Time-Varying Systems and Computations

On Reflections and Generalized Rotations

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Exemplary Case

QR decomposition as a standard computational tool

$$A = Q \cdot R \qquad A \in \mathcal{R}^{m \times n}$$

$$Q^T Q = 1$$

$$R : \text{upper triangular}$$

Readily available in numerical packages such as Matlab, NumPy, etc. (why not QL, or RQ?)

Standard QR Decomposition

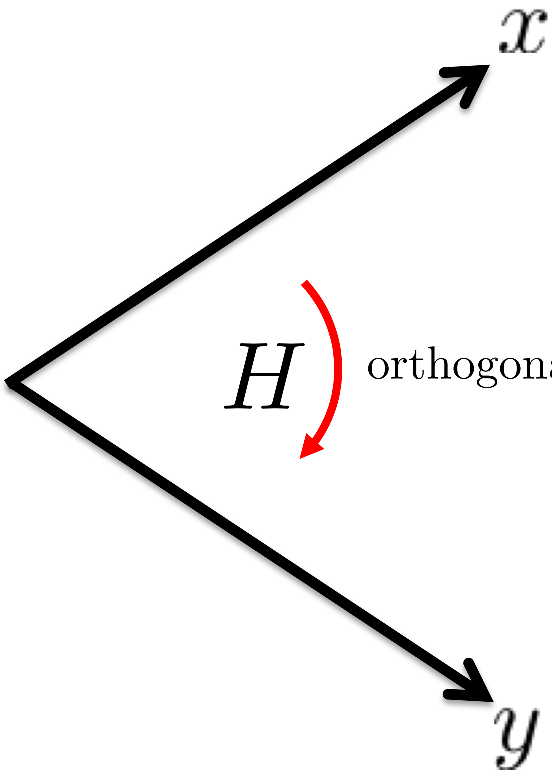
Column-wise elimination process

$$
\underbrace{\begin{bmatrix} \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{bmatrix}}_{A}
\overset{H_1}{\rightarrow}
\begin{bmatrix} \star & \star & \star & \star \\ 0 & \cdot & \cdot & \cdot \\ 0 & \cdot & \cdot & \cdot \\ 0 & \cdot & \cdot & \cdot \\ 0 & \cdot & \cdot & \cdot \\ 0 & \cdot & \cdot & \cdot \end{bmatrix}
\overset{H_2}{\rightarrow}
\begin{bmatrix} \star & \star & \star & \star \\ 0 & \star & \star & \star \\ 0 & 0 & \cdot & \cdot \\ 0 & 0 & \cdot & \cdot \\ 0 & 0 & \cdot & \cdot \\ 0 & 0 & \cdot & \cdot \end{bmatrix}
\overset{H_3}{\rightarrow}
\begin{bmatrix} \star & \star & \star & \star \\ 0 & \star & \star & \star \\ 0 & 0 & \star & \star \\ 0 & 0 & 0 & \cdot \\ 0 & 0 & 0 & \cdot \\ 0 & 0 & 0 & \cdot \end{bmatrix}
\overset{H_4}{\rightarrow}
\begin{bmatrix} \star & \star & \star & \star \\ 0 & \star & \star & \star \\ 0 & 0 & \star & \star \\ 0 & 0 & 0 & \star \\ \hline 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} R
$$

$$Q = H_1 \cdot H_2 \cdot H_3 \cdot H_4$$

Note: Changed Matlab interface

Elementary Step



orthogonal transformation

$$x = \begin{bmatrix} x_1 \\ \hline | \\ x_2 \\ | \end{bmatrix}$$

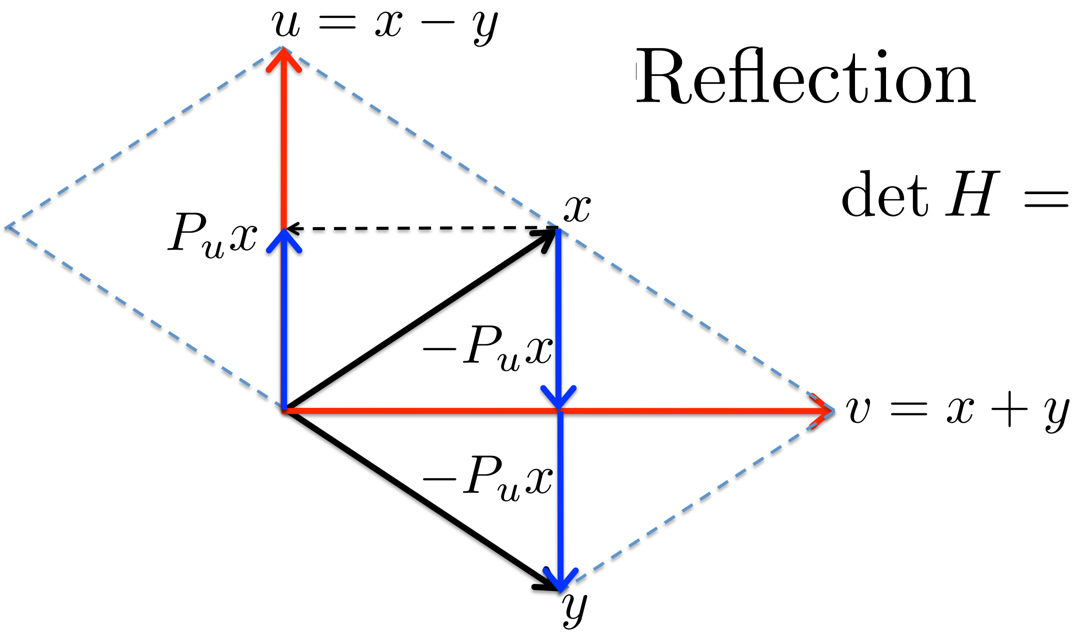
$$\|x\|^2 = x_1^2 + x_2^T x_2$$

$$y = H \cdot x$$

$$y = \begin{bmatrix} \|x\| \\ \hline 0 \\ \vdots \\ 0 \end{bmatrix}$$

$$\|x\|^2 = \|y\|^2 = 1$$

Standard Householder Construction



Reflection

$$\det H = -1$$

$$y = \underbrace{(1 - 2 \cdot P_u)}_{H} x \qquad P_u = u(u^T u)^{-1} u^T$$

Quest for Rotations

Computer Vision, Robotics, Machine Intelligence

We need Q to be a rotation!

- Clifford Algebra ☹ ☹
- Exponential map of skew-symmetric matrices ☹ ☹
- 2 reflections give a rotation ☹
- Quick post-fixes possible (e.g. for SVD) ☹
- Use rotations for eliminiation → Givens rotations ☺

But: Householder is twice as efficient as Givens! ☹

QR Decomposition Methods

What do the textbooks say?

Gram- Schmidt	– not recommended (Golub,vLoan)
Householder QR	– Reflections (Matlab, NumPy)
Givens QR	– Rotations

<u>Question:</u>

Can we <u>construct</u> a Householder-like rotation?

Constructing a Householder Rotation

$$x = \begin{bmatrix} x_1 \\ \hline | \\ x_2 \\ | \end{bmatrix}$$

$$y = \begin{bmatrix} 1 \\ \hline | \\ 0 \\ | \end{bmatrix}$$

Find T

$$y = T \cdot x \quad \begin{cases} T \in \mathcal{R}^{n \times n} \\ T^T T = 1 \\ \det T = 1 \end{cases}$$

Constructing a Householder Rotation

Start with a partially specified matrix

$$T = \begin{bmatrix} x_1 & x_2^T \\ \star & K \end{bmatrix}$$

Orthogonality condition

$$T^T T = \begin{bmatrix} x_1^2 + \star^T \star & x_1 x_2^T + \star^T K \\ x_1 x_2 + K^T \star & K^T K + x_2 x_2^T \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Constructing a Householder Rotation

From orthogonality condition …

$$T^T T = \begin{bmatrix} x_1^2 + \star^T \star & x_1 x_2^T + \star^T K \\ x_1 x_2 + K^T \star & K^T K + x_2 x_2^T \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

… we can read off

$$\star = \pm x_2 \qquad K^T K = 1 - x_2 x_2^T$$

Constructing a Householder Rotation

Rank-one Perturbation of Identity with Parameter k

$$K = 1 - k x_2 x_2^T$$

Plugged into Orthogonality (22-Block of $T^T T$)

$$(1 - k x_2 x_2^T) \cdot (1 - k x_2 x_2^T) = 1 - x_2 x_2^T$$

Quadratic Equation for Parameter k

$$k^2(1 - x_1^2) - 2k + 1 = 0 \implies \quad k_1 = \frac{1}{1 - x_1} \quad k_2 = \frac{1}{1 + x_1}$$

Constructing a Householder Rotation

Resolve sign ambiguity

$$\begin{bmatrix} x_1 & x_2^T \\ \pm x_2 & K \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1^2 + x_2^T x_2 \\ \pm x_2 x_1 + K x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$k_1 = \frac{1}{1 - x_1} \implies T_1 = \begin{bmatrix} x_1 & x_2^T \\ x_2 & K \end{bmatrix}$$

$$k_2 = \frac{1}{1 + x_1} \implies T_2 = \begin{bmatrix} x_1 & x_2^T \\ -x_2 & K \end{bmatrix}$$

Constructing a Householder Rotation

Two candidate matrices (check determinant)

$$T_1 = \begin{bmatrix} x_1 & x_2^T \\ x_2 & 1 - \frac{x_2 x_2^T}{1 - x_1} \end{bmatrix} \implies \det T_1 = -1$$

Reflection

$$T_2 = \begin{bmatrix} x_1 & x_2^T \\ -x_2 & 1 - \frac{x_2 x_2^T}{1 + x_1} \end{bmatrix} \implies \det T_2 = 1$$

Rotation

Householder Rotation

Rotation is constructed from elementary concepts of Linear Algebra (linear map, orthogonality, …)

It is very similar to reflections and leads to the same computational complexity than reflection-based QRD

Rotation-based QRD is not covered in textbooks and/or manuals