Quaternion-Valued MIMO Transmission

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Outline

Complex numbers $\mathbb{C}$:

$$c = \underbrace{\mathrm{Re}\{c\}}_{\in\mathbb{R}} + \underbrace{\mathrm{Im}\{c\}}_{\in\mathbb{R}}\ \mathrm{i}$$

- field extension of $\mathbb{R}$
- imaginary unit $\mathrm{i} = \sqrt{-1}$

Quaternions $\mathbb{H}$:

$$\begin{aligned} q &= \underbrace{(\mathrm{Re}\{q^{\{1\})}\} + \mathrm{Im}\{q^{\{1\}}\}\,\mathrm{i})}_{q^{\{1\}}\in\mathbb{C}} \quad + \quad \underbrace{(\mathrm{Re}\{q^{\{2\}}\} + \mathrm{Im}\{q^{\{2\}}\}\,\mathrm{i})}_{q^{\{2\}}\in\mathbb{C}}\ \mathrm{j} \\ &= \quad \underbrace{q^{(1)}}_{\in\mathbb{R}} + \underbrace{q^{(2)}}_{\in\mathbb{R}}\ \mathrm{i} \quad + \quad \underbrace{q^{(3)}}_{\in\mathbb{R}}\ \mathrm{j} + \underbrace{q^{(4)}}_{\in\mathbb{R}}\ \mathrm{k} \end{aligned}$$

- extension of $\mathbb{C}$
- imaginary units i, j, and $\mathrm{k} = \mathrm{i} \cdot \mathrm{j}$
- multiplication is non-commutative (*skew field*)

	i	j	k
i	-1	k	$-$j
j	$-$k	-1	i
k	j	$-$i	-1

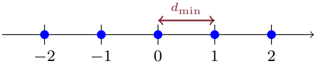
Definition of a lattice $\boldsymbol{\Lambda} = \{\boldsymbol{\lambda}\}$

- *infinite* set of points (vectors) over N-dimensional Euclidean space
- Abelian group with respect to addition:
  - closure:
    $$\boldsymbol{\lambda}_1 + \boldsymbol{\lambda}_2 \in \boldsymbol{\Lambda}$$
  - associativity:
    $$\boldsymbol{\lambda}_1 + (\boldsymbol{\lambda}_2 + \boldsymbol{\lambda}_3) = (\boldsymbol{\lambda}_1 + \boldsymbol{\lambda}_2) + \boldsymbol{\lambda}_3$$
  - commutativity:
    $$\boldsymbol{\lambda}_1 + \boldsymbol{\lambda}_2 = \boldsymbol{\lambda}_2 + \boldsymbol{\lambda}_1$$
  - identity element:
    $$\mathbf{0} \in \boldsymbol{\Lambda}\,, \quad \text{where} \quad \boldsymbol{\lambda}_1 + \mathbf{0} = \boldsymbol{\lambda}_1$$
  - inverse element:
    $$-\boldsymbol{\lambda}_1 \in \boldsymbol{\Lambda}\,, \quad \text{where} \quad -\boldsymbol{\lambda}_1 + \boldsymbol{\lambda}_1 = \mathbf{0}$$
- created by a generator matrix $\boldsymbol{G} = [\boldsymbol{g}_1, \ldots, \boldsymbol{g}_M]$, $\quad \boldsymbol{\Lambda} = \boldsymbol{G}\mathbb{Z}^M$
  - N-dimensional basis vectors $\boldsymbol{g}_1, \ldots, \boldsymbol{g}_M$ span lattice of rank M
  - generator matrix of particular lattice is not unique
    $\rightarrow$ lattice basis reduction via unimodular integer transformation matrix

1D/2D Lattices

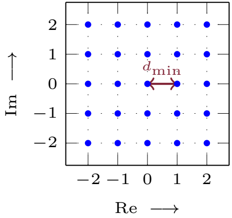
Real-valued (1D) integer lattice $\mathbb{Z}$:

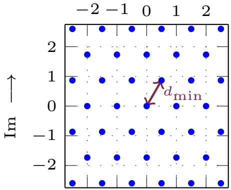
- Euclidean ring
  - division with small remainder possible
  - Euclidean algorithm well-defined
- two nearest neighbors
- squared minimum distance $d^2_{\min} = 1$



Complex-Valued (2D) lattices:

- Gaussian integers $\mathbb{G} = \mathbb{Z} + \mathbb{Z}\,\mathrm{i}$
  - Euclidean ring
  - four nearest neighbors
  - squared minimum distance $d^2_{\min} = 1$
  - isomorphic to 2D real-valued integer lattice $\mathbb{Z}^2$
- Eisenstein integers $\mathbb{E} = \mathbb{Z} + \mathbb{Z}\,\omega$
  - $\omega = \mathrm{e}^{\mathrm{i}\frac{2\pi}{3}}$ *Eisenstein unit* (sixth root of unity)
  - Euclidean ring
  - six nearest neighbors
  - squared minimum distance $d^2_{\min} = 1$
  - isomorphic to 2D hexagonal lattice $\boldsymbol{A}_2$





Lipschitz integers $\mathcal{L}$:

- quaternions with components

$$l = \underbrace{l^{(1)}}_{\in\mathbb{Z}} + \underbrace{l^{(2)}}_{\in\mathbb{Z}}\,\mathrm{i} + \underbrace{l^{(3)}}_{\in\mathbb{Z}}\,\mathrm{j} + \underbrace{l^{(4)}}_{\in\mathbb{Z}}\,\mathrm{k}$$

- isomorphic to 4D integer lattice $\mathbb{Z}^4$

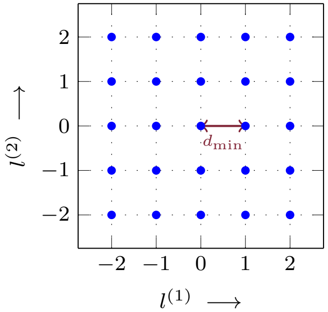
Important properties: [CS99, CS03]

- eight nearest neighbors
- squared minimum distance

$$d_{\min}^2 = 1^2 + 0^2 + 0^2 + 0^2 = 1$$

- $\mathcal{L}$ forms a *non-Euclidean* ring
  - no division with small remainder
  - Euclidean algorithm not defined

2D projection



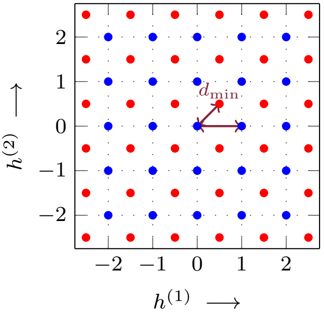
Hurwitz integers $\mathcal{H}$:

- quaternions with components

$$h = h^{(1)} + h^{(2)}\,\mathrm{i} + h^{(3)}\,\mathrm{j} + h^{(4)}\,\mathrm{k}\ ,$$
$$(h^{(1)}, h^{(2)}, h^{(3)}, h^{(4)}) \in \mathbb{Z}^4 \cup (\mathbb{Z}+1/2)^4$$

- two subsets $\mathcal{H}_1 = \mathcal{L}$ and $\mathcal{H}_2 = \mathcal{L} + (1+\mathrm{i}+\mathrm{j}+\mathrm{k})/2$
- isomorphic to 4D checkerboard lattice $\boldsymbol{D}_4$

2D projection



Important properties: [CS99, CS03]

- 24 nearest neighbors
- squared minimum distance

$$d_{\min}^2 = (\tfrac{1}{2})^2 + (\tfrac{1}{2})^2 + (\tfrac{1}{2})^2 + (\tfrac{1}{2})^2 = 1$$

- $\mathcal{H}$ forms a *Euclidean* ring
  - division with small remainder
  - Euclidean algorithm well-defined

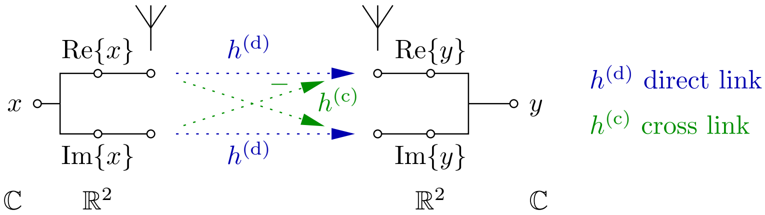
SISO fading (ECB domain; two orthogonal components)

$$\underbrace{y}_{\mathrm{Re}\{y\}+\mathrm{Im}\{y\}\,\mathrm{i}} = \underbrace{h}_{\mathrm{Re}\{h\}+\mathrm{Im}\{h\}\,\mathrm{i}} \cdot \underbrace{x}_{\mathrm{Re}\{x\}+\mathrm{Im}\{x\}\,\mathrm{i}}$$

- equivalently described by real-valued matrix equation

$$\begin{bmatrix}\mathrm{Re}\{y\}\\ \mathrm{Im}\{y\}\end{bmatrix} = \begin{bmatrix}\mathrm{Re}\{h\} & -\mathrm{Im}\{h\}\\ \mathrm{Im}\{h\} & \mathrm{Re}\{h\}\end{bmatrix}\begin{bmatrix}\mathrm{Re}\{x\}\\ \mathrm{Im}\{x\}\end{bmatrix} = \begin{bmatrix}h^{(\mathrm{d})} & -h^{(\mathrm{c})}\\ h^{(\mathrm{c})} & h^{(\mathrm{d})}\end{bmatrix}\begin{bmatrix}\mathrm{Re}\{x\}\\ \mathrm{Im}\{x\}\end{bmatrix}$$

- realized via quadrature up/down-mixing



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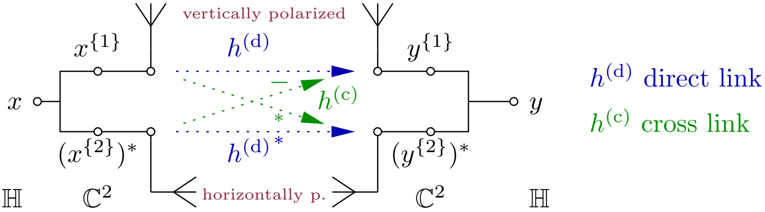
Quaternion-valued SISO fading (four orthogonal components)

$$\underbrace{y}_{y^{\{1\}}+y^{\{2\}}\,\mathrm{j}} = \underbrace{h}_{h^{\{1\}}+h^{\{2\}}\,\mathrm{j}} \cdot \underbrace{x}_{x^{\{1\}}+x^{\{2\}}\,\mathrm{j}}$$

- equivalently described by complex-valued matrix equation

$$\begin{bmatrix} y^{\{1\}} \\ (y^{\{2\}})^* \end{bmatrix} = \begin{bmatrix} h^{\{1\}} & -h^{\{2\}} \\ (h^{\{2\}})^* & (h^{\{1\}})^* \end{bmatrix} \begin{bmatrix} x^{\{1\}} \\ (x^{\{2\}})^* \end{bmatrix} = \begin{bmatrix} h^{(\mathrm{d})} & -h^{(\mathrm{c})} \\ (h^{(\mathrm{c})})^* & (h^{(\mathrm{d})})^* \end{bmatrix} \begin{bmatrix} x^{\{1\}} \\ (x^{\{2\}})^* \end{bmatrix}$$

- realized via RF mixing and dual-polarized antennas [CLF14, LBX16]



Quaternion-valued system equation [IS95, WWS06]

$$y = hx + n$$

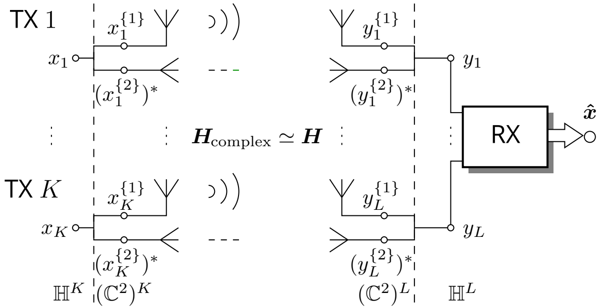
- transmit symbol $x = (\underbrace{x^{(1)} + x^{(2)}\,\mathrm{i}}_{\text{vertical}}) + (\underbrace{x^{(3)} - x^{(4)}\,\mathrm{i}}_{\text{horizontal}})^*\,\mathrm{j}$
  $\rightarrow$ drawn from 4D signal constellation
- fading factor $h = (\underbrace{h^{(1)} + h^{(2)}\,\mathrm{i}}_{\text{direct gain}}) + (\underbrace{h^{(3)} - h^{(4)}\,\mathrm{i}}_{\text{cross-polar gain}})^*\,\mathrm{j}$
  $\rightarrow$ four i.i.d. real-valued Gaussian coefficients
- additive noise $n = (\underbrace{n^{(1)} + n^{(2)}\,\mathrm{i}}_{\text{vertical noise}}) + (\underbrace{n^{(3)} - n^{(4)}\,\mathrm{i}}_{\text{horizontal noise}})^*\,\mathrm{j}$
  $\rightarrow$ four i.i.d. real-valued Gaussian coefficients
- receive symbol $y = (\underbrace{y^{(1)} + y^{(2)}\,\mathrm{i}}_{\text{vertical}}) + (\underbrace{y^{(3)} - y^{(4)}\,\mathrm{i}}_{\text{horizontal}})^*\,\mathrm{j}$

Quaternion-valued vector/matrix system equation

$$\boldsymbol{y} = \boldsymbol{H}\boldsymbol{x} + \boldsymbol{n}\,, \quad \text{with}$$

$$\boldsymbol{H} = \left[h_{l,k}\right]_{\substack{l=1,\dots,L \\ k=1,\dots,K}} \in \mathbb{H}^{L\times K}$$

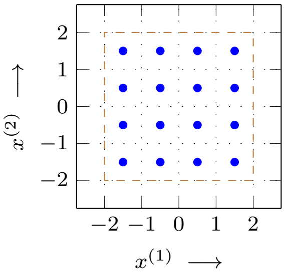
K-user MIMO multiple-access channel with L *pairs of antennas:* [SF18]



Constellations based on Lipschitz integers $\mathcal{L}$

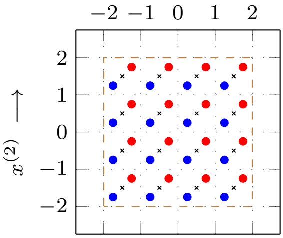
- QAM in each polarization
- aka polarization-multiplexed QAM
- cardinalities $M = 4^2 = 16, 16^2 = 256, \ldots$
- 4D Gray labeling possible
- separable into ASK per component

2D projection



Constellations based on Hurwitz integers $\mathcal{H}$ [SF18]

- subsets $\mathcal{L}_1$ and $\mathcal{L}_2$ with offset $\pm\frac{1}{4}(1 + \mathrm{i} + \mathrm{j} + \mathrm{k})$
- enable one additional bit
  $\rightarrow M = 2 \cdot 16 = 32, 2 \cdot 256 = 512, \ldots$
- same boundaries and minimum distance
- 4D Gray labeling not possible



$\rightarrow$ coded modulation via two-stage approach [SFFF19]

1. offset stage (one bit selects offset/subset)
2. subset stage (conventional schemes, e.g., BICM)

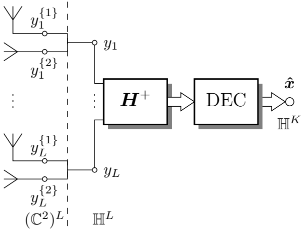
MIMO Equalization: Linear

$L \times K$ MIMO multiple-access channel

- *quaternion-valued* linear equalization
- MMSE criterion via

$$\boldsymbol{H}^{+} = (\boldsymbol{H}^{\mathsf{H}}\boldsymbol{H} + \zeta \boldsymbol{I})^{-1}\boldsymbol{H}^{\mathsf{H}} \in \mathbb{H}^{K \times L}$$

  - regularized pseudoinverse
  - ζ inverse SNR



Diversity order (i.i.d. Gaussian channel)

- slope of (uncoded) error curve $\rightarrow \Delta$ decades per 10 dB SNR
- $h_{l,k} = \mathrm{Re}\{h_{l,k}\} + \mathrm{Im}\{h_{l,k}\}\,\mathrm{i} \in \mathbb{C}$
  $\rightarrow$ two independent gains; i.e., diversity

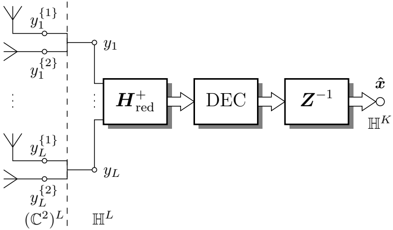
$$\Delta_{\mathrm{c,LIN}} = L - K + 1 \qquad \text{(i.e., } \Delta_{\mathrm{c,LIN}} = 1 \text{ for } L = K\text{)}$$

- $h_{l,k} = h_{l,k}^{(1)} + h_{l,k}^{(2)}\,\mathrm{i} + h_{l,k}^{(3)}\,\mathrm{j} + h_{l,k}^{(4)}\,\mathrm{k} \in \mathbb{H}$
  $\rightarrow$ four independent gains; i.e., diversity

$$\Delta_{\mathrm{q,LIN}} = 2\,(L - K + 1) \qquad \text{(i.e., } \Delta_{\mathrm{q,LIN}} = 2 \text{ for } L = K\text{)}$$

Lattice-reduction-aided (LRA) linear equalization [YW02, WF03]:

- factorization $\boldsymbol{H} = \boldsymbol{H}_{\text{red}}\boldsymbol{Z}$
- $\boldsymbol{Z} \in \boldsymbol{\Lambda}^{K\times K}$ *integer matrix* $\rightarrow$ integer lattice $\boldsymbol{\Lambda}$
- non-integer equalization via $\boldsymbol{H}_{\text{red}}^{+}$
- integer equalization via $\boldsymbol{Z}^{-1}$



Choice of integer matrix/lattice

- $\boldsymbol{H} \in \mathbb{R}^{L\times K}$: $\boldsymbol{\Lambda} = \mathbb{Z}$
- $\boldsymbol{H} \in \mathbb{C}^{L\times K}$: $\boldsymbol{\Lambda} = \mathbb{G} = \mathbb{Z} + \mathbb{Z}\,\mathrm{i}$
- $\boldsymbol{H} \in \mathbb{H}^{L\times K}$:
  - Lipschitz-integer lattice $\mathcal{L}$?
  - Hurwitz-integer lattice $\mathcal{H}$?

  } How to solve factorization task?

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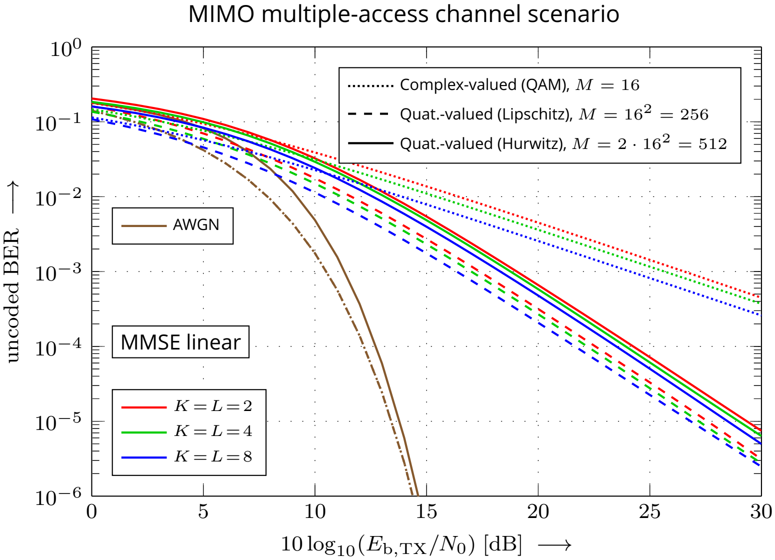
Lenstra-Lenstra-Lovász (LLL) reduction

- reduction criterion/algorithm with polynomial complexity
- *generalization of Euclidean algorithm* w.r.t. matrices [N96]
  $\rightarrow$ division with *small* remainder required
- available variants:
  - (R)LLL for $\mathbf{\Lambda} = \mathbb{Z}$
  - CLLL for $\mathbf{\Lambda} = \mathbb{G}$ [GLM09]
  - QLLL for ~~$\mathbf{\Lambda} = \mathcal{L}$~~ $\rightarrow$ not a Euclidean ring!
  - QLLL for $\mathbf{\Lambda} = \mathcal{H}$ [SF18] $\rightarrow$ inherently contains $\mathcal{L} \subset \mathcal{H}$

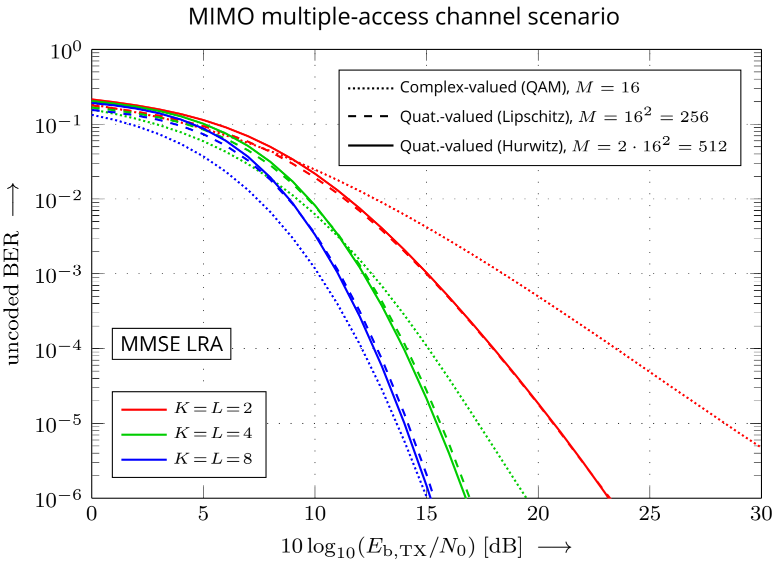
Diversity order (i.i.d. Gaussian channel)

- LRA linear equalization achieves full receive diversity [TMK07]
- complex-valued transmission: $\Delta_{\mathrm{c,LRA}} = L$
- quaternion-valued transmission: $\Delta_{\mathrm{q,LRA}} = 2\,L$

MIMO Channel: Numerical Results (I)



MIMO Channel: Numerical Results (II)



Quaternions

- algebraic structure with four orthogonal components
- multiplication non-commutative
- related rings Lipschitz and Hurwitz integers

Quaternion-valued transmission

- realized via dual-polarized antennas
- doubled bandwidth efficiency
- same power efficiency

} w.r.t. complex-valued case

Quaternion-valued MIMO transmission

- diversity orders doubled w.r.t. complex-valued case
- LRA equalization via QLLL (based on Hurwitz integers)

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