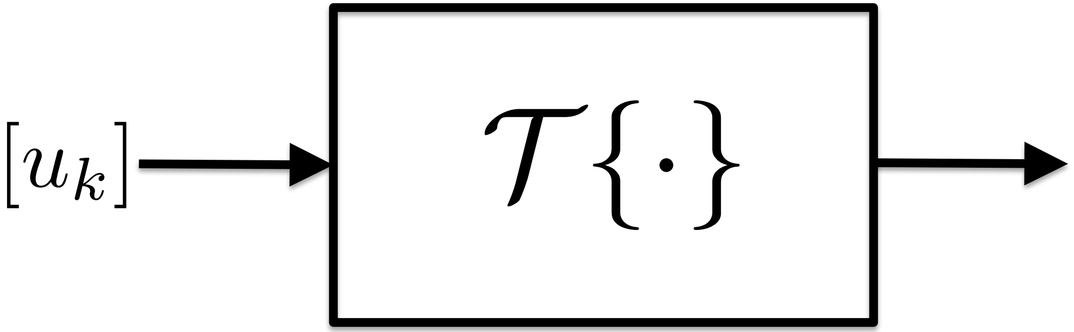
kldi

Linear Time-Invariant Systems

Input-Output Description

Linear Time-Invariant System



Convolution Operator

$$[y_k] = \mathcal{T}\{[u_k]\}$$

$$k = \ldots, -1, 0, 1, 2, \ldots$$

$$\begin{aligned}[y_k] &= [t_k] \star [u_k] \\ &= \sum_{i=-\infty}^{\infty} [t_{k-i}] \cdot u_i\end{aligned}$$

Linear Time-Invariant System

Convolution Operator

$$[u_k] \longrightarrow \boxed{\mathcal{T}\{\cdot\}} \longrightarrow [y_k] = \mathcal{T}\{[u_k]\}$$

$$k = \ldots, -1, 0, 1, 2, \ldots$$

$$\vec{u} = \begin{bmatrix} \vdots \\ u_{-1} \\ u_0 \\ u_1 \\ u_2 \\ \vdots \end{bmatrix} \qquad \vec{t} = \begin{bmatrix} \vdots \\ t_{-1} \\ t_0 \\ t_1 \\ t_2 \\ \vdots \end{bmatrix} \qquad \vec{y} = \begin{bmatrix} \vdots \\ y_{-1} \\ y_0 \\ y_1 \\ y_2 \\ \vdots \end{bmatrix}$$

Convolution Operator: Toeplitz Matrix

$$\vec{y} = \vec{t} \star \vec{u} = T \cdot \vec{u}$$

$$[y_k] = \mathcal{T}\{[u_k]\}$$

$$k = \ldots, -1, 0, 1, 2, \ldots$$

Convolution Operator

$$\underbrace{\begin{bmatrix} \vdots \\ y_{-1} \\ y_0 \\ y_1 \\ y_2 \\ \vdots \end{bmatrix}}_{\vec{y}} = \underbrace{\begin{bmatrix} \ddots & \vdots & \vdots & \vdots & \vdots & \\ \ddots & t_{-1} & t_{-2} & t_{-3} & t_{-4} & \\ \ddots & t_0 & t_{-1} & t_{-2} & t_{-3} & \ddots \\ \ddots & t_1 & t_0 & t_{-1} & t_{-2} & \ddots \\ & t_2 & t_1 & t_0 & t_{-1} & \ddots \\ & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}}_{T} \cdot \underbrace{\begin{bmatrix} \vdots \\ u_{-1} \\ u_0 \\ u_1 \\ u_2 \\ \vdots \end{bmatrix}}_{\vec{u}}$$

→ Infinite-dimensional Toeplitz Matrix

$$T = \begin{bmatrix} \ddots & \vdots & \vdots & \vdots & \vdots & \\ \ddots & t_{-1} & t_{-2} & t_{-3} & t_{-4} & \\ \ddots & t_0 & t_{-1} & t_{-2} & t_{-3} & \ddots \\ \ddots & t_1 & t_0 & t_{-1} & t_{-2} & \ddots \\ & t_2 & t_1 & t_0 & t_{-1} & \ddots \\ & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

Need for efficient

- Represenation
- Computation

Toeplitz Matrix exhibits special (shift) structure

Shift Operator

$$u_k = \mathcal{Z}\{u_{k+1}\}$$

Shift Operator – Time Delay

$$Z \cdot \begin{bmatrix} \vdots \\ u_{-1} \\ \boxed{u_0} \\ u_1 \\ u_2 \\ \vdots \end{bmatrix} = \begin{bmatrix} \vdots \\ u_{-2} \\ \boxed{u_{-1}} \\ u_0 \\ u_1 \\ \vdots \end{bmatrix},$$

Pre-multiply with Shift Matrix $\Rightarrow u \downarrow$

→ Shifting all column vector entries down by one position

Shift Matrix

$$Z = \begin{bmatrix} \ddots & & & & & \\ \ddots & 0 & & & & \\ & 1 & 0 & & & \\ & & 1 & 0 & & \\ & & & 1 & 0 & \\ & & & & \ddots & \ddots \end{bmatrix}$$

Infinite dimensional orthogonal matrix

$$Z'Z = ZZ' = 1$$

$$Z' = Z^{-1}$$

Building the Toeplitz Matrix

$$T = \begin{bmatrix} & & | & | & | & | & & \\ \cdots & & Z^{-1}\bar{t} & Z^{0}\bar{t} & Z^{1}\bar{t} & Z^{2}\bar{t} & & \cdots \\ & & | & | & | & | & & \end{bmatrix}$$

Convolution

$$\begin{bmatrix} \vdots \\ y_{-1} \\ y_0 \\ y_1 \\ y_2 \\ \vdots \end{bmatrix} = \begin{bmatrix} & | & | & | & | & \\ \dots & Z^{-1}\vec{t} & Z^{0}\vec{t} & Z^{1}\vec{t} & Z^{2}\vec{t} & \dots \\ & | & | & | & | & \end{bmatrix} \cdot \begin{bmatrix} \vdots \\ u_{-1} \\ u_0 \\ u_1 \\ u_2 \\ \vdots \end{bmatrix}$$

$$\vec{y} = \sum_{i=-\infty}^{\infty} \left(Z^i \vec{t} \right) u_i$$

Superposition of Diagonals

$$T = \begin{bmatrix} \ddots & \ddots & & & \\ \ddots & t_0 & t_{-1} & & \\ & t_1 & t_0 & t_{-1} & \\ & & t_1 & t_0 & \ddots \\ & & & \ddots & \ddots \end{bmatrix} =$$

$$+ \begin{bmatrix} \ddots & \ddots & & & \\ & \ddots & t_{-1} & & \\ & & \ddots & t_{-1} & \\ & & & \ddots & \ddots \\ & & & & \ddots \end{bmatrix} + \begin{bmatrix} \ddots & & & & \\ & t_0 & & & \\ & & t_0 & & \\ & & & t_0 & \\ & & & & \ddots \end{bmatrix} + \begin{bmatrix} \ddots & & & & \\ \ddots & \ddots & & & \\ & t_1 & \ddots & & \\ & & t_1 & \ddots & \\ & & & \ddots & \ddots \end{bmatrix} +$$

Shifting of Diagonals

$$T =$$

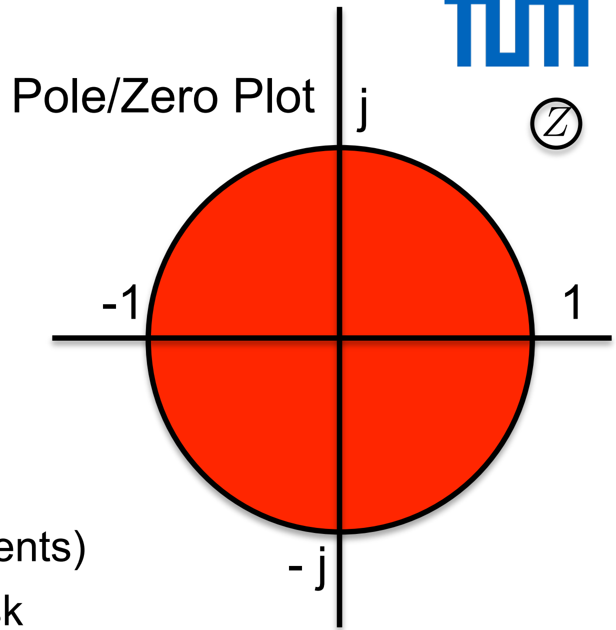
$$\cdots + Z^{-1} \cdot \begin{bmatrix} \ddots & & & \\ & t_{-1} & & \\ & & t_{-1} & \\ & & & \ddots \end{bmatrix} + \begin{bmatrix} \ddots & & & \\ & t_0 & & \\ & & t_0 & \\ & & & \ddots \end{bmatrix} + Z \cdot \begin{bmatrix} \ddots & & & \\ & t_1 & & \\ & & t_1 & \\ & & & \ddots \end{bmatrix} + \cdots$$

$$\cdots + Z^{-1} diag(t_{-1}) + Z^{0} diag(t_0) + Z^{1} diag(t_1) + \ldots$$

$$T = \sum_{i-\infty}^{\infty} Z^i \cdot diag(t_i) = \sum_{i-\infty}^{\infty} Z^i \cdot \bar{t}_i \qquad \bar{t}_i := diag(t_i)$$

Z-Transformation

$$T(z) = \sum_{i=-\infty}^{\infty} t_i z^i$$



$T(z)$ is complex valued transfer function

z is complex-valued scalar variable

Positive exponents of z (normally we use negative exponents)

→ Stability region – poles must be outside unit disk

t_i are scalar sample values

Method for handling infinite dimensional Toeplitz matrices

Works for time-invariant systems

Diagonal Expansion

$$T = \sum_{i=-\infty}^{\infty} Z^i \cdot \bar{t}_i$$

T is an infinite-dimensional Toeplitz matrix

Z is the real-valued shift matrix

Positive exponents of Z

$\bar{t}_i$ infinite-dimensional diagonal matrix

Convenient method for directly using computational tools from linear algebra

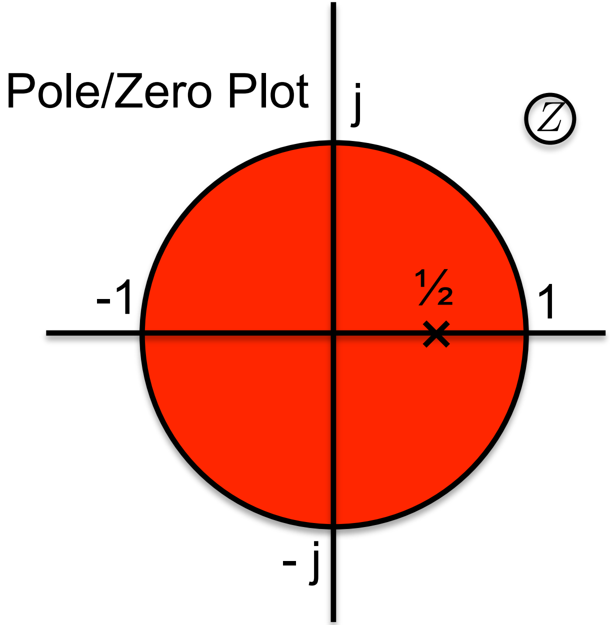
Inversion of Toeplitz Operator – Example 1

$$T(z) = 1 - 2z \quad \Rightarrow \quad \boxed{T^{-1}(z) = \frac{1}{1-2z}}$$

$$T = \begin{bmatrix} \ddots & & & & & \\ \ddots & 1 & & & & \\ & -2 & 1 & & & \\ & & -2 & 1 & & \\ & & & -2 & 1 & \\ & & & & \ddots & \ddots \end{bmatrix} \quad \Rightarrow \quad T^{-1} = ?$$

Inversion of Toeplitz Operator – Example 1

$$T^{-1}(z) = \frac{1}{1-2z} = 1 + 2z + 4z^2 + 8z^3 + 16z^4 + \ldots$$



Note: $1 + q + q^2 + q^3 + \cdots = \dfrac{1}{1-q}$

$|q| < 1$

→ pole inside unit disc at $z = \dfrac{1}{2}$
→ Instability
→ Series does not converge

Inversion of Toeplitz Operator – Example 1

$$T = \bar{1} - Z \cdot \bar{2}$$

$$\Rightarrow T^{-1} = \frac{1}{1 - Z \cdot \bar{2}}$$

$$T = \begin{bmatrix} \ddots & & & & & \\ \ddots & 1 & & & & \\ & -2 & 1 & & & \\ & & -2 & 1 & & \\ & & & -2 & 1 & \\ & & & & \ddots & \ddots \end{bmatrix} \Rightarrow \; T^{-1} = ?$$

Inversion of Toeplitz Operator – Example 1

$$T^{-1} = \frac{1}{1 - Z \cdot \bar{2}} \quad = \bar{1} + (Z \cdot \bar{2}) + (Z \cdot \bar{2})^2 + (Z \cdot \bar{2})^3 + \dots$$

$$T^{-1} = \begin{bmatrix} \ddots & & & & & \\ \ddots & 1 & & & & \\ \ddots & 2 & 1 & & & \\ \ddots & 4 & 2 & 1 & & \\ \ddots & 8 & 4 & 2 & 1 & \\ & \ddots & \ddots & \ddots & \ddots & \ddots \end{bmatrix}$$

Inverse Matrix is unbounded

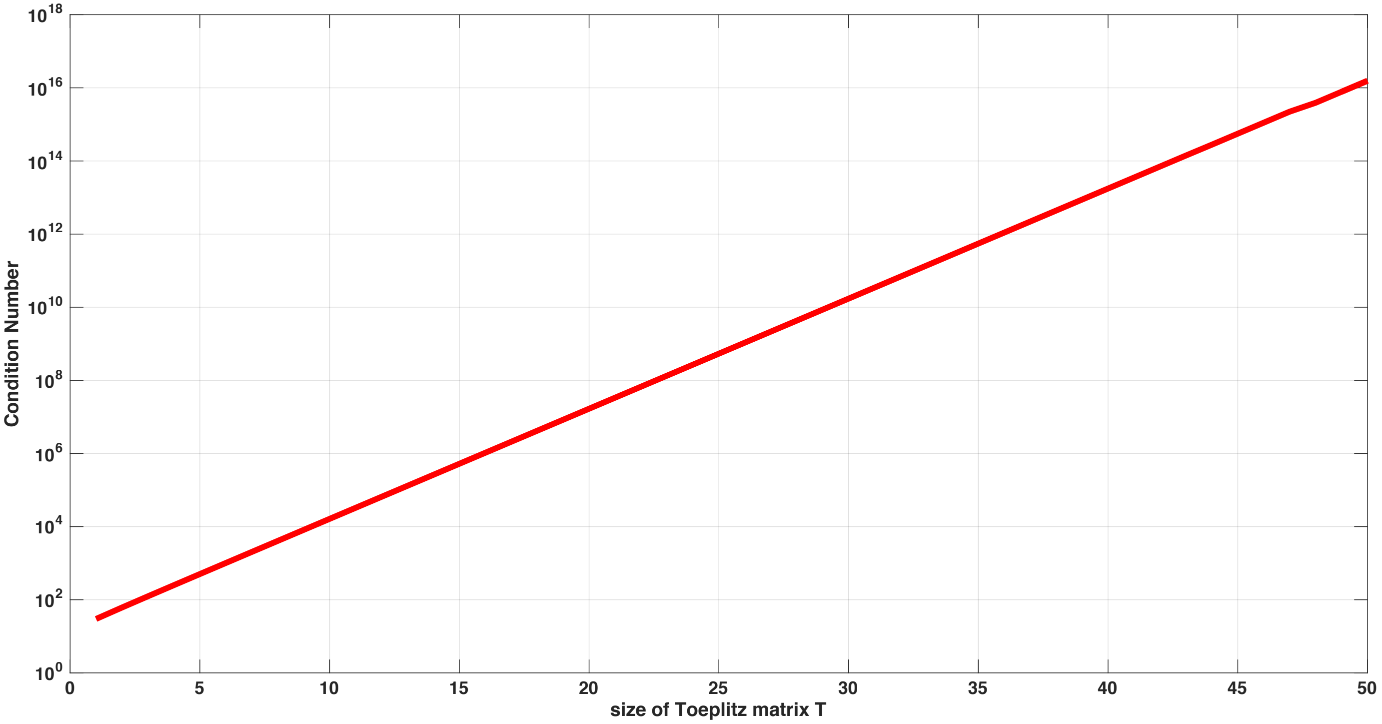
Finite 6 × 6 – Matrix – Example 1

$$T = \begin{bmatrix} 1 & & & & & \\ -2 & 1 & & & & \\ & -2 & 1 & & & \\ & & -2 & 1 & & \\ & & & -2 & 1 & \\ & & & & -2 & 1 \end{bmatrix}$$

Ask Matlab:

$$T^{-1} = \begin{bmatrix} 1 & & & & & \\ 2 & 1 & & & & \\ 4 & 2 & 1 & & & \\ 8 & 4 & 2 & 1 & & \\ 16 & 8 & 4 & 2 & 1 & \\ 32 & 16 & 8 & 4 & 2 & 1 \end{bmatrix}$$

Condition number of T – Example 1



Inversion of Toeplitz Operator – Example 2

$$T = \begin{bmatrix} \ddots & & & & & \\ \ddots & 1 & & & & \\ & -1/2 & 1 & & & \\ & & -1/2 & 1 & & \\ & & & -1/2 & 1 & \\ & & & & \ddots & \ddots \end{bmatrix}$$

$$T = \bar{1} - Z \cdot \frac{\bar{1}}{2} \quad \Rightarrow \quad T^{-1} = \frac{1}{\bar{1} - Z \cdot \frac{\bar{1}}{2}}$$

Inversion of Toeplitz Operator – Example 2

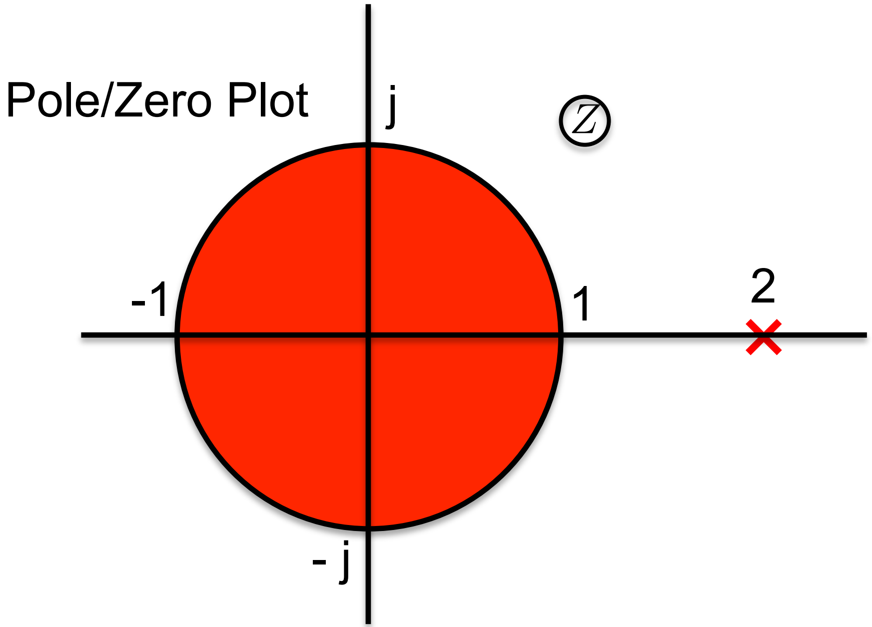
$$T^{-1} = \frac{1}{\bar{1} - Z \cdot \frac{\bar{1}}{2}} \quad \Rightarrow = \bar{1} + (Z \cdot \frac{\bar{1}}{2}) + (Z \cdot \frac{\bar{1}}{2})^2 + (Z \cdot \frac{\bar{1}}{2})^3 + \ldots$$

$$T^{-1} = \begin{bmatrix} \ddots & & & & & \\ \ddots & 1 & & & & \\ \ddots & 1/2 & 1 & & & \\ \ddots & 1/4 & 1/2 & 1 & & \\ \ddots & 1/8 & 1/4 & 1/2 & 1 & \\ & \ddots & \ddots & \ddots & \ddots & \ddots \end{bmatrix}$$

Inverse Matrix is bounded

Inversion of Toeplitz Operator – Example 2

$$T(z)^{-1} = \frac{1}{1-\frac{1}{2}z} = 1 + \frac{1}{2}z + \frac{1}{4}z^2 + \frac{1}{8}z^3 + \ldots$$



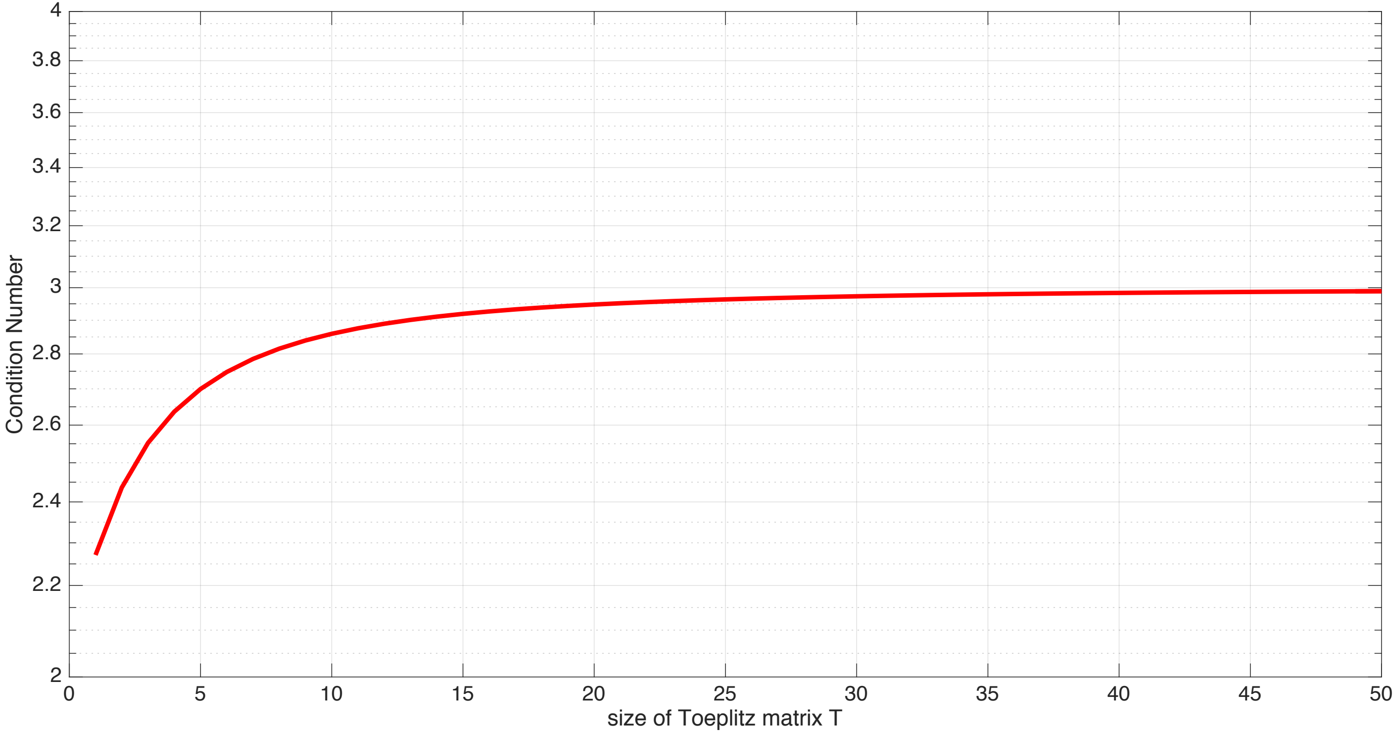
→ Stable pole outside unit disk at $z = 2$
→ Stability of transfer function
→ Inverse Fktn. converges

Finite 6 × 6 – Matrix – Example 2

$$T = \begin{bmatrix} 1 & & & & & \\ -1/2 & 1 & & & & \\ & -1/2 & 1 & & & \\ & & -1/2 & 1 & & \\ & & & -1/2 & 1 & \\ & & & & -1/2 & 1 \end{bmatrix}$$

Ask Matlab: $T^{-1} = \begin{bmatrix} 1 & & & & & \\ 1/2 & 1 & & & & \\ 1/4 & 1/2 & 1 & & & \\ 1/8 & 1/4 & 1/2 & 1 & & \\ 1/16 & 1/8 & 1/4 & 1/2 & 1 & \\ 1/32 & 1/16 & 1/8 & 1/4 & 1/2 & 1 \end{bmatrix}$

Condition number of T – Example 2



Non-Toeplitz Matrix – Example 3

$$T = \left[\begin{array}{ccc|cccc}
\ddots & & & & & & \\
\ddots & 1 & & & & & \\
 & -2 & 1 & & & & \\
 & & -2 & 1 & & & \\
\hline
 & & & -1/2 & 1 & & \\
 & & & & -1/2 & 1 & \\
 & & & & & \ddots & \ddots
\end{array}\right]$$

$$T = \bar{1} - Z \cdot \bar{t} \quad \Rightarrow \quad T^{-1} = ?$$

Non-Toeplitz Matrix – Example 3

$$T = \bar{1} - Z \cdot \bar{t} \qquad \Rightarrow \quad T^{-1} = (\bar{1} - Z \cdot \bar{t})^{-1}$$

$$T^{-1} = \bar{1} + (Z \cdot \bar{t})^1 + (Z \cdot \bar{t})^2 + (Z \cdot \bar{t})^3 + \dots$$

$$\bar{t} = \left[\begin{array}{ccc|ccc} \ddots & & & & & \\ & 2 & & & & \\ & & 2 & & & \\ \hline & & & 1/2 & & \\ & & & & 1/2 & \\ & & & & & \ddots \end{array}\right]$$

Note:

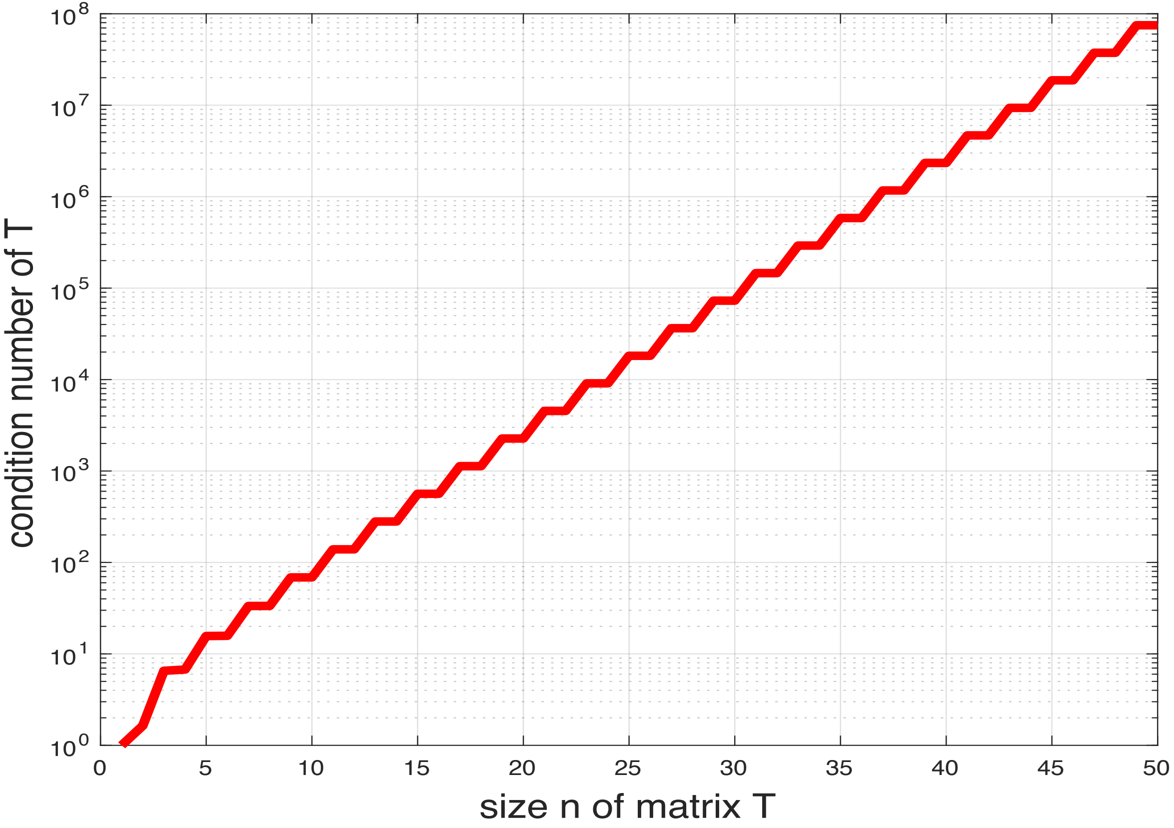
$$(Z \cdot \bar{t})^2 = Z \cdot \bar{t} \cdot Z \cdot \bar{t}$$
$$\neq Z^2 \cdot \bar{t}^2$$

Non-Toeplitz Matrix – Example 3

$$T^{-1} = \bar{1} + (Z \cdot \bar{t})^1 + (Z \cdot \bar{t})^2 + (Z \cdot \bar{t})^3 + \ldots$$

$$T^{-1} = \left[\begin{array}{cccc|cccc} 1 & & & & & & & \\ 2 & 1 & & & & & & \\ 4 & 2 & 1 & & & & & \\ 8 & 4 & 2 & 1 & & & & \\ \hline 4 & 2 & 1 & 1/2 & 1 & & & \\ 2 & 1 & 1/2 & 1/4 & 1/2 & 1 & & \\ 1 & 1/2 & 1/4 & 1/8 & 1/4 & 1/2 & 1 & \end{array}\right]$$

Condition number of Non-Toeplitz T – Ex. 3



Summary

linear convolution of two sequences → matrix-vector mulitplication using Toeplitz matrix

complex variable z → real-valued Shift matrix

Pole/zero plot in complex plane → eigenvalues of Toeplitz matrix

Instability of transfer function → unboundedness of Toeplitz matrix

Linear time-invariant systems → infinite dimensional Toeplitz matrices

Finite dimensional matrices → linear time-variant systems