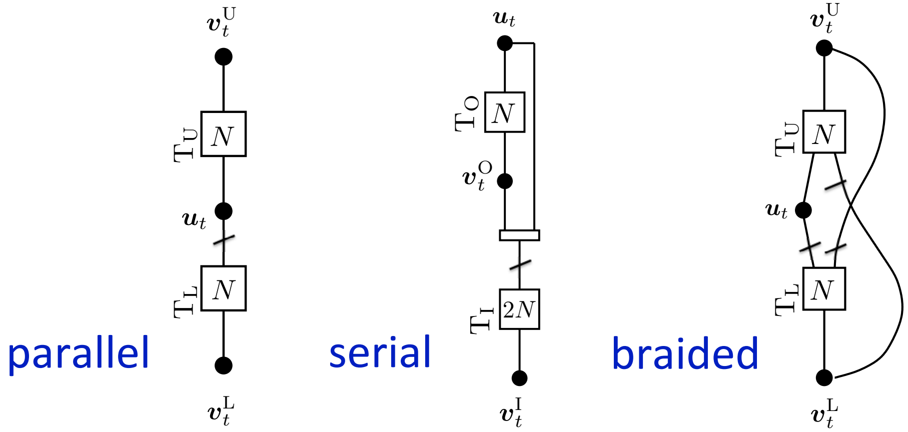
	ϵ_{BP}	ϵ_{MAP}	ϵ_{SC}		
			$m=1$	$m=3$	$m=5$
SC-PCCs	0.64282	0.65538	0.65538	0.65538	0.65538
SC-SCCs	0.61184	0.66154	0.65190	0.66140	0.66153

[6] M. Lentmaier, S. Moloudi, and A. Graell i Amat,
"Braided Convolutional Codes – A Class of Spatially Coupled Turbo-Like Codes," SPCOM, 2014.
[7] S. Moloudi, M. Lentmaier, and A. Graell i Amat, "Spatially coupled turbo codes,"
in Proc. 8th International Symposium on Turbo Codes & Iterative Information Processing, 2014.

Proof of Threshold Saturation

- Turbo-like codes as scalar admissible systems

Idea: formulate DE equations as a recursion in the form below



Definition [8]: A scalar admissible system (f, g), is defined by the recursion

$$x^{(i)} = f\Big(g(x^{(i-1)}); \varepsilon\Big),$$

where $f : [0,1] \times [0,1] \to [0,1]$ and $g : [0,1] \to [0,1]$ satisfy the following conditions:

- f is increasing in both arguments $x, \varepsilon \in (0,1]$
- g is increasing in $x \in (0,1]$
- $f(0;\varepsilon) = f(x;0) = g(0) = 0$
- f and g have continuous second derivatives

[8] A.Yedla, Y.-Y. Jian, P.S. Nguyen, and H.D. Pfister, "A simple proof of threshold saturation for coupled scalar recursions," in Proc. Intern. Symp. on Turbo Codes & Iterative Inform. Proc. 2012.

Proof of Threshold Saturation

- Potential function:

$$U(x;\varepsilon) = \int_0^x \big(z - f(g(x);\varepsilon)\big) g'(z) dz$$
$$= x g(x) - G(x) - F(g(x);\varepsilon),$$

where $F(x;\varepsilon) = \int_0^x f(z;\varepsilon)dz$ and $G(x) = \int_0^x g(z)dz$.

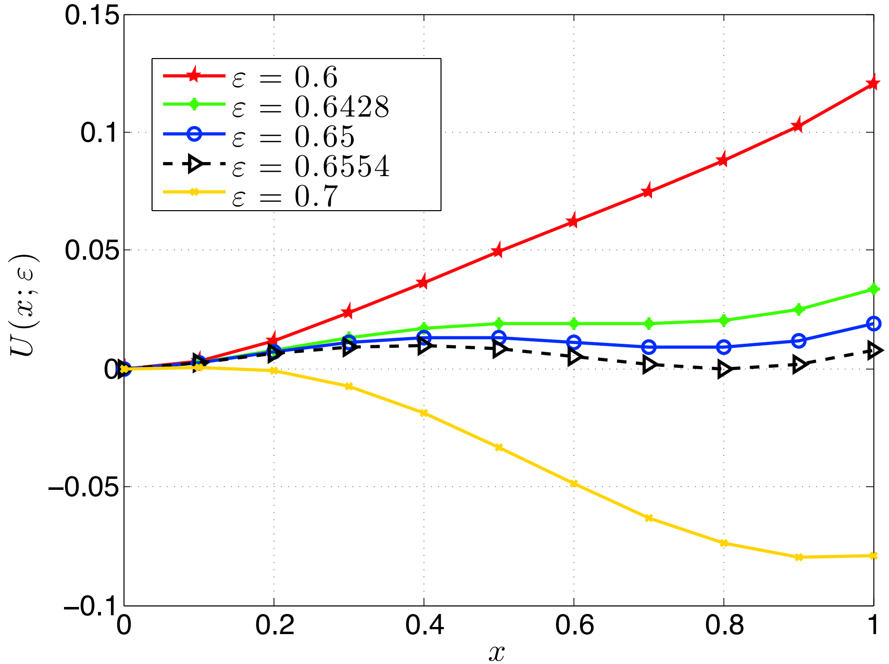
- Property:
a fixed point of the DE recursion ($x = f(g(x);\varepsilon)$) corresponds to a stationary point of the corresponding potential function $U(x;\varepsilon)$

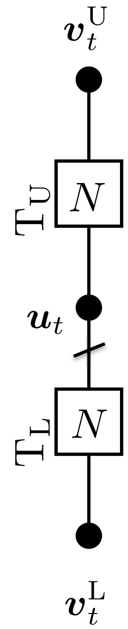
- Threshold saturation can be proved using the potential function [9], following the steps of Yedla, Pfister et al. in [8].

[9] S. Moloudi, M. Lentmaier, and A. Graell i Amat, "Threshold Saturation for Spatially Coupled Turbo-like Codes over the Binary Erasure Channel," to appear in Proc. ITW 2015.

Potential Function

Example: parallel concatenation, G=(1,5/7)





BP threshold:

$$\varepsilon^{\mathrm{BP}} = \sup\left\{\varepsilon \in [0,1] | U'(x;\varepsilon) > 0,\ \forall x \in (0,1]\right\}$$

Potential threshold:

$$\varepsilon^{*} = \sup\left\{\varepsilon \in [0,1] | U(x;\varepsilon) \geq 0,\ \forall x \in [0,1]\right\}.$$

As expected, the values coincide with the BP and MAP thresholds calculated by DE and the area theorem

Conclusions

- We considered various spatially coupled turbo-like codes:
  - Parallel concatenated codes
  - Serial concatenated codes
  - Braided convolutional codes

- Exact density evolution is feasible for the BEC
- Threshold saturation is observed for all three classes

- Optimizing component codes for iterative decoding does not necessarily optimize the strength of the overall code
- Spatial coupling can overcome this discrepancy due to threshold saturation: new design criteria possible