On finite geometry codes with locality

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Motivation

Codes with locality or Locally Repairable Codes (LRCs) allow to recover one erased code symbol using only a few other symbols.

LRCs can be applied in

- distributed storage systems for repairing multiple disk failures
- and also for management of hot data.

Outline

1. Codes and locality
2. Connection with majority logic decoding
3. Finite geometry
4. New codes

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Codes

A code $\mathcal{C}$ is a set of codewords $c = (c_1, c_2, \dots, c_n)$ over a finite field $\mathbb{F}_q$.

Code distance d is the minimum Hamming distance between different codewords.

The code $\mathcal{C}$ is *linear* $[n, k]$-code if it is $\mathbb{F}_q$-linear subspace of $\mathbb{F}_q^n$.

Generator matrix G of a linear $[n, k]$-code is a $k \times n$ matrix over $\mathbb{F}_q$, such that rows of G form a basis of $\mathcal{C}$.

Encoding: $c = uG$, where u is an information vector of length k.

Systematic encoding using systematic $G = (I_k, P)$, then $c = uG = (u, p)$.

A parity check matrix is $H = (-P^T, I_{n-k})$ and for all $c \in \mathcal{C}$ holds $cH^T = 0$.

Codes with locality

A code symbol c_i has *repair locality* r if it can be recovered by accessing at most r other symbols. The set of indexes of those symbols is called repair set $\mathcal{R}$. E.g. for $r = 2$ and $\mathcal{R} = \{2, 3\}$, $i = 1$:

$$c_1 = c_2 + c_3.$$

A linear code has *information locality* r if every information symbol has locality r. A single erasure can be recovered. What if more erasures?

A code symbol c_i has $(r,\ t)$-*locality* if there exist t disjoint repair sets $\mathcal{R}_j$ for c_i, each containing at most r symbols. E.g. for $r = 2$, $t = 2$, and $\mathcal{R}_1 = \{2, 3\}$, $\mathcal{R}_2 = \{4, 5\}$:

$$\begin{aligned} c_1 &= c_2 + c_3, \\ c_1 &= c_4 + c_5. \end{aligned} \tag{1}$$

A linear code has *information locality* (r, t) if every information symbol has locality (r, t). This code can correct up to t erasures and hence

Upper bound for the code distance

Lemma

The distance d of $[n,\ k]$ code with information $(r,\ t)$-locality satisfies

$$d \geq t + 1.$$

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Connection with majority logic decoding

Orthogonal parity checks for c_1, $t = 2$ checks:

$$\begin{aligned} c_1 &= c_2 + c_3, \\ c_1 &= c_4 + c_5, \\ c_1 &= c_1. \end{aligned} \tag{2}$$

Symbol can be corrected by *majority logic decoder* if there were up to t erasures or up to $t/2$ errors.

If every (information) symbol has at least t orthogonal parity checks then the code has distance $d \geq t + 1$. The code corrects up to t erasures or up to $t/2$ errors by majority logic decoder.

Some known codes with majority logic decoding: Reed-Muller codes, codes based on finite geometries.

We will use *finite geometries* to design codes with locality.

Upper bound for the code distance

To simplify decoding we require that each repair set contains a single parity symbol only.

Lemma ([RPDV16])

Let $\mathcal{C}$ be an $[n, k]$ code with information (r, t)-locality such that each repair set contains a single parity symbol. Then the code distance is bounded by

$$d \leq n - k - \left\lceil \frac{kt}{r} \right\rceil + t + 1. \tag{3}$$

Can we reach this bound?

[RPDV16] Rawat, A.S., Papailiopoulos, D.S., Dimakis, A.G., Vishwanath, S.: Locality and availability in distributed storage. IEEE Transactions on Information Theory 62(8), 4481–4493, 2016.

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Partial geometry

Definition ([CD96])

A (finite) *partial geometry* is an incidence structure $\mathcal{S} = (\mathcal{P}, \mathcal{L}, I)$ in which $\mathcal{P}$ is a set of *points* p, $\mathcal{L}$ is a set of *lines* ℓ and I is a symmetric point-line incidence relation satisfying the following axioms:

1. Each point p is incident with $u+1$ lines$(u \geq 1)$, and two distinct points are incident with at most one line.
2. Each line ℓ is incident with $s+1$ points$(s \geq 1)$, and two distinct lines are incident with at most one point.
3. Given a point p not incident with a line ℓ, there are exactly α lines incident with p and also with some point of ℓ.

Such a partial geometry will be denoted by $\mathrm{PG}(s+1,\ u+1,\ \alpha)$. Parameter α is called *connection number*.

[CD96] Colbourn, C.J., Dinitz, J.H.: Handbook of combinatorial designs. CRC Press (1996).

Fano plane

Well known examples of partial geometries are Euclidean and projective geometries over finite fields.
The Fano plane [F1892] is the finite projective geometry with the smallest number of points and lines: 7 points and 7 lines, with 3 points on every line and 3 lines through every point.

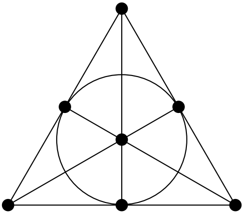


Figure 1: Partial geometry PG(3,3,3)

[F1892] Gino Fano, (1892), "Sui postulati fondamentali della geometria proiettiva", Giornale di Matematiche, 30: 106–132.

New partial geometry [DLLB2015]

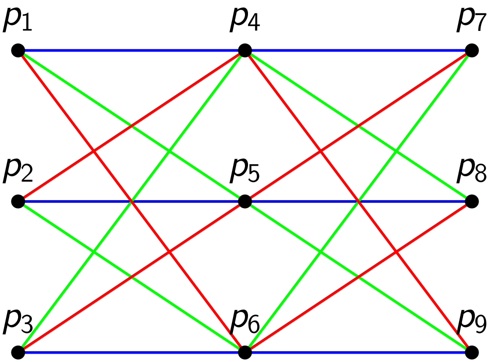


Figure 2: Partial geometry PG(3,3,2)

[DLLB2015] Qiuju Diao,Juane Li, Shu Lin, Ian Blake, New Classes of Partial Geometries and Their Associated LDPC Codes, arXiv: 1503.06900v1 [cs.IT] 24 Mar 2015

Number of points and lines

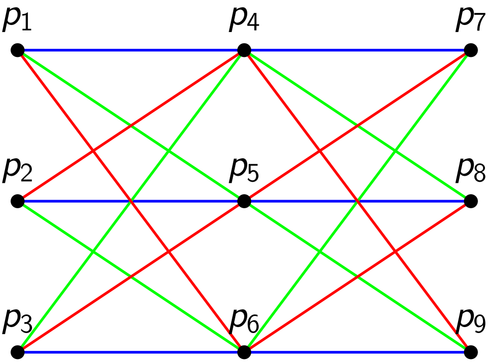
Let $P = |\mathcal{P}|$ and $L = |\mathcal{L}|$, then [CD96]

$$P = \frac{(s+1)(su+\alpha)}{\alpha}, \quad L = \frac{(u+1)(su+\alpha)}{\alpha}. \tag{4}$$

[CD96] Colbourn, C.J., Dinitz, J.H.: Handbook of combinatorial designs. CRC Press (1996).

Incidence matrix N of PG(3,3,2)

$$
N = \begin{array}{c} \\ \ell_1 \\ \ell_2 \\ \ell_3 \\ \ell_4 \\ \ell_5 \\ \ell_6 \\ \ell_7 \\ \ell_8 \\ \ell_9 \end{array}
\begin{array}{c}
\begin{array}{ccccccccc} p_1 & p_2 & p_3 & p_4 & p_5 & p_6 & p_7 & p_8 & p_9 \end{array} \\
\begin{pmatrix}
1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\
0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 \\
1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\
0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0
\end{pmatrix}
\end{array}. \qquad (5)
$$



N was used as parity check matrix H.

Dual geometry. Some known geometries.

The dual of a partial geometry $\mathrm{PG}(s+1,\ u+1,\ \alpha)$ is obtained by exchanging the set of points and the set of lines, which is also a partial geometry $\mathrm{PG}(u+1,\ s+1,\ \alpha)$ with incidence matrix $N^{\perp} = N^T$.

Up to duality, parameters of some known partial geometries are [CD96]:

Type 0: $s = w,\ u = w^{m-1} - 1,\ \alpha = w$, with $m \geq 2$ and w is a power of prime;

Type 1: $s = 2^h - 2^m,\ u = 2^h - 2^{h-m},\ \alpha = (2^{h-m} - 1)(2^m - 1)$, $1 \leq m \leq h$;

Type 2: $s = 2^h - 1,\ u = (2^h + 1)(2^m - 1),\ \alpha = 2^m - 1$, $1 \leq m \leq h$;

Type 3: $s = 2^{2h-1} - 1,\ u = 2^{2h-1},\ \alpha = 2^{2h-2}$, $1 < h$;

Type 4: $s = 3^{2m} - 1,\ u = (3^{4m} - 1)/2,\ \alpha = (3^{2m} - 1)/2,\ m \geq 1$;

[CD96] Colbourn, C.J., Dinitz, J.H.: Handbook of combinatorial designs. CRC Press (1996).

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New codes

Given $L \times P$ incidence matrix N of $\mathrm{PG}(s+1,\ u+1,\ \alpha)$, we define a code $\mathcal{C}_{(s+1,\ u+1,\ \alpha)}$ over $\mathbb{F}_q$ by the following binary systematic generator matrix

$$G = [I_L | N]. \tag{6}$$

Theorem

The q-ary linear code $\mathcal{C}_{(s+1,\ u+1,\ \alpha)}$ over $\mathbb{F}_q$ is an $[n,\ k]$ locally repairable code with information $(r,\ t)$-locality, where $n = L + P$, $k = L$, $r = u+1$, $t = s+1$. The code distance $d = s+2$ reaches the upper bound (3).

Remark: The q-ary code over a field $\mathbb{F}_q$ of characteristic 2, $q = 2^m$, is the interleaving of m binary codes, since both G and H are binary matrices.

Example of a new code

Let N be the incidence matrix of $\mathrm{PG}(3,3,2)$ see (5). Then the code $\mathcal{C}_{(3,3,2)}$ has the parity check matrix $H = (N^T, -I_L)$, defined by (6), as follows:

$$H = \left(\begin{array}{ccccccccc|ccccccccc}
1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\
0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1
\end{array}\right)$$

and we have $[n = 18,\ k = 9]$ locally recoverable code over $\mathbb{F}_q$ with information $(r = 3,\ t = 3)$-locality and distance $d = 4$.

For the first information symbol the repair relations with a single parity symbol are: $c_1 = c_{10} - c_4 - c_7 = c_{13} - c_6 - c_8 = c_{16} - c_5 - c_9$. Every repair set includes only one parity check symbol.

New optimal codes

Theorem

1. *For partial geometry of Type 0 the code* $\mathcal{C}_{(s+1,\ u+1,\ \alpha)}$ *is* $[n = w^{m-1}(w^{m-1} + w + 1),\ k = w^{2(m-1)},\ d = w + 2]$ *with information* $(r = w^{m-1},\ t = w + 1)$*-locality.*
2. *For Type 1 we get* $[n = 2(2^h + 1)(2^h - 2^{m-1} - 2^{h-m-1} + 1),\ k = (2^h + 1)(2^h - 2^{h-m} + 1),\ d = 2^h - 2^m + 2]$ *code with information* $(r = 2^h - 2^{h-m} + 1,\ t = 2^h - 2^m + 1)$*-locality.*
3. *Type 2:* $[n = 2^{m+2h}(2^h + 1),\ k = 2^{m+2h}(2^h - 2^{h-m} + 1),\ d = 2^h + 1]$ *code with information* $(r = 2^m(2^h - 2^{h-m} + 1),\ t = 2^h)$*-locality.*
4. *Type 3: we get* $[n = 2^{4h} - 1,\ k = (2^{2h-1} + 1)(2^{2h} - 1),\ d = 2^{2h-1} + 1]$ *code with information* $(r = 2^{2h-1} + 1,\ t = 2^{2h-1})$*-locality.*
5. *Type 4:* $[n = \frac{3^{4m}(3^{2m}+1)^2}{2},\ k = \frac{3^{4m}(3^{4m}+1)}{2},\ d = 3^{2m} + 1]$ *code with information* $(r = \frac{3^{4m}+1}{2},\ t = 3^{2m})$*-locality.*

All codes reach the bound (3) *and have rates* $k/n = \frac{1}{1+t/r}$.

Conclusions

- From partial geometries, we constructed a class of q-ary locally recoverable codes (LRC) with new parameters, which are optimal with respect to the upper bound (3).
- New geometries will give new codes.
- Future work: Design efficient decoders correcting more than $d/2$ errors (and erasures) using interleaved structure of the codes.

Thank you!