Combination / Interaction of Coded Modulation and Precoding in MIMO Systems

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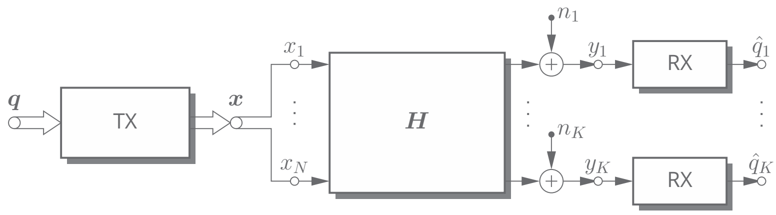
Introduction

Situation: point-to-multipoint transmission, *broadcast channel*

- K non-cooperating single-antenna users
- central base station with N transmit antennas
  ⇒ *joint generation of the TX signals*



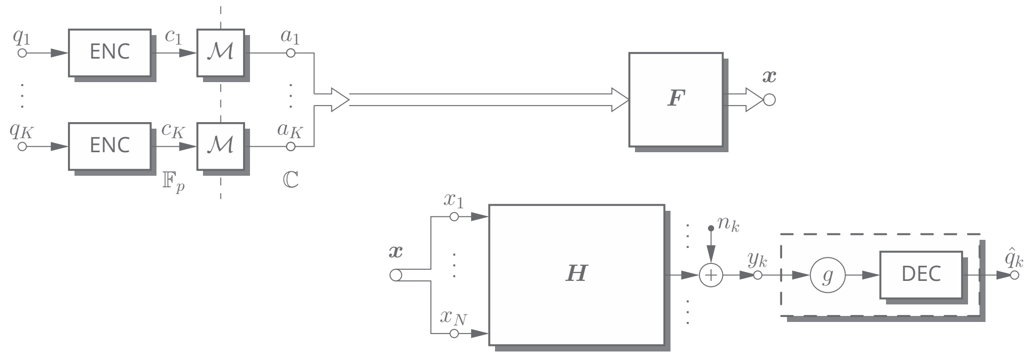
Precoding Schemes:

- linear preequalization
- Tomlinson–Harashima precoding (THP)
- lattice-reduction-aided (LRA) / integer-forcing (IF) precoding
- vector precoding

Question: *How to combine precoding schemes with coded modulation?*

Precoding

Linear Preequalization:



- simple but poor performance
- equalization and channel coding completely decoupled
- MMSE linear preequalization

$$\boldsymbol{F} = \frac{1}{g}\boldsymbol{H}^{\mathsf{H}}\left(\boldsymbol{H}\boldsymbol{H}^{\mathsf{H}} + \frac{\sigma_n^2}{\sigma_a^2}\boldsymbol{I}\right)^{-1}$$

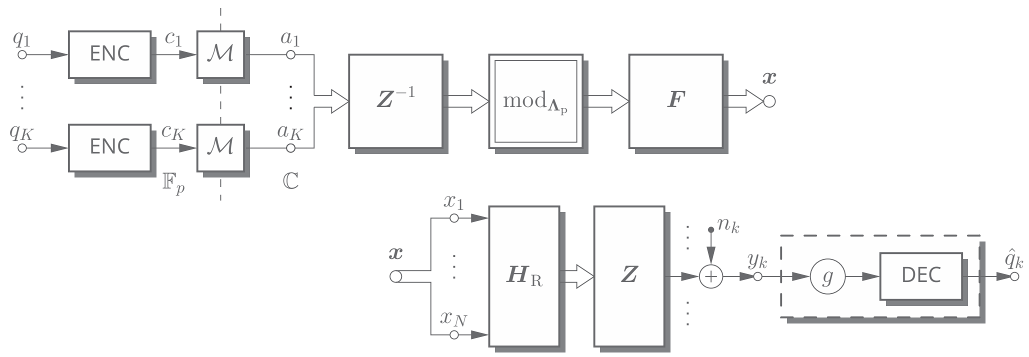
$\sigma_a^2 = \mathrm{E}\{|a_k|^2\}$; $\sigma_n^2 = \mathrm{E}\{|n_k|^2\}$

g chosen to meet sum power constraint

Precoding (II)

Lattice-Reduction-Aided Precoding:

[YW'02], [WF'03], [WFH'04]



- idea: ***factorize the channel matrix*** — *integer matrix* $\boldsymbol{Z} \in \mathbb{G}^{K\times K} = (\mathbb{Z} + \mathrm{j}\mathbb{Z})^{K\times K}$

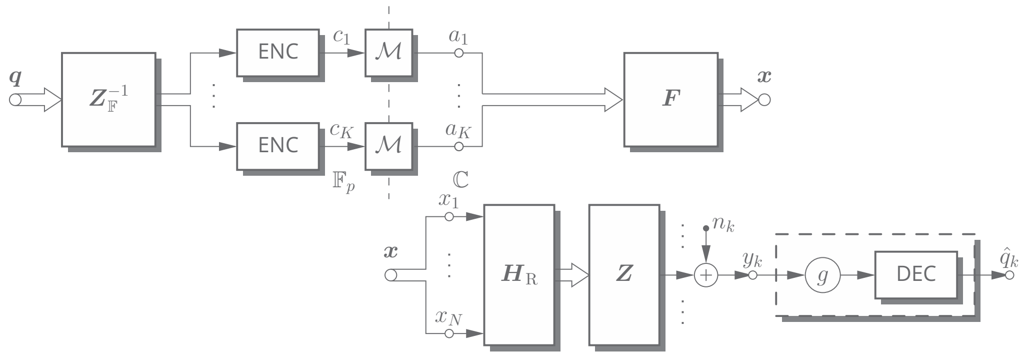
$$\boldsymbol{H} = \boldsymbol{Z}\boldsymbol{H}_{\mathrm{R}}$$

- linearly equalize the *reduced part* $\boldsymbol{H}_{\mathrm{R}}$
- equalize the integer part via ***modulo precoding***

⇒ ***constraints on the signal constellation / coding scheme***

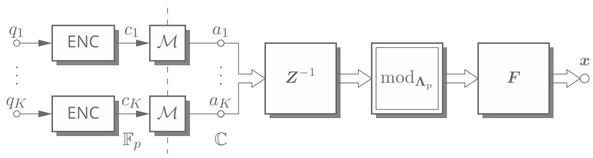
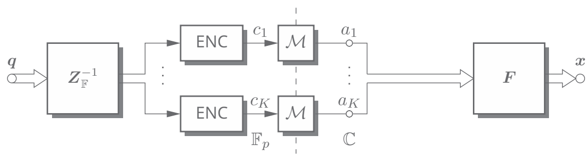
Precoding (III)

Integer-Forcing Precoding:



- dual to *integer-forcing equalization* [ZNEG'14], [HNS'14]
- tightly related to *compute-and-forward* strategy in relaying [NG'11], [FSK'13]
- and *reverse compute-and-forward* in distributed antenna systems [HC'13]
- usually studied for one-dim. p-ary constell., p prime; [ZNEG'14], [NG'11]
  or two-dim. p^2-ary constellations; [HC'13]
  general case by imposing *algebraic structure* on the signal constell. [FSK'13]

Precoding (IV)

Lattice-Reduction-Aided Precoding	Integer-Forcing Precoding
	
treat integer interference over	
$\mathbb{Z}$ or $\mathbb{G}$	$\mathbb{F}_p$
constraint on signal constellation	
periodic continuation required	match between arithmetic in $\mathbb{R}$ or $\mathbb{C}$ and $\mathbb{F}_p$ one-dim. p-ary constellation, p a prime
factorization	
$\boldsymbol{H} = \boldsymbol{Z}\,\boldsymbol{H}_{\rm R}$ $\lvert\det(\boldsymbol{Z})\rvert = 1$ lattice reduction shortest basis problem	$\boldsymbol{Z} = \underset{\substack{\boldsymbol{Z}=[\boldsymbol{z}_1\cdots\boldsymbol{z}_N]\in\mathbb{G}^{N\times N}\\ \mathrm{rank}(\boldsymbol{Z})=N}}{\operatorname{argmin}} \max_m \boldsymbol{z}_m^{\mathsf{H}}\left(\boldsymbol{H}\boldsymbol{H}^{\mathsf{H}} + \frac{\sigma_n^2}{\sigma_a^2}\boldsymbol{I}\right)^{-\frac{1}{2}}$ noise enhancement shortest independent vector problem
usually treated uncoded	incorporation of coding

Signal Constellations

Conventional Design:

- signal point lattice

  $$\mathbf{\Lambda}_\mathrm{a}$$

  here:

  $$\mathbf{\Lambda}_\mathrm{a} = \mathbb{G} = \mathbb{Z} + \mathrm{j}\,\mathbb{Z}$$

  Gaussian Integers

- precoding lattice

  $$\mathbf{\Lambda}_\mathrm{p}$$

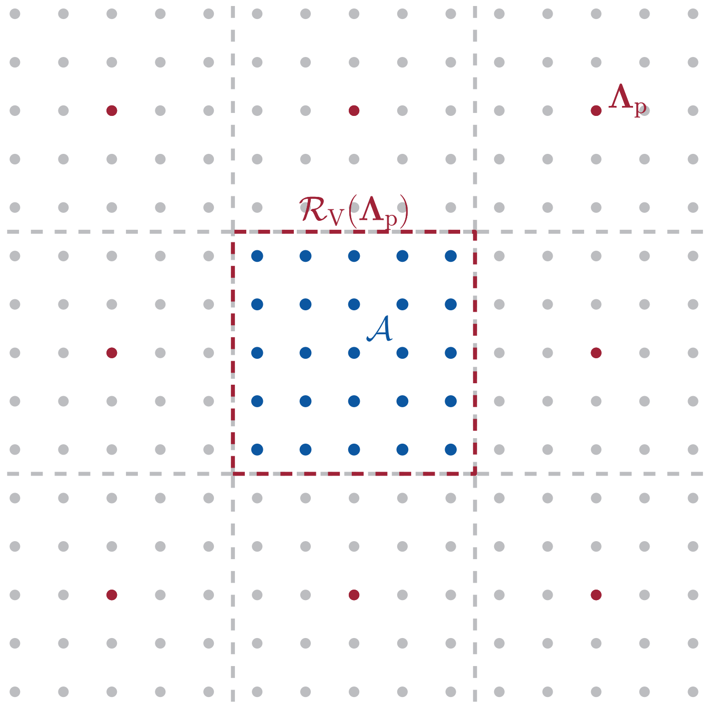
  sublattice of $\mathbf{\Lambda}_\mathrm{a}$

- Voronoi region

  $$\mathcal{R}_\mathrm{V}(\mathbf{\Lambda}_\mathrm{p})$$

- signal constellation

  $$\mathcal{A} = \mathbf{\Lambda}_\mathrm{a} \cap \mathcal{R}_\mathrm{V}(\mathbf{\Lambda}_\mathrm{p})$$



Signal Constellations (II)

Conventional Design:

- signal point lattice

  $$\mathbf{\Lambda}_\mathrm{a}$$

  here: ($\omega = \mathrm{e}^{\mathrm{j}2\pi/3}$)

  $$\mathbf{\Lambda}_\mathrm{a} = \mathbb{E} = \mathbb{Z} + \omega\,\mathbb{Z}$$

  Eisenstein Integers

- precoding lattice

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  sublattice of $\mathbf{\Lambda}_\mathrm{a}$

- Voronoi region

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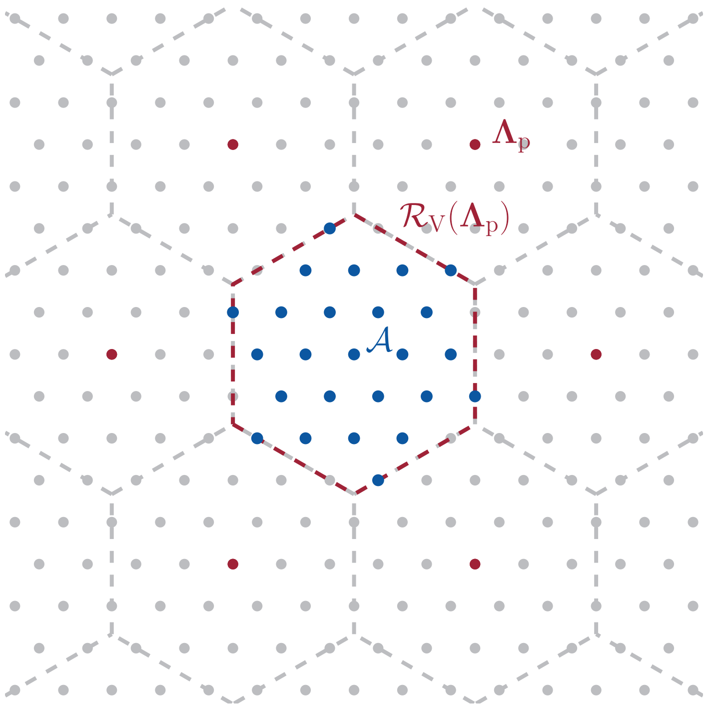


Signal Constellations (III)

8

How to choose Λ_{p}:

- usually

$$\boldsymbol{\Lambda}_{\mathrm{p}} = \Theta \cdot \boldsymbol{\Lambda}_{\mathrm{a}}$$

with $\Theta \in \mathbb{R}$

- new degree of freedom

$$\Theta = a + \mathrm{j}\, b \in \mathbb{G}$$

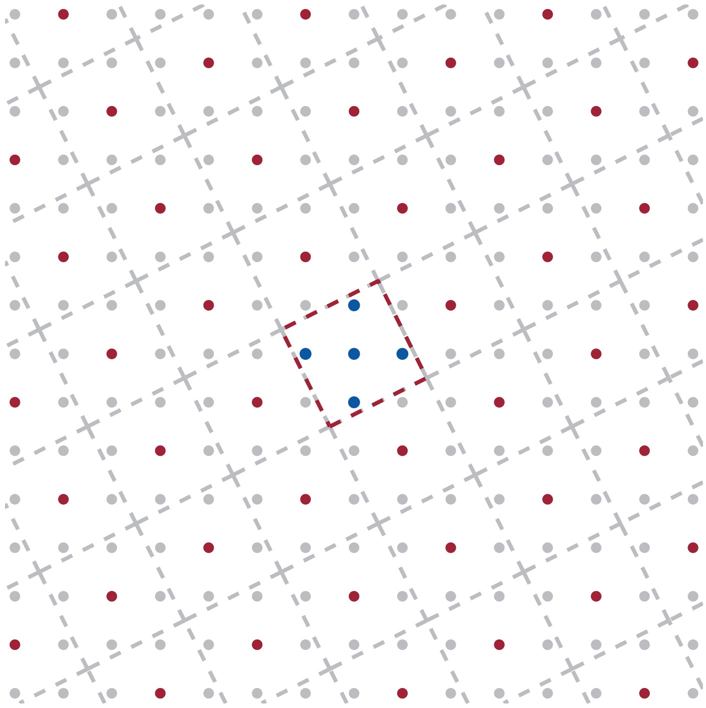
- for primes p, $\mathrm{rem}_4(p) = 1$, ***Gaussian primes*** Θ exits

$$p = a^2 + b^2 = |\Theta|^2$$

⇒ algebraic property [H'94]

$$\mathcal{A} \bmod \boldsymbol{\Lambda}_{\mathrm{p}} \;\simeq\; \mathbb{F}_p$$

- $|\mathcal{A}| = p = |\Theta|^2$
- $p = 5, \;\; \Theta = 2 + \mathrm{j}$



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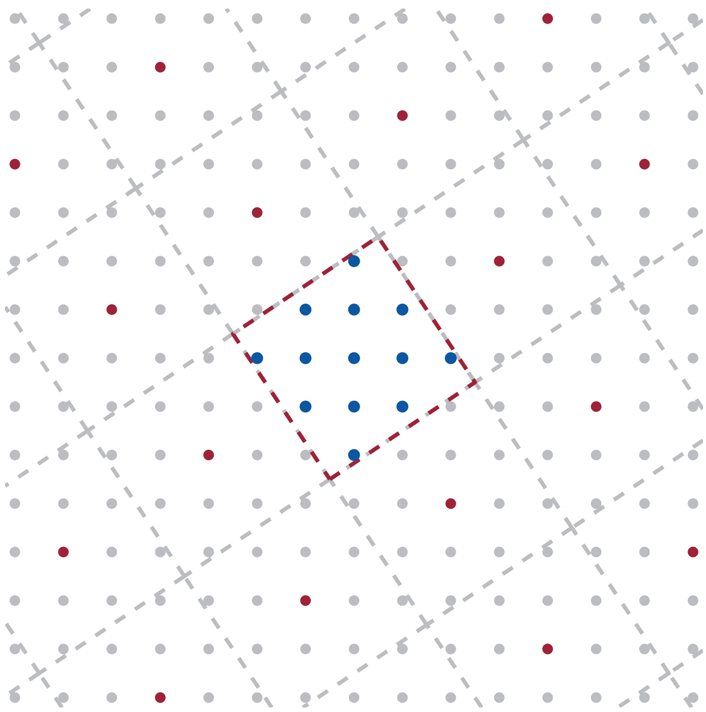
- $p = 13, \;\; \Theta = 3 + \mathrm{j}\,2$



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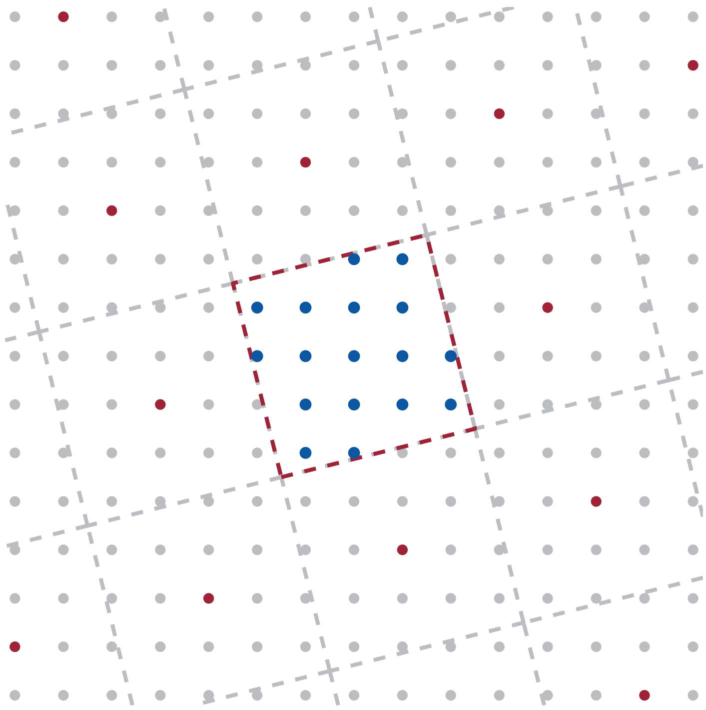
- for primes p, $\mathrm{rem}_4(p) = 1$, ***Gaussian primes*** Θ exits
$$p = a^2 + b^2 = |\Theta|^2$$
⇒ algebraic property [H'94]
$$\mathcal{A} \bmod \boldsymbol{\Lambda}_\mathrm{p} \;\simeq\; \mathbb{F}_p$$
- $|\mathcal{A}| = p = |\Theta|^2$
- $p = 17,\;\; \Theta = 4 + \mathrm{j}$



Signal Constellations (III)

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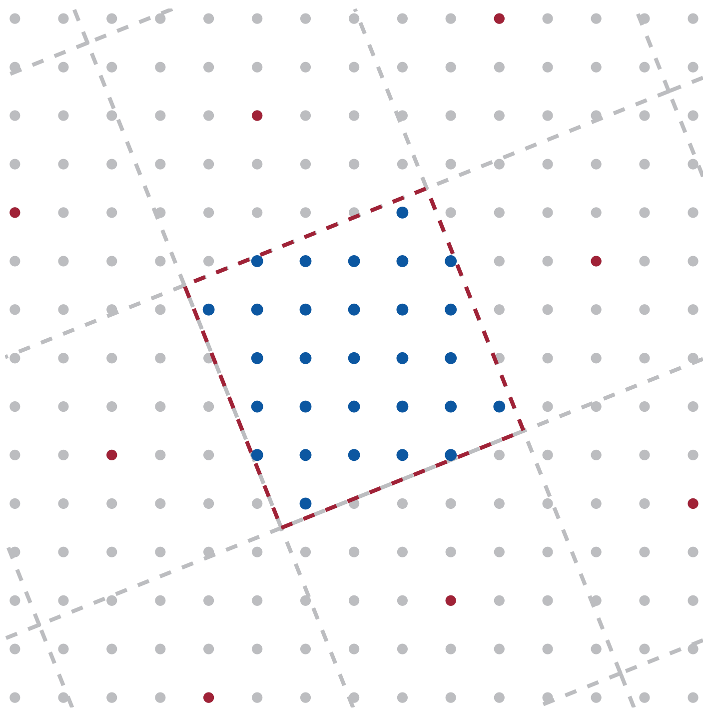
⇒ algebraic property [H'94]
$$\mathcal{A} \bmod \boldsymbol{\Lambda}_\mathrm{p} \;\simeq\; \mathbb{F}_p$$
- $|\mathcal{A}| = p = |\Theta|^2$
- $p = 29,\;\; \Theta = 5 + \mathrm{j}\,2$



Signal Constellations (III)

How to choose $\Lambda_{\rm p}$:

- usually

$$\boldsymbol{\Lambda}_{\rm p} = \Theta \cdot \boldsymbol{\Lambda}_{\rm a}$$

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- new degree of freedom

$$\Theta = a + {\rm j}\, b \in \mathbb{G}$$

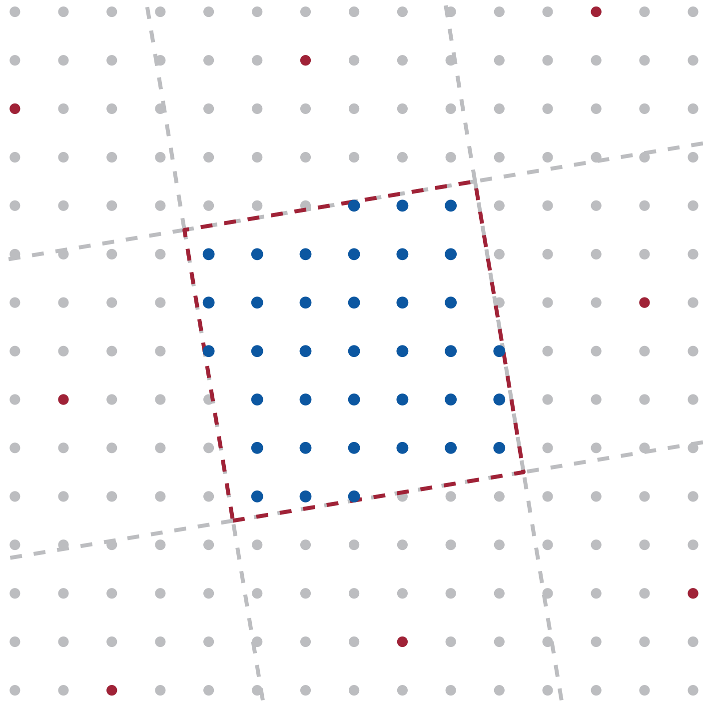
- for primes p, $\text{rem}_4(p) = 1$, ***Gaussian primes*** Θ exits

$$p = a^2 + b^2 = |\Theta|^2$$

  ⇒ algebraic property [H'94]

$$\mathcal{A} \bmod \boldsymbol{\Lambda}_{\rm p} \;\simeq\; \mathbb{F}_p$$

- $|\mathcal{A}| = p = |\Theta|^2$
- $p = 37,\;\; \Theta = 6 + {\rm j}$



Signal Constellations (IV)

How to choose $\Lambda_{\rm p}$:

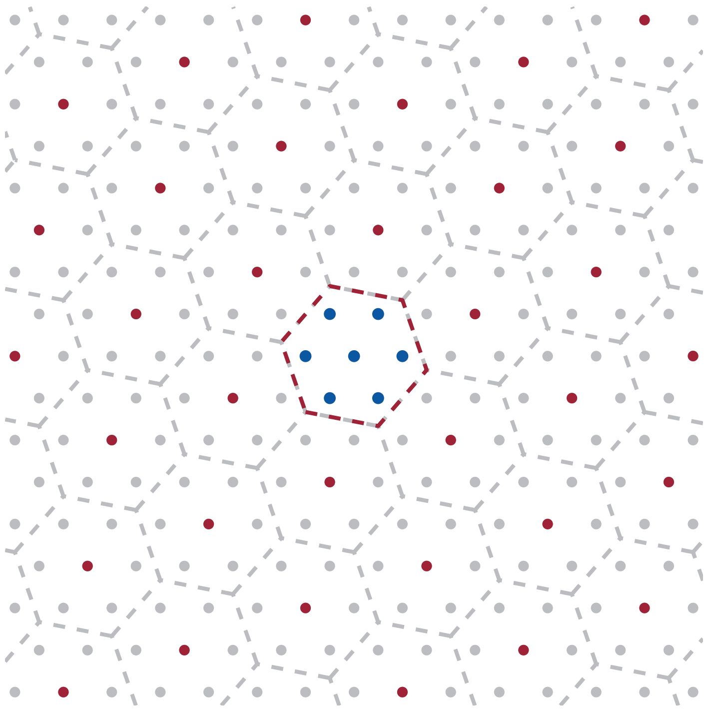
- for primes p, $\text{rem}_6(p) = 1$, ***Eisenstein primes*** Θ exits

$$p = |a + \omega\, b|^2 = |\Theta|^2$$

  ⇒ algebraic property

$$\mathcal{A} \bmod \boldsymbol{\Lambda}_{\rm p} \;\simeq\; \mathbb{F}_p$$

- $|\mathcal{A}| = p = |\Theta|^2$
- $p = 7,\;\; \Theta = 3 + \omega$



Signal Constellations (IV)

How to choose Λ_p:

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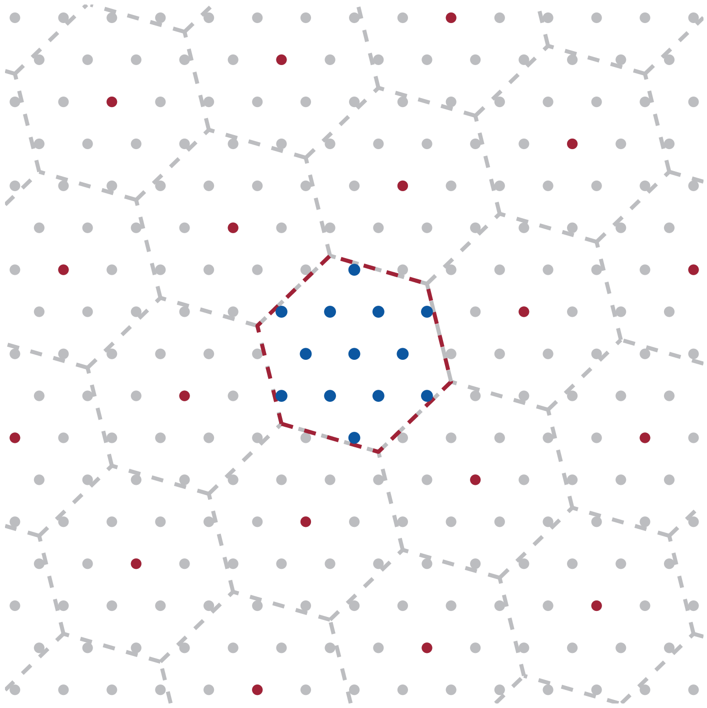

Signal Constellations (IV)

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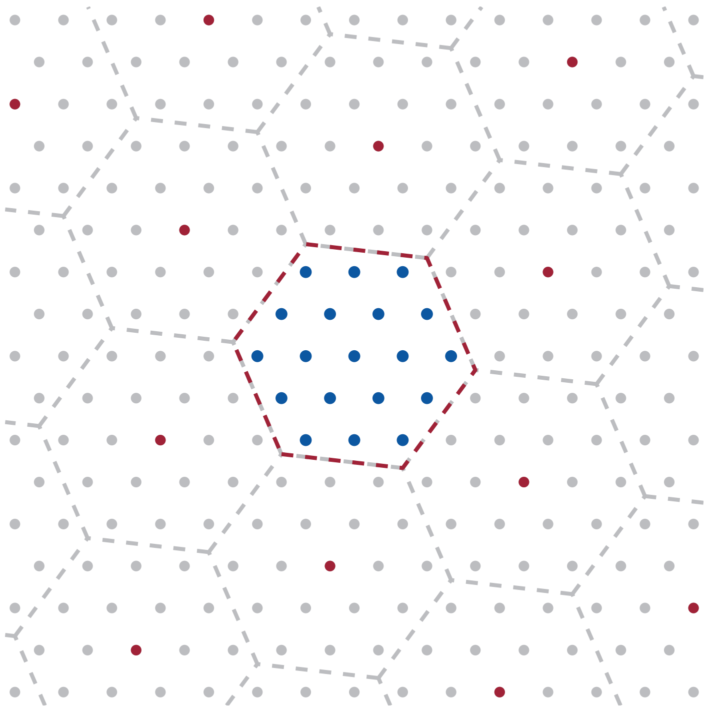
  ⇒ algebraic property

  $$\mathcal{A} \bmod \boldsymbol{\Lambda}_\mathrm{p} \;\simeq\; \mathbb{F}_p$$

- $|\mathcal{A}| = p = |\Theta|^2$
- $p = 19,\;\; \Theta = 5 + \omega\, 2$



Signal Constellations (IV)

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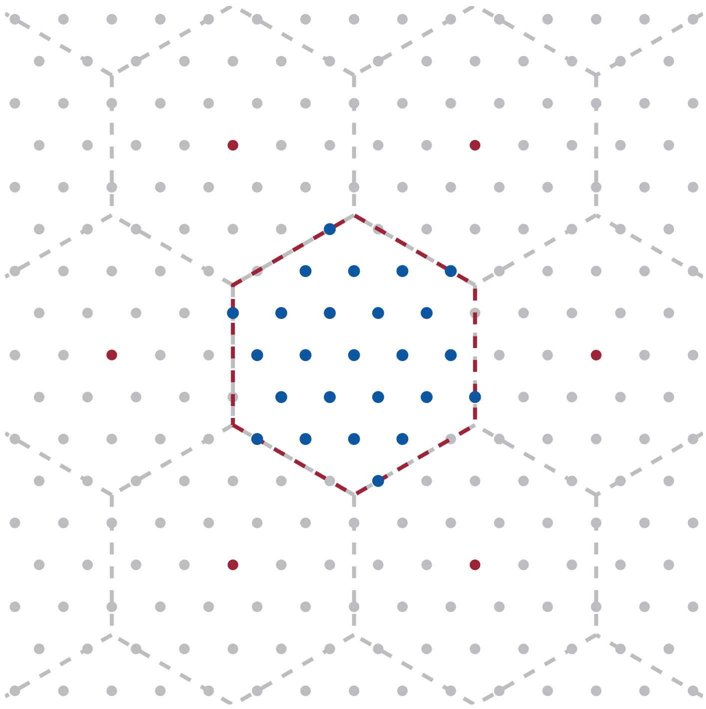
  $$p = |a + \omega\, b|^2 = |\Theta|^2$$

  ⇒ algebraic property

  $$\mathcal{A} \bmod \mathbf{\Lambda}_{\mathrm{p}} \;\simeq\; \mathbb{F}_p$$

- $|\mathcal{A}| = p = |\Theta|^2$
- $p = 25,\;\; \Theta = 5$



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How to choose Λ_{p}:

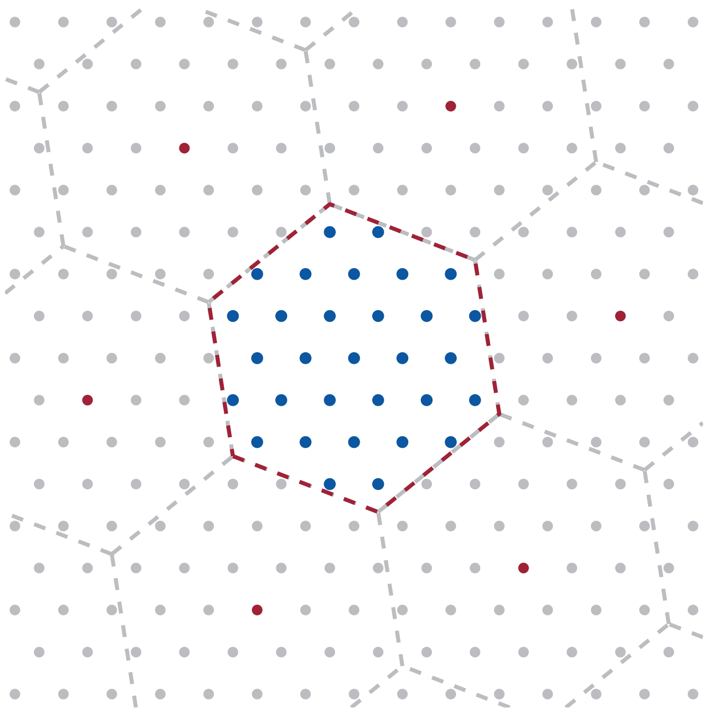
- for primes p, $\mathrm{rem}_6(p) = 1$,
  Eisenstein primes Θ exits

  $$p = |a + \omega\, b|^2 = |\Theta|^2$$

  ⇒ algebraic property

  $$\mathcal{A} \bmod \mathbf{\Lambda}_{\mathrm{p}} \;\simeq\; \mathbb{F}_p$$

- $|\mathcal{A}| = p = |\Theta|^2$
- $p = 31,\;\; \Theta = 6 + \omega$



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  $$\mathcal{A} \bmod \mathbf{\Lambda}_\mathrm{p} \;\simeq\; \mathbb{F}_p$$

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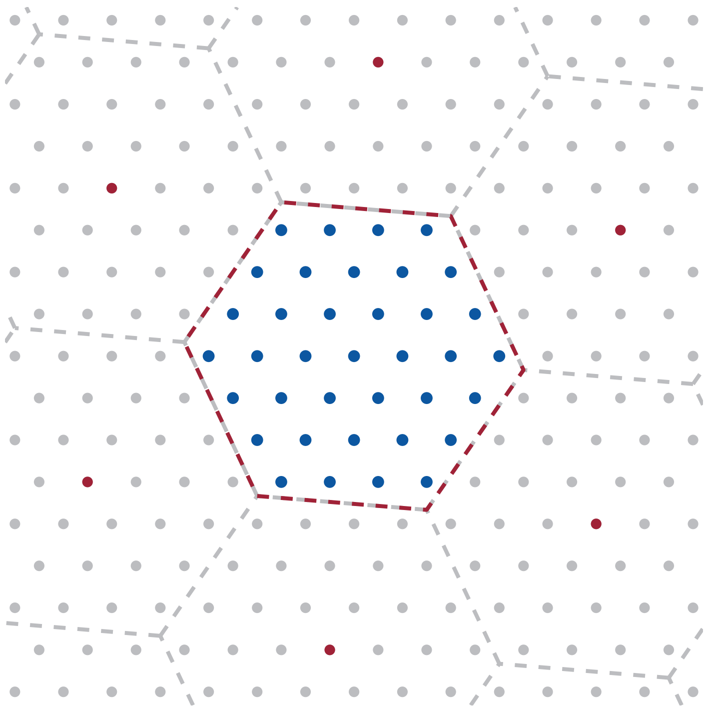
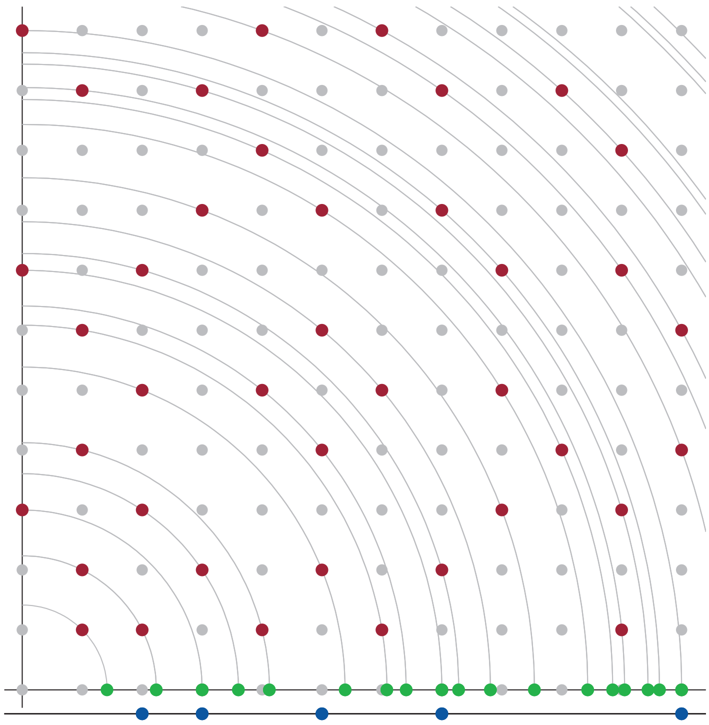
- $p = 37,\;\; \Theta = 7 + \omega\, 3$



Flexibility:

- Gaussian primes
  $|\mathcal{A}| = p = |\Theta|^2$
- cardinality per dim.
  $\sqrt{|\mathcal{A}|} = |\Theta|$
- possible cardinalities
- reference:

  $$\mathbb{Z}_{\sqrt{|\mathcal{A}|}}\,, \quad \sqrt{|\mathcal{A}|}\ \text{prime}$$

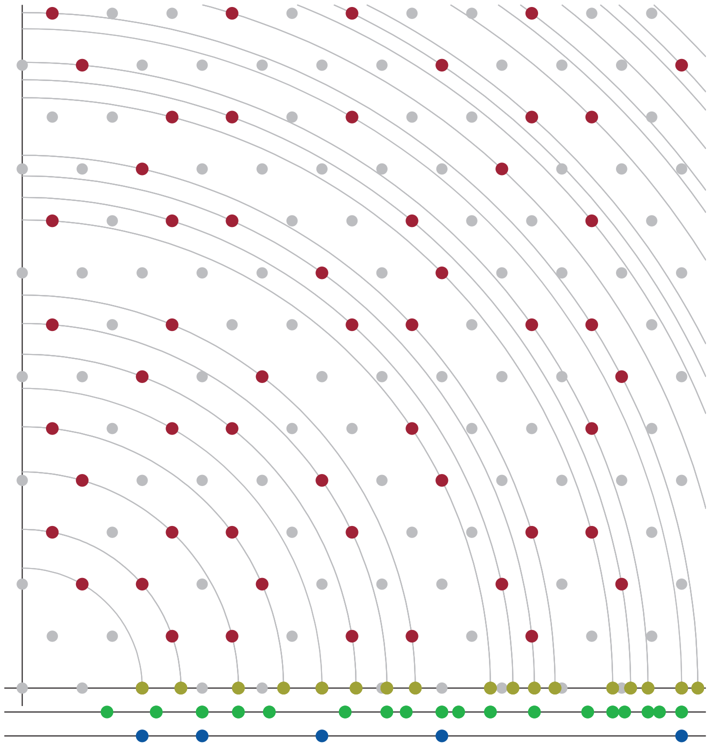


Signal Constellations (V)

Flexibility:

- Gaussian primes
  $|\mathcal{A}| = p = |\Theta|^2$
- cardinality per dim.
  $\sqrt{|\mathcal{A}|} = |\Theta|$
- possible cardinalities
- reference:

  $\mathbb{Z}_{\sqrt{|\mathcal{A}|}}$, $\sqrt{|\mathcal{A}|}$ prime
- Eisenstein primes



Code Requirements

Signal Constellations for LRA/IF Precoding: $\mathcal{A} \bmod \mathbf{\Lambda}_\mathrm{p} \simeq \mathbb{F}_p$

- Gaussian prime constellations
- Eisenstein prime constellations

  ⇒ *cardinality is a prime number*

Codes for LRA/IF Precoding:

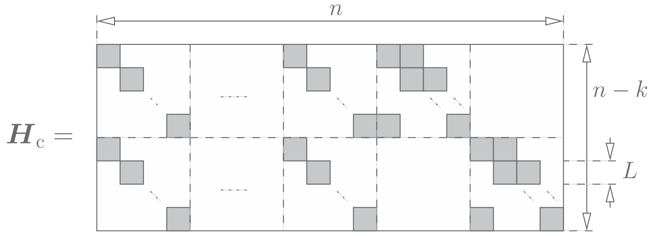
- linear codes (integer linear combinations)
- code symbols from $\mathbb{F}_p$, p prime

Non-Binary Codes:

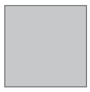
- algebraic codes over Gaussian integers [H'94]
- protograph-based irregular repeat-accumulate codes over $\mathbb{Z}_7$ or $\mathbb{Z}_{251}$ [HC'11]
- non-binary LDPC codes
  however, typically $q = 2^r$ [DM'98], [DF'07], [ZLTSLA'08], [YCMWSW'11], [CMLPC'12]

Code Construction: [YCMW'11]

- quasi-cyclic LDPC code with systematic quasi-cyclic generator matrix
- random construction of parity-check matrix with specific structure



- $L \times L$ subblocks in left part



$$= a \cdot \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & & \\ & & & \ddots & \\ 0 & & & & 1 \\ 1 & 0 & & \cdots & 0 \end{bmatrix}^{s}$$

with $a \in \mathbb{F}_p \setminus \{0\}$ and $s \in \{0, \ldots, L-1\}$ randomly chosen

Non-Binary LDPC Codes

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Code Construction: [YCMW'11]

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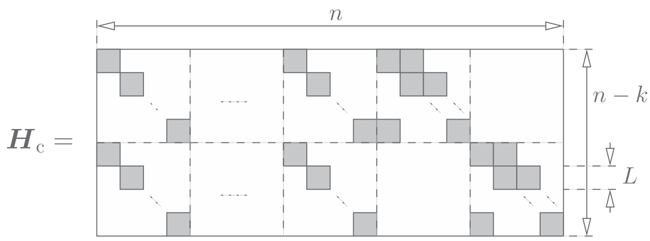

- $L \times 2L$ subblocks in right part

$$= \left[a_{i,1} \cdot \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & & \\ & & & \ddots & \\ 0 & & & & 1 \\ 1 & 0 & & \cdots & 0 \end{bmatrix}^{s_{i,1}} \quad a_{i,2} \cdot \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & & \\ & & & \ddots & \\ 0 & & & & 1 \\ 1 & 0 & & \cdots & 0 \end{bmatrix}^{s_{i,2}} \right]$$

with $a_{i,j} \in \mathbb{F}_p \setminus \{0\}$ and $s_{i,j} \in \{0, \ldots, L-1\}$ randomly chosen and

$$\sum_i (s_{i,1} - s_{i,2}) \equiv 0 \bmod L \qquad \prod_i a_{i,1}^{-1} a_{i,2} \neq 1$$

Fischer, Stern: *Combination / Interaction of Coded Modulation and Precoding in MIMO Systems*

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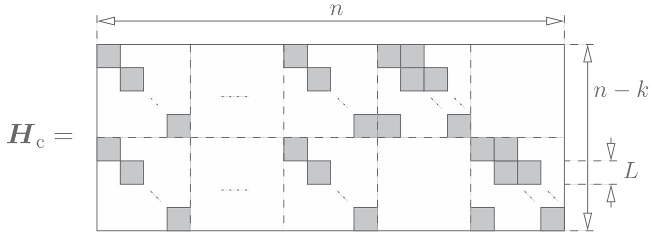
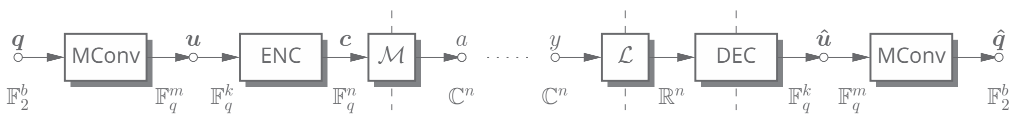


- non-singular parity-check matrix guaranteed
- ultra-sparse quasi-cyclic parity-check matrix
- sparse systematic generator matrix can easily be given; low-complexity encoding
- restrictions on the code parameters n, k, L

Numerical Results

Parameters:

- signal constellations / codes

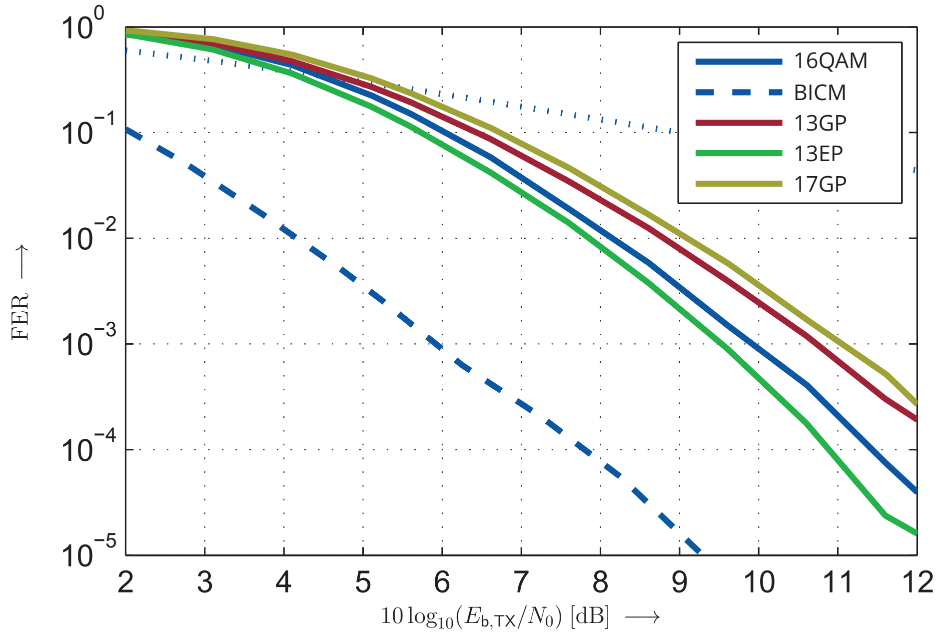


Scenario	Λ_a	Λ_p	q	n	k	R_c	# bits	b	m	$2^b/q^m$
16QAM	$\mathbb{G}$	$4\mathbb{G}$	2	64800	21600	0.333	21600	—	—	—
			2^4	16480	5056	0.307	20224	4	1	1
13GP	$\mathbb{G}$	$(3+j\,2)\mathbb{G}$	13	16480	5480	0.333	20276	37	10	0.997
13EP	$\mathbb{E}$	$(3+\omega)\mathbb{E}$								
17GP	$\mathbb{G}$	$(4+j)\mathbb{G}$	17	16480	4944	0.300	20208	421	103	0.994

- belief-propagation decoding — metric calculation $\bmod \Lambda_p$
- CLLL for $\mathbb{G}$ — adapted LLL for $\mathbb{E}$, i.e., $Z \in \mathbb{E}^{K\times K}$
- mapping from $\mathbb{F}_2$ to $\mathbb{F}_q$ symbols via *modulus conversion* [F'02]
- comparison: BICM with DVB S2 LDPC code

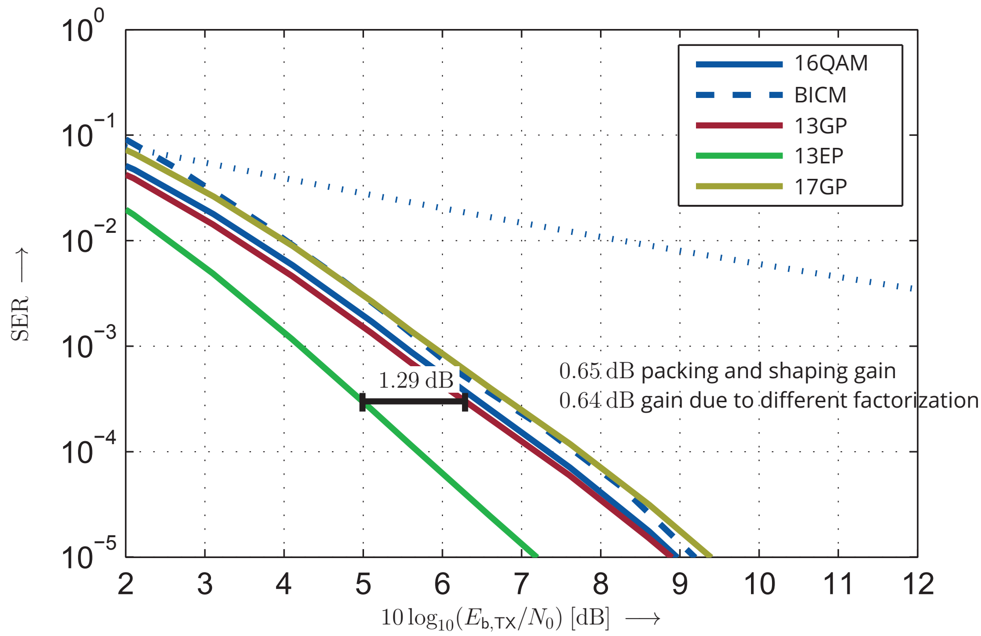
Numerical Results

Numerical Example: frame error rate (FER) — $N = 8$, i.i.d. block-fading channel



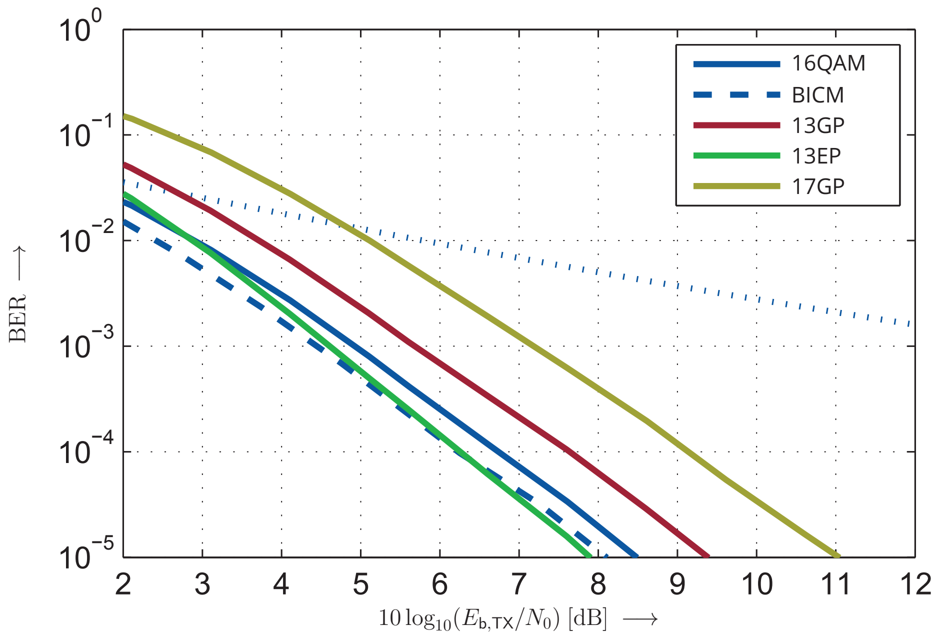
Numerical Results (II)

Numerical Example: symbol error rate (SER) — $N = 8$, i.i.d. block-fading channel



Numerical Results (III)

Numerical Example: bit error rate (BER) — $N = 8$, i.i.d. block-fading channel



Summary and Conclusions

Gaussian and Eisenstein Prime Constellations:

- have algebraic structure
- inherent modulo periodicity
- are well-suited for LRA/IF equalization/precoding schemes

Eisenstein Prime Constellations:

- offer packing and shaping gain over Gaussian Prime Constellations
- the channel factorization has to be adapted — $\boldsymbol{Z} \in \mathbb{E}^{K \times K}$

  ⇒ *additional gain achieved*

Non-Binary / Non-2^r Channel Coding:

- required when employing Gaussian/Eisenstein Prime Constellations
- little is known about these code classes

  ⇒ *research is required*

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