Probabilistic Amplitude Shaping for Higher-Order Modulation

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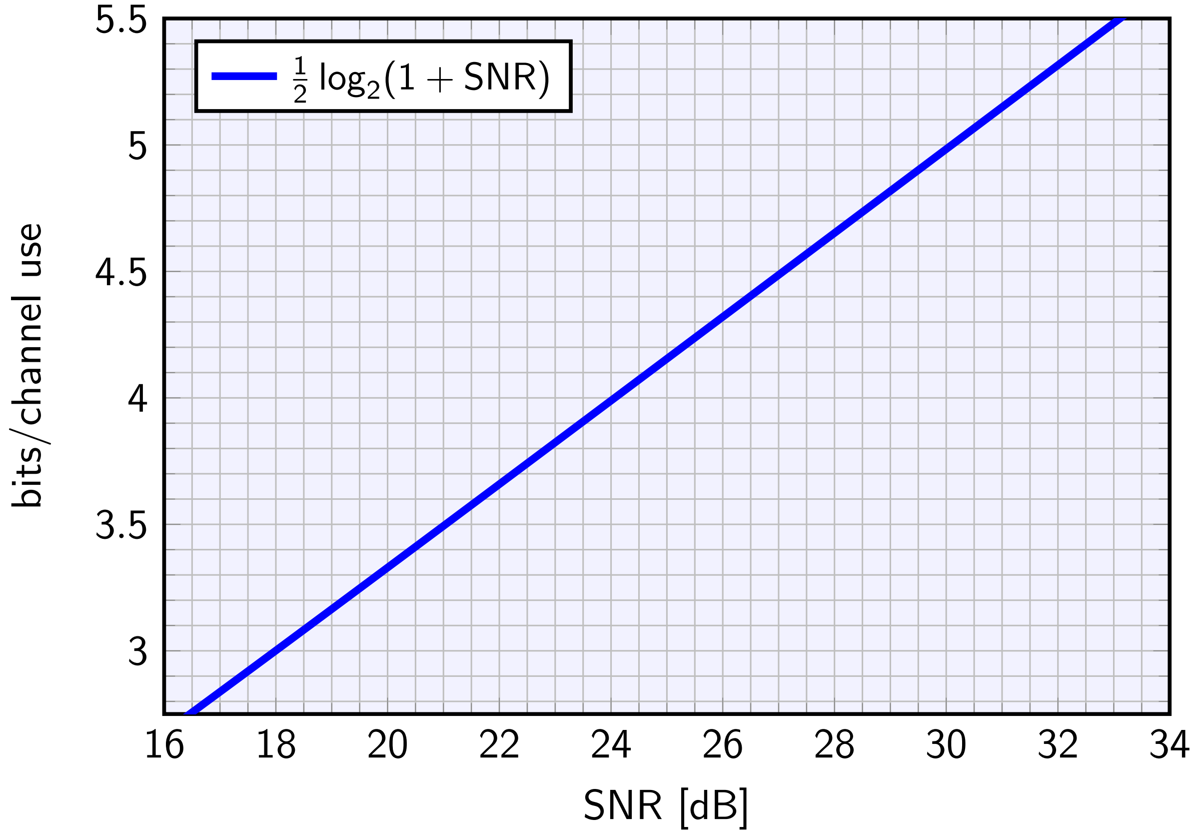
Outline

- The Problem: Achieve $\frac{1}{2}\log_2(1 + \text{SNR})$
- The Solution: Probabilistic Amplitude Shaping[1]
  - Signaling
  - Encoding
  - Decoding
- Why PAS is practical
- Outlook

[1] G. Böcherer, P. Schulte, F. Steiner Bandwidth-Efficient and Rate-Matched Low-Density Parity-Check Coded Modulation, http://arxiv.org/abs/1502.02733, submitted to Com-Trans., under revision.

Bandwidth-Limited Communication



Bi-Polar Amplitude Shift Keying (ASK)

- **Equidistant 2^m-ASK constellation:**

$$\mathcal{X} = \{\pm 1, \pm 3, \dots, \pm(2^m - 1)\}.$$

- **I/O-relation:**

$$Y = \Delta \cdot X + Z.$$

  - Noise Z is zero mean Gaussian, variance one.
  - Input X with distribution P_X on $\mathcal{X}$.
  - Δ scales the constellation.

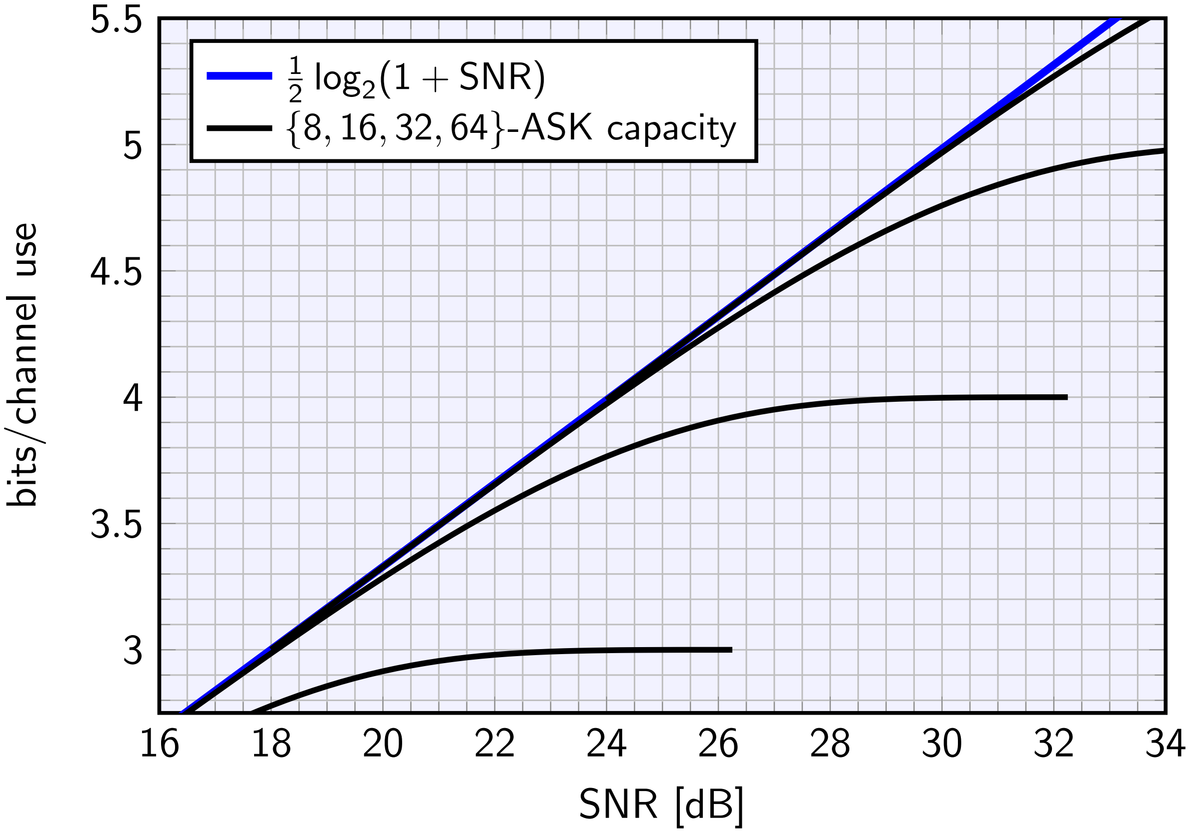
ASK Capacity

- **ASK capacity:**

$$C_{ask}(P) = \max_{\Delta, P_X:\ E[|\Delta X|^2] \le P} I(X; \Delta X + Z).$$

- Capacity-achieving distribution P_{X^*}.

Bandwidth-Limited Communication



Problem 1: Signaling

$P^n_{X^*}$ is a non-uniform distribution on $\mathcal{X}^n \Rightarrow$ variable rate.

- Choose signal set $\mathcal{S} \subset \mathcal{X}^n$.
- Constant rate $\Rightarrow$ uniform distribution on $\mathcal{S}$, i.e.,

$$P_{X^n}(x^n) = \frac{1}{|\mathcal{S}|}, \quad \forall x^n \in \mathcal{S}.$$

- We want

$$\frac{1}{n} I(X^n; Y^n) \stackrel{!}{=} C_{\text{ask}}.$$

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Sign–Amplitude Factorization

- P_{X^*} is **symmetric**:

$$P_{X^*}(x) = P_{X^*}(-x).$$

- P_{X^*} **factorizes:**

$$P_{X^*}(x) = P_A(|x|) P_S(\text{sign}(x))$$

  where $A := |X|$ and $S := \text{sign}(X)$.
- The sign S is **uniform**:

$$P_S(-1) = P_S(1) = \frac{1}{2}.$$

Probabilistic Amplitude Shaping

- Use $S^n \in \{-1,1\}^n$ for sign sequence.
- Use A^n in type class $\mathcal{T}^n_{P_A}$ for amplitude sequence.[2]
- **Example:** 4-ASK, $\mathcal{A} = \{1,3\}$, $n = 4$,
  $P_A(1) = 1 - P_A(3) = \frac{3}{4}$. Signaling set for amplitudes is

$$\mathcal{T}^4_{(\frac{3}{4},\frac{1}{4})} = \{(1,1,1,3),\\ (1,1,3,1),\\ (1,3,1,1),\\ (3,1,1,1)\}.$$

[2] $a^n \in \mathcal{T}^n_{P_A} \Leftrightarrow$ each amplitude α occurs $nP_A(\alpha)$ times in a^n.

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Probabilistic Amplitude Shaping

- Transmitted signal is X^n with

$$X_i = S_i \cdot A_i, \qquad i = 1, \dots, n$$

- How large is $\frac{1}{n} I(X^n; Y^n)$?
- **Problem:**
  inputs $X_1, X_2, \dots, X_n$ are dependent because $A^n \in \mathcal{T}^n_{P_A}$.

Tool: Mismatched Mutual Information Lower Bound

- Channel $p_{Y|X}$, auxiliary channel $q_{Y|X}$.
- Auxiliary output density $q_Y(y) = \sum_{a\in\mathcal{X}} P_X(a)p_{Y|X}(y|a)$.
- Mismatched mutual information lower bound:

$$I(X;Y) = E\left[\log_2 \frac{p_{Y|X}(Y|X)}{p_Y(Y)}\right] \geq E\left[\log_2 \frac{q_{Y|X}(Y|X)}{q_Y(Y)}\right]$$

Signaling Analysis

- Auxiliary channel

$$q_{Y^n|X^n}(y^n|x^n) = \frac{\prod_{i=1}^n P_X(x_i)}{P_{X^n}(x^n)} \prod_{i=1}^n p_{Y|X}(y_i|x_i).$$

- We can show

$$q_{Y^n}(y^n) \leq \prod_{i=1}^n p_Y(y_i).$$

- Thus

$$\frac{1}{n}I(X^n;Y^n) \geq \frac{1}{n}E\left[\log_2 \frac{\prod_{i=1}^n p_{Y|X}(Y_i|X_i)}{\prod_{i=1}^n p_Y(Y_i)}\right] - \frac{1}{n}D(P_{X^n}\|P_X^n)$$

$$= \underbrace{I(X;Y)}_{\mathsf{C}_{\text{ask}}} - \frac{1}{n}D(P_{A^n}\|P_A^n).$$

Signaling Analysis

- Use $0 \mapsto -1, 1 \mapsto 1$.
- Constant Composition Distribution Matching (CCDM)[3] efficiently indexes $\mathcal{T}_{P_A}$ with $k = \log_2 |\mathcal{T}_{P_A}|$ bits.
- $\frac{1}{n} D(P_{A^n} \| P_A^n) \overset{n \to \infty}{\to} 0$.

$\Rightarrow$ $\frac{1}{n} I(X^n; Y^n) \overset{n \to \infty}{\to} \mathsf{C}_{\text{ask}}$.

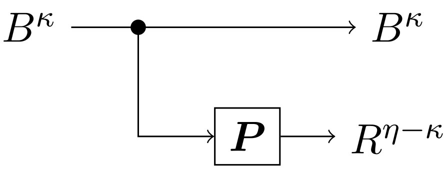
[3] P. Schulte, G. Böcherer Constant Composition Distribution Matching, http://arxiv.org/abs/1503.05133, submitted to IT-Trans, under revision.

Problem 2: Encoding

- We have to add redundancy.

$\Rightarrow$ Encode to a sub-set of signal set $\mathcal{S}$.

Shaping and Error Correction

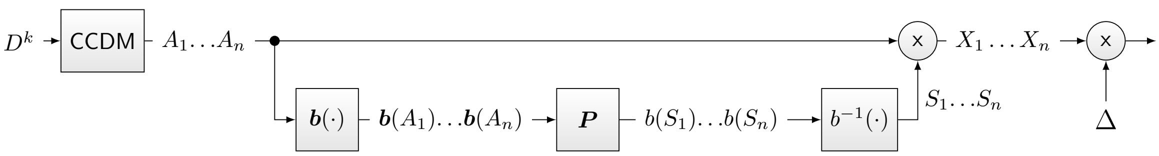


- Binary systematic $\kappa \times \eta$ generator matrix $\boldsymbol{G} = [\boldsymbol{I}|\boldsymbol{P}]$.
- Binary information B^κ arbitrarily distributed.
- Uniform check bit assumption[4]:

$$R_1, R_2, \ldots \text{ iid} \sim \text{Bernoulli}(1/2).$$

[4] G. Böcherer Capacity Achieving Probabilistic Shaping for Noisy and Noiseless Channels, PhD thesis, 2012.

PAS Encoder



- Map Bernoulli(1/2) data bits D^k to amplitude sequence A^n.
- Represent amplitudes by $(m-1)$ bits: $A \mapsto \boldsymbol{b}(A)$.
- Represent signs by 1 bit: $S \mapsto b(S)$.
- Multiply amplitude labels by parity matrix $\boldsymbol{P}$ to get sign labels.
- Code rate $c = (m-1)/m$.

- Marginals are P_{X^*}!

Problem 3: Decoding

- Our mismatched lower bound

$$\mathsf{C}_{\text{ask}} - \frac{1}{n} D(P_{A^n} \| P_A^n)$$

corresponds to decoding metric

$$\hat{x}^n = \operatorname*{argmax}_{x^n \in \mathcal{C}} \prod_{i=1}^{n} P_X(x_i) p_{Y|X}(y_i|x_i).$$

- We can pull the same trick once again to do bit-metric decoding (BMD).

Bit-Metric Decoding (BMD)

- Label the ASK constellation by m bits $\boldsymbol{B} = B_1 \cdots B_m$.
- Use auxiliary channel

$$q_{Y|\boldsymbol{B}}(y|\boldsymbol{b}) = \frac{\prod_{i=1}^{m} P_{B_i}(b_i)}{P_{\boldsymbol{B}}(\boldsymbol{b})} \prod_{i=1}^{m} p_{Y|B_i}(y_i|b_i).$$

- BMD achievable rate is[5]

$$\mathsf{R}_{\text{bmd}} = \sum_{i=1}^{m} I(B_i; Y) - D(P_{\boldsymbol{B}} \| \prod_{i=1}^{m} P_{B_i}).$$

[5] G. Böcherer Achievable Rates for Shaped Bit-Metric Decoding, http://arxiv.org/abs/1410.8075, submitted to IT-Tran., revised.

Bit-Metric Decoding

- The corresponding decoding metric is

$$\hat{x}^n = \operatorname*{argmax}_{x^n \in \mathcal{C}} \prod_{i=1}^{n} \prod_{j=1}^{m} P_{B_j}(b_{ji}) p_{Y|B_j}(y_i|b_{ji}).$$

- Use bit-wise demapper

$$L_{ji} = \log_2 \frac{p_{Y|B_j}(y_i|0)}{p_{Y|B_j}(y_i|1)} + \log_2 \frac{P_{B_j}(0)}{P_{B_j}(1)}, \quad j = 1, 2, \dots, m.$$

- Pass the L_{ji} to a binary iterative decoder.
- No iterative demapping!

Transmission Experiment

- CCDM implementation from `www.beam.to/ccdm`.
- Blocklength 64800 binary LDPC code[6].
- 64-ASK constellation.
- Losses compared to $\frac{1}{2}\log_2(1 + \text{SNR})$:

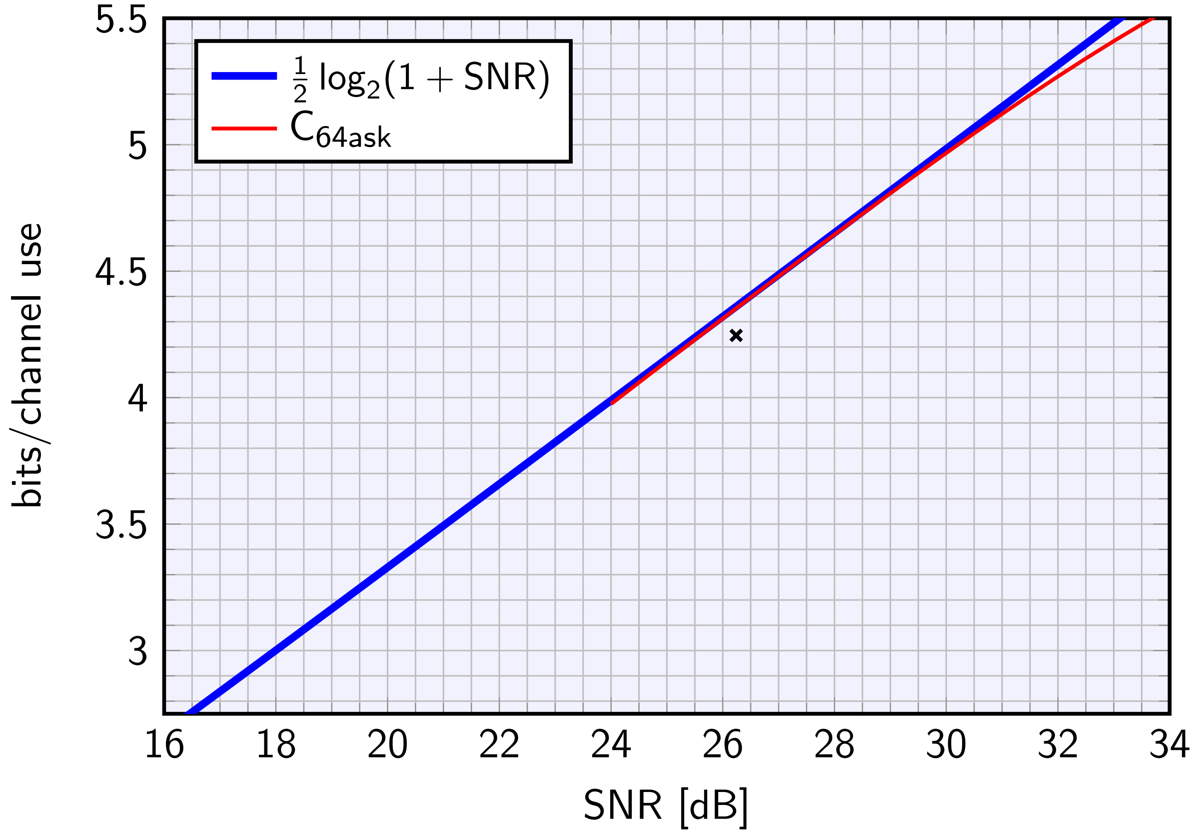
CCDM: $\frac{1}{n} D(P_{A^n} \| P_A^n) = 0.01$ bits $\Rightarrow 0.06$ dB

BMD: 0.0032 dB

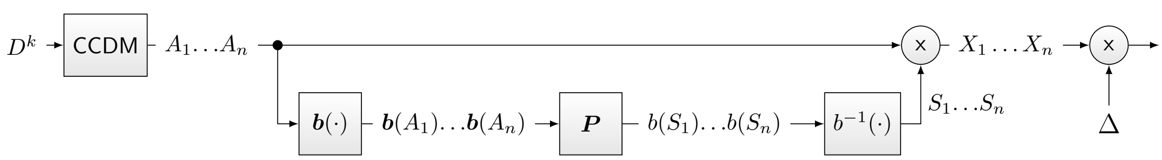
iterative decoding threshold: 0.35 dB.

[6] F. Steiner, G. Böcherer, G. Liva Protograph-Based LDPC Code Design for Shaped Bit-Metric Decoding, `http://arxiv.org/abs/1504.03628`, submitted to JSAC.

FER 10^{-3} within 0.69 dB of Capacity



Why PAS is Practical



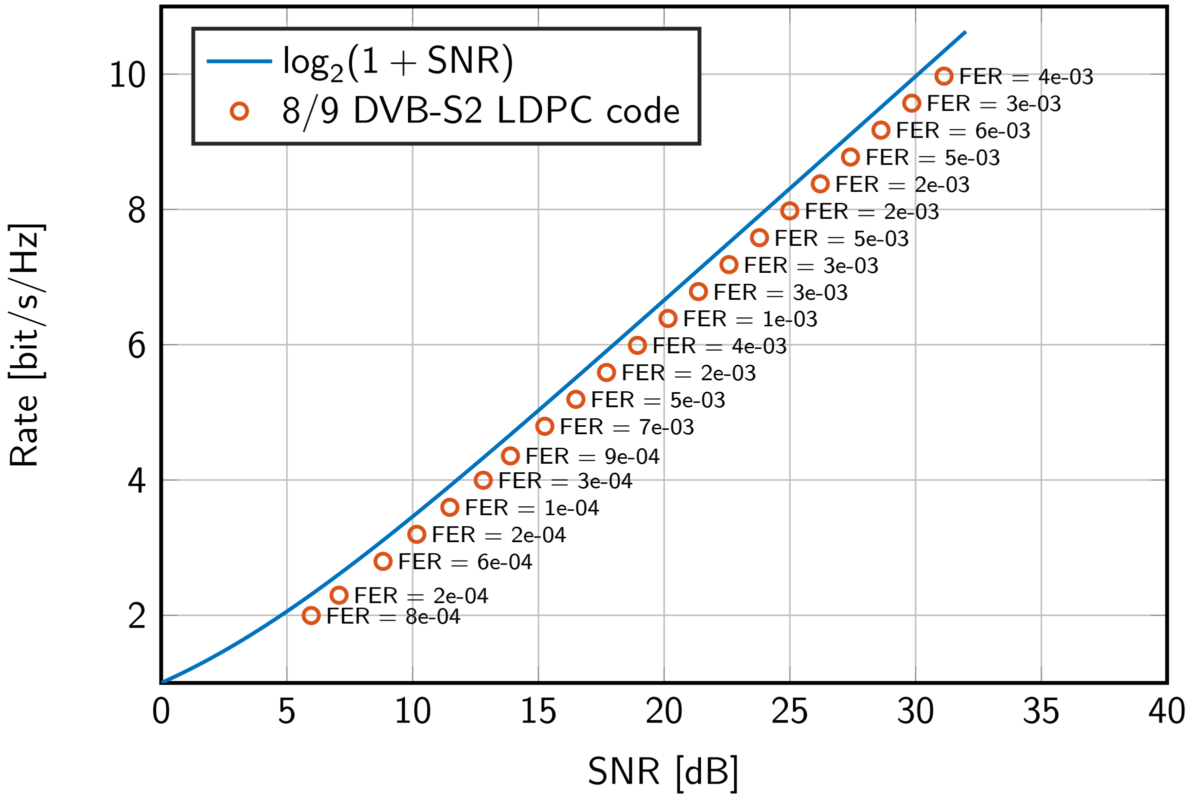
- Rate is $k/n \approx H(P_A)$ $\left[\frac{\text{bits}}{\text{channel use}}\right]$.

$\Rightarrow$ Control rate via P_A.

$\Rightarrow$ Control power via Δ.

$\Rightarrow$ PAS enables **rate-matched** transmission!

Rate-Matched QAM Transmission



Outlook

- Optical Experiment with PAS.
- Chip Implementation of PAS.
- PAS for short block-length. Challenges:
  - CCDM loss $\frac{1}{n}D(P_{A^n} \| P_A^n)$ is large.
  - Iterative decoding is not good.

Literatur

- G. Böcherer, P. Schulte, and F. Steiner, "Bandwidth efficient and rate-matched low-density parity-check coded modulation," *arXiv preprint*, 2015, submitted to IEEE Trans. Commun., under revision. [Online]. Available: http://arxiv.org/abs/1502.02733
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