Lecture 2

System Identification and Canonical Forms

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Overview

- Fast recap on systems and identification
- Isometric, co-isometric and unitary inners
- Numerics: RQ-factorization
- External canonical forms
- Outer-inner factorization
- A "simple" numerical example
- What shall be next?

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Dynamical System Theory: a definition?

The theory that describes the evolution of a system as time progresses

Key notion #1: the STATE of the system: "what the system remembers from its past"

Key notion #2: the EVOLUTION of the state (i.e. its dynamics)

Key notion #3: the BEHAVIOR of the system (i.e. how the system looks from the outside)

Basic notions

- the STATE: a time dependent vector
- the EVOLUTION of the state: a difference equation
- REACHABILITY: how a state can be reached by past inputs (important for control)
- OBSERVABILITY: how one can estimate the state of a system by observing it (important for estimation)
- MINIMALITY: no superfluous states!

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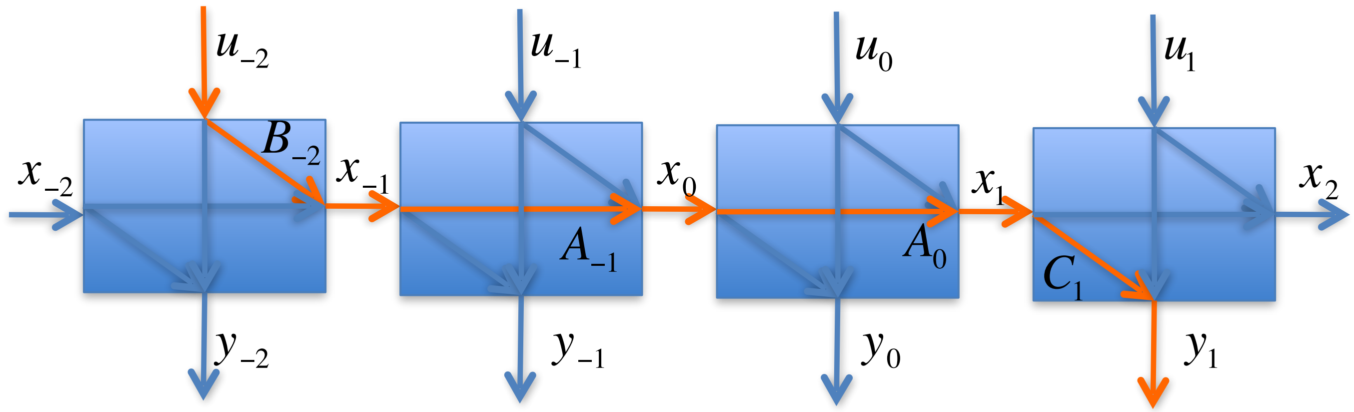
the input-output operator (causal)

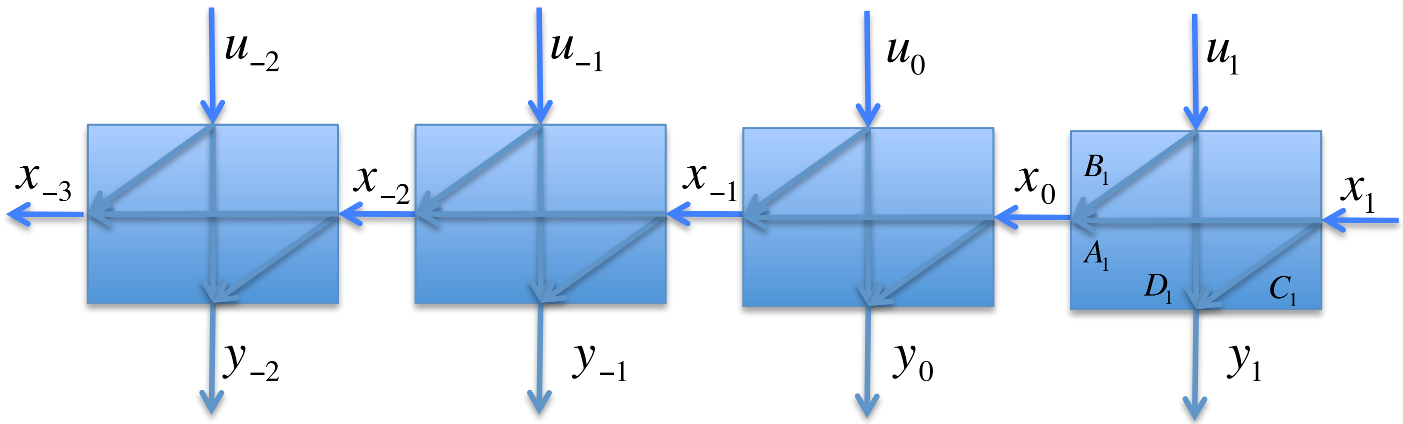


$$\begin{bmatrix} \vdots \\ y_{-2} \\ y_{-1} \\ y_0 \\ y_1 \\ \vdots \end{bmatrix} = \begin{bmatrix} \ddots & & 0 & \\ & & & \\ & & D_0 & \\ C_1 A_0 A_{-1} B_{-2} & & & \ddots \end{bmatrix} \begin{bmatrix} \vdots \\ u_{-2} \\ u_{-1} \\ u_0 \\ u_1 \\ \vdots \end{bmatrix}$$

-2

$$y = Tu$$

input-output anti-causal



$$T = \begin{bmatrix} \ddots & \ddots & \ddots & \ddots & \iddots \\ & D_{-1} & C_{-1}B_0 & C_{-1}A_0B_1 & \ddots \\ & & D_0 & C_0B_1 & \ddots \\ & 0 & & D_1 & \ddots \\ & & & & \ddots \end{bmatrix}$$

Representations

Linear Time-Invariant:

$$\begin{cases} U(z) = \cdots + u_{-1}z^{-1} + u_0 + u_1 z + \dots \\ Y(z) = \cdots + y_{-1}z^{-1} + y_0 + y_1 z + \cdots \\ T(z) = D + C(I - zA)^{-1} zB \end{cases}$$

Time-variant: define *block diagonal operators*

instantaneous:

$$= \begin{bmatrix} \ddots & & & \\ & A_{-1} & & & \\ & & \boxed{A_0} & & \\ & & & A_1 & \\ & & & & \ddots \end{bmatrix}, B = \begin{bmatrix} \ddots & & & \\ & B_{-1} & & & \\ & & \boxed{B_0} & & \\ & & & B_1 & \\ & & & & \ddots \end{bmatrix}$$

etc...

shifts, causal:

$$Z = \begin{bmatrix} \ddots & & & & \\ \ddots & 0 & & & \\ & I & \boxed{0} & & \\ & & I & 0 & \\ & & & \ddots & \ddots \end{bmatrix}$$

anti-causal:

$$Z' = \begin{bmatrix} \ddots & \ddots & & & \\ & 0 & I & & \\ & & 0 & I & \\ & & & \ddots & \ddots \end{bmatrix}$$

Resulting transfer operators:

$$T = D + C(I - ZA)^{-1} ZB \qquad T = D + C(I - Z'A)^{-1} Z'B$$

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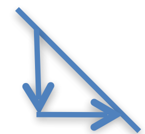
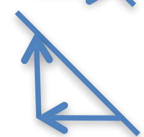
Stability?

The state evolves as:

$$(I - ZA)^{-1} = I + ZA + ZAZA + \cdots$$

Define diagonal shifts:

$$A^{\langle +1 \rangle} = ZAZ'$$  *(forward)*

$$A^{\langle -1 \rangle} = Z'AZ$$  *(backward)*

$$(I - ZA)^{-1} = I + ZA + Z^2 A^{\langle -1 \rangle} A + Z^3 A^{\langle -2 \rangle} A^{\langle -1 \rangle} A + \cdots$$

continuous product should decrease exponentially
(called "u.e.s."= uniform exponentially stable)

example of unstable:

$$\begin{bmatrix} 1 & & & & \\ -2 & 1 & & & \\ & -2 & 1 & & \\ & & -2 & 1 & \\ & & & -2 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & & & & \\ 2 & 1 & & & \\ 4 & 2 & 1 & & \\ 8 & 4 & 2 & 1 & \\ 16 & 8 & 4 & 2 & 1 \end{bmatrix}$$

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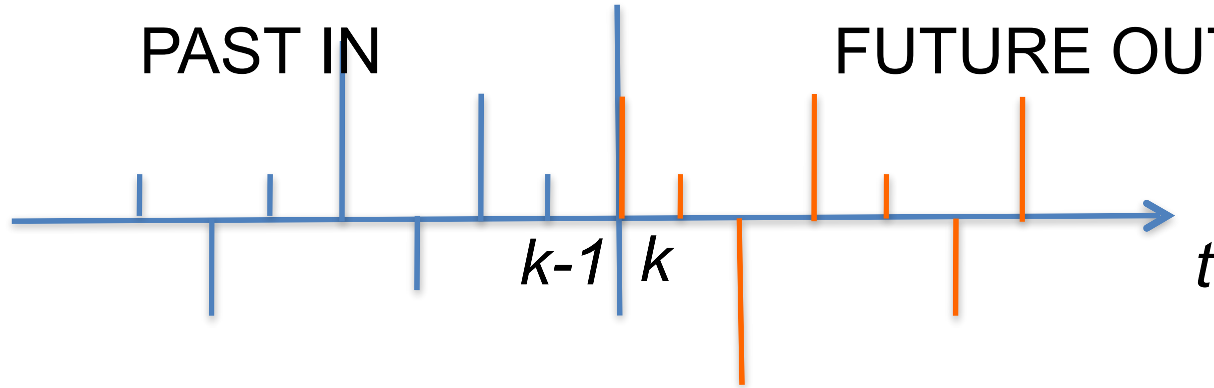
Past-present-future: the Hankel operator

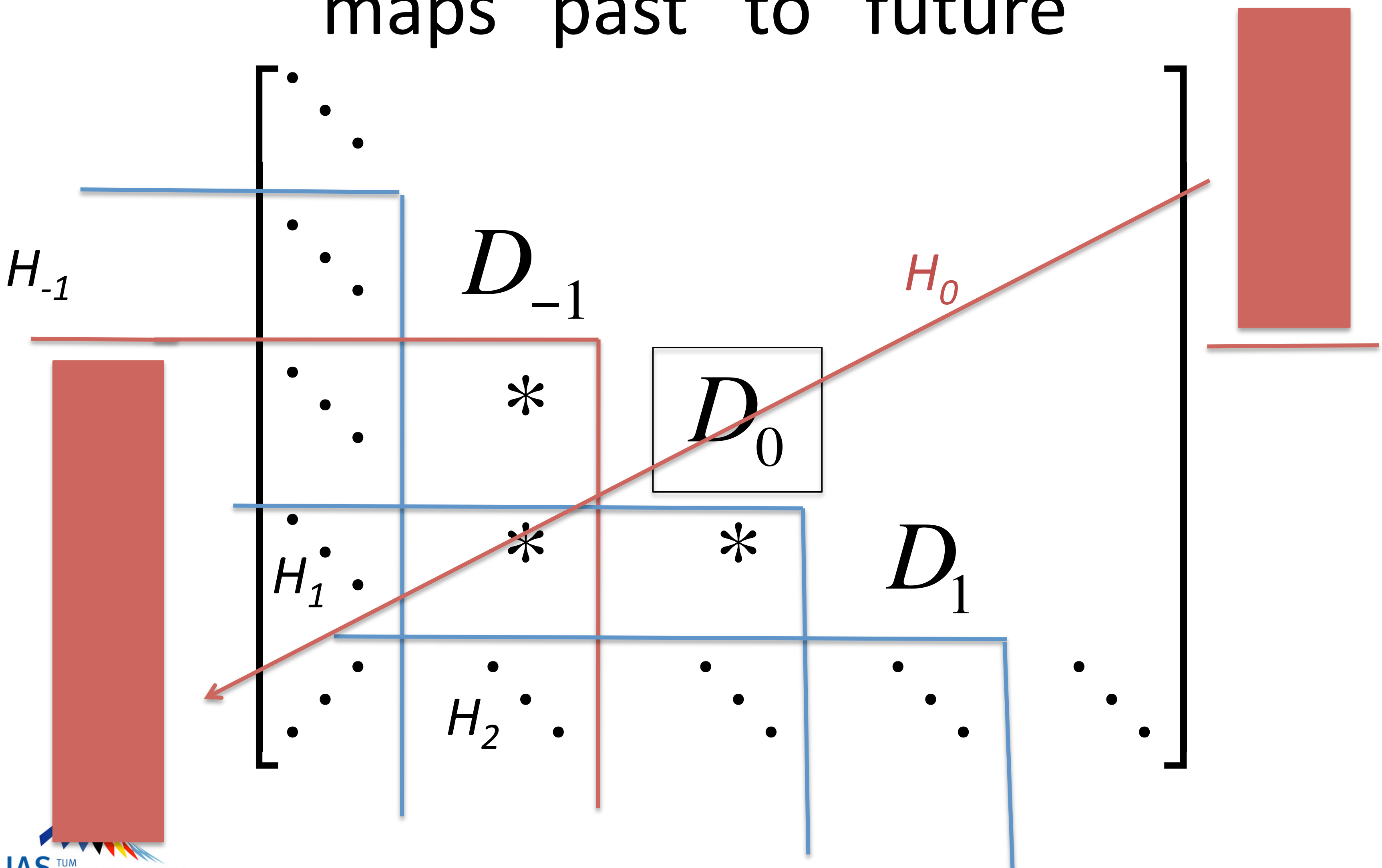
the causal case: H_k maps past inputs up to *k-1* to future outputs from *k* on:



$$\begin{bmatrix} & & \ddots & & \\ \cdots & C_k A_{k-1} B_{k-2} & C_k B_{k-1} & D_k & \\ \cdots & C_{k+1} A_k A_{k-1} B_{k-2} & C_{k+1} A_k B_{k-1} & & \ddots \\ \cdots & C_{k+2} A_{k+1} A_k A_{k-1} B_{k-2} & C_{k+2} A_{k+1} A_k B_{k-1} & & \\ \cdot^{\cdot^{\cdot}} & \vdots & \vdots & & \end{bmatrix}$$

$$H_k = \begin{bmatrix} T_{k,k-1} & T_{k,k-2} & \cdots \\ T_{k+1,k-1} & T_{k+1,k-2} & \cdots \\ \vdots & \vdots & \ddots \end{bmatrix}$$

the (causal) Hankel operator: maps "past" to "future"



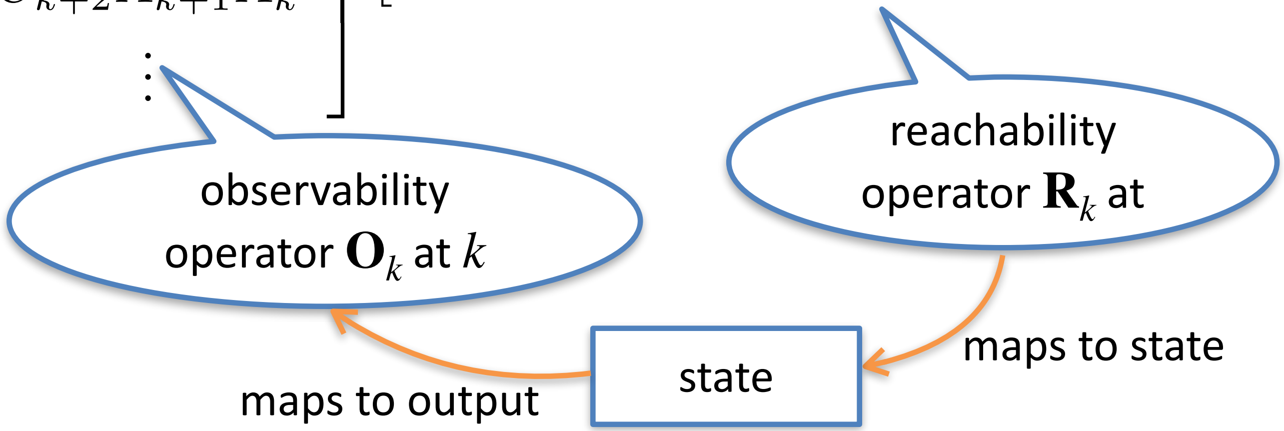
System identification: factoring the Hankel operator

Given T, what is a minimal realization $\{A, B, C, D\}$?

The answer: it is given by a minimal factorization of **each** Hankel operator H_{Tk}:

(generalized Kronecker theorem)

$$H_{\text{Tk}} = \begin{bmatrix} C_k \\ C_{k+1}A_k \\ C_{k+2}A_{k+1}A_k \\ \vdots \end{bmatrix} \begin{bmatrix} \cdots & A_{k-1}A_{k-2}B_{k-3} & A_{k-1}B_{k-2} & B_{k-1} \end{bmatrix}$$



minimal factorization ≡ choosing complementary bases

Identification (2) and normal forms

Remark: $\mathbf{O}_k = \begin{bmatrix} C_k \\ \mathbf{O}_{k+1} A_k \end{bmatrix}$, $\mathbf{R}_k = \begin{bmatrix} A_{k-1}\mathbf{R}_{k-1} & B_{k-1} \end{bmatrix}$

hence: $C_k = [\mathbf{O}_k]_k,\ B_k = [\mathbf{R}_{k+1}]_k,\ A_k = \mathbf{O}_{k+1}^{+}[\mathbf{O}_k]_{k+1:\infty}$

Input normal form: choose an orthonormal basis for all R_k

then $\begin{bmatrix} A_k & B_k \end{bmatrix}$ is co-isometric: $A_k A_k' + B_k B_k' = I$

Output normal form: choose an orthonormal basis for all O_k

then $\begin{bmatrix} A_k \\ C_k \end{bmatrix}$ will be isometric: $A_k' A_k + C_k' C_k = I$

Change of basis (causal case -- notice: R_k is **very** different from $\mathbf{R}_k$!):

$$x_k = R_k \hat{x}_k \Rightarrow \begin{bmatrix} A_k & B_k \\ C_k & D_k \end{bmatrix} \mapsto \begin{bmatrix} R_{k+1}^{-1} A_k R_k & R_{k+1}^{-1} B_k \\ C_k R_k & D_k \end{bmatrix}$$

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How to obtain a normalized from any minimal factorization?

Solve an RQ factorizstion for the base transformation to obtain the Output Normal Form -- this is called a *Lyapunov-Stein equation*:

$$R_{k+1}R'_{k+1} = A_k R_k R'_k A'_k + B_k B'_k$$ (a forward recursion)

Best method: R-Q factorization (square root algorithm):

$$\begin{bmatrix} A_k R_k & B_k \end{bmatrix} = \begin{bmatrix} 0 & R_{k+1} \end{bmatrix} Q_k$$

Compress columns

Unitary matrix

example: $$\begin{bmatrix} \sqrt{2} & 1 & 3 \\ 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & -2 & 2\sqrt{2} \\ 0 & 0 & \sqrt{2} \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & 1/2 & -1/2 \\ -1/\sqrt{2} & 1/2 & -1/2 \\ 0 & 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$$

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An algebraic approach to canonical forms

The start: isometric, co-isometric and unitary operators
(an alternative theory works with polynomials, see later)

T is isometric if $T'T = I$ (some care needed with infinite matrices!)
We consider quasi-separable *T*'s (i.e., having realizations.)

Central property: an *isometric T* has a u.e.s. *isometric realization* and conversely $\begin{bmatrix} A_k' & C_k' \\ B_k' & D_k' \end{bmatrix} \begin{bmatrix} A_k & B_k \\ C_k & D_k \end{bmatrix} = I$

Caution: "u.e.s." is essential in the statement in case of infinitely indexed systems (u.e.s. = uniformly exponentially) stable)

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isometries (ctn'd)

An isometric realization of an isometry is in *Output Normal Form ONF)*

a co-isometric realization of a co-isometry is in *Input Normal Form (INF)*

a *unitary* quasi-separable operator has a *unitary realization* that is also u.e.s and conversely,
however:
unitary realizations may not correspond to unitary operators!
Example, when e.g. for large *k>0*:

$$\begin{bmatrix} A_k & B_k \\ C_k & D_k \end{bmatrix} = \begin{bmatrix} \sqrt{1-\frac{1}{k^2}} & \frac{1}{k} \\ -\frac{1}{k} & \sqrt{1-\frac{1}{k^2}} \end{bmatrix}$$

energy disappears at infinity!

the basic ingredient: R-Q factorization

The situation: suppose a matrix $T : \mathcal{U} \rightarrow \mathcal{Y}_1 \oplus \mathcal{Y}_2$

$$T = \begin{bmatrix} T_1 \\ T_2 \end{bmatrix}$$

and we wish a basis for the range (columns) of T_2 *first,* and then generate a full basis for *T*:

$$T = \begin{bmatrix} 0 & R_{11} & R_{12} \\ 0 & 0 & R_{22} \end{bmatrix} \bullet \begin{bmatrix} Q_1 \\ Q_2 \\ Q_3 \end{bmatrix}$$

orthogonal

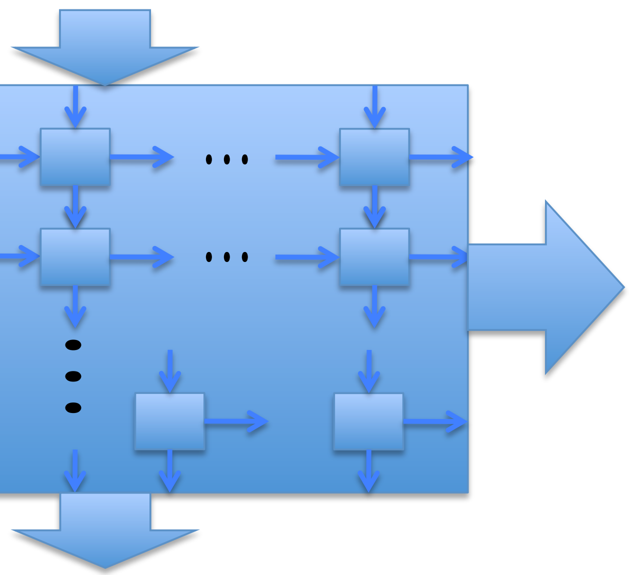
compress the columns

an elementary algorithm

use Jacobi (Givens) rotations:

$$[a \quad b] = \left[0 \quad \sqrt{|a|^2 + |b|^2}\right] \bullet \begin{bmatrix} \frac{b}{\sqrt{|a|^2+|b|^2}} & \frac{a^*}{\sqrt{|a|^2+|b|^2}} \\ \frac{-a}{\sqrt{|a|^2+|b|^2}} & \frac{b^*}{\sqrt{|a|^2+|b|^2}} \end{bmatrix}$$

and a Gentlemen-Kung array, processing the rows in sequence:



(and there are nice alternatives!)

External factorization
(characterizes the dynamics - poles)

We start out with a realization in input normal form:

$$T \sim_c \begin{bmatrix} A & B \\ C & D \end{bmatrix} \text{ with } AA' + BB' = I$$

and assume the system to be u.e.s. as well. Let

$$V \sim_c \begin{bmatrix} A & B \\ C_V & D_V \end{bmatrix}$$ be a unitary completion of $\begin{bmatrix} A & B \end{bmatrix}$

consider $TV' = \Delta'$, then

Lemma: partial fraction decomposition

$$
\begin{aligned}
\Delta' &= [D + C(I - ZA)^{-1}ZB][D'_V + B'Z'(I - A'Z')^{-1}C'_V] \\
&= [DD'_V + CC'_V] + [DB' + CA']Z'(I - A'Z')^{-1}C'_V
\end{aligned}
$$

because

$$(I - ZA)^{-1}ZBB'Z'(I - A'Z')^{-1} = (I - ZA)^{-1}ZA + I + A'Z'(I - A'Z')^{-1}$$

External factorization (2)

hence a canonical factorization: $T = \Delta' V = \Delta' (V')^{-1}$

interpretation: *V* characterizes the dynamics of *T* (in the LTI-case, the poles)

LTI example: $\dfrac{z-2}{1-(1/3)z} = \dfrac{z-2}{z-3} \bullet \dfrac{z-3}{1-(1/3)z}$

State space formula: $\Delta =_c \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} I & C' \\ 0 & D' \end{bmatrix}$

This was a "right" factorization, there is of course also a "left" factorization, based on the observability data:

$$T = \Delta_r' V_r = V_\ell \Delta_\ell'$$

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dynamics of the inverse system?

T^{-1}, even when it exists, is not necessarily causal
In first instance, we look at the anti-causal dynamics

The trick is: outer-inner factorization $T = T_o V$,
in which V is co-isometric and T_o *left* invertible

V' is the largest (anti-causal) isometry that can be applied
to T without destroying causality

Geometric interpretation: generalized "Beurling-Lax theorem"
to be discussed in a further lecture.

Let's determine V algebraically!

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Outer-Inner by a square-root algorithm

We want TV' still causal, with *V* maximal (the bigger *V*, the more more it pushes T back into the past). Let

$$V \sim_c \left[\begin{array}{cc} A_V & B_V \\ C_V & D_V \end{array}\right]$$ be a co-isometric realization for *V*

then

$$TV' = [D + C(I - ZA)^{-1}ZB][D_V' + B_V' Z'(I - A_V' Z')^{-1}C_V']$$

causal terms: $DD_V' + C(I - ZA)^{-1}ZBD_V'$

mixed term: $C(I - ZA)^{-1}ZBB_V' Z'(I - A_V' Z')^{-1}C_V'$

anti-causal term: $DB_V' Z'(I - A_V' Z')^{-1}$

a partial fraction decomposition is needed!

This is provided by a generalized partial fraction decomposition:

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Lemma: partial fraction decomposition again

$$C(I-ZA)^{-1}ZBB_V'Z'(I-A_V'Z')^{-1}C_V' =$$
$$C(I-ZA)^{-1}ZAY + Y + YA_V'Z'(I-A_V'Z')^{-1}C_V'$$

in which the new *Y* satisfies the Lyapunov-Stein equation

$$Z'YZ = BB_V' + AYA_V'$$

i.e., $Y_{k+1} = [BB_V' + AYA_V']_k$

a forward equation, which always has a unique solution provided A is u.e.s.

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outer-inner sq.r. algorithm (2)

require the anti-causal terms to add up to zero:

$$CYA'_V + DB'_V = 0$$

and define the remainder:

$$T_o = [CYC'_V + DD'_V] + CI_AZ)^{-1}[AYC'_V + BD'_V]$$

hence:

$$T_o \sim_c \left[\begin{array}{cc} A & B_o \\ C & D_o \end{array} \right] = \left[\begin{array}{cc} A & BD'_V + AYC'_V \\ C & DD'_V + CYC'_V \end{array} \right]$$

Putting the four equations together:

$$\left[\begin{array}{cc} AY & B \\ CY & D \end{array} \right]_k \left[\begin{array}{cc} A'_V & C'_V \\ B'_V & D'_V \end{array} \right]_k = \left[\begin{array}{cc} Y_{k+1} & B_{ok} \\ 0 & D_{ok} \end{array} \right]$$

isometric

left invertible

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the square-root algorithm (3)

(complete RQ-factorization: embed the isometry)

$$\begin{bmatrix} AY & B \\ CY & D \end{bmatrix}_k = \begin{bmatrix} 0 & Y_{k+1} & B_{ok} \\ 0 & 0 & D_{ok} \end{bmatrix} \begin{bmatrix} C_n & D_n \\ A_V & B_V \\ C_V & D_V \end{bmatrix}_k$$

R-factor

Q-factor

interpretations:
- V characterizes the anti-causal dynamics of T
- $W = D_n + C_n(I - ZA_V)^{-1}ZB_V$ defines the kernel of "T *acting* on causal signals"
- D_{ok} and Y_{k+1} are compressed to left invertible
- Y is the cross-correlation between the reachability ops. of T and V
- Proofs are based on ranges and kernels

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some more remarks on O-I (4)

$$\begin{bmatrix} AY & B \\ CY & D \end{bmatrix}_k = \begin{bmatrix} 0 & Y_{k+1} & B_{ok} \\ 0 & 0 & D_{ok} \end{bmatrix} \begin{bmatrix} C_n & D_n \\ A_V & B_V \\ C_V & D_V \end{bmatrix}_k$$

- the outer factor T_o is causally left-invertible
- Y can disappear completely (when T is already outer) the algorithm can be used to show whether a transfer function is indeed causally invertible
- T_o as obtained from the algorithm is not necessarily minimal (e.g. when T is already inner)

this is the most important algorithm in system theory

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Applications

In the following talks I shall show:

- how the Kalman estimation filter is nothing but an outer-inner factorization, and how this insight gives an easy and simple proof
- how outer-inner succeeds in providing a stable algorithm for LU or spectral factorization
- how outer-inner and inner-outer succeeds in computing the Moore-Penrose inverse of semi-separable systems
- how outer-inner or inner-outer can be used in control applications (aside from the Kalman filter)
- how the theory generalizes to non-linear, and which consequences that has

A simple example

Let's try to compute the Moore-Penrose inverse of

$$T = \begin{bmatrix} \boxed{1} & & & \\ -2 & 1 & & \\ 0 & -2 & 1 & \\ \vdots & \ddots & \ddots & \ddots \end{bmatrix}$$

T is not invertible, co-kernel: $\mathrm{span}[1 \ 1/2 \ 1/4 \ \cdots]$

it has a left inverse:

$$\begin{bmatrix} \boxed{0} & -1/2 & -1/4 & -1/8 & \cdots \\ & 0 & -1/2 & -1/4 & \ddots \\ & & 0 & -1/2 & \ddots \\ & & & 0 & \ddots \\ & & & & \ddots \end{bmatrix}$$

but this is not the Moore-Penrose inverse.

Moore-Penrose inverse (2)

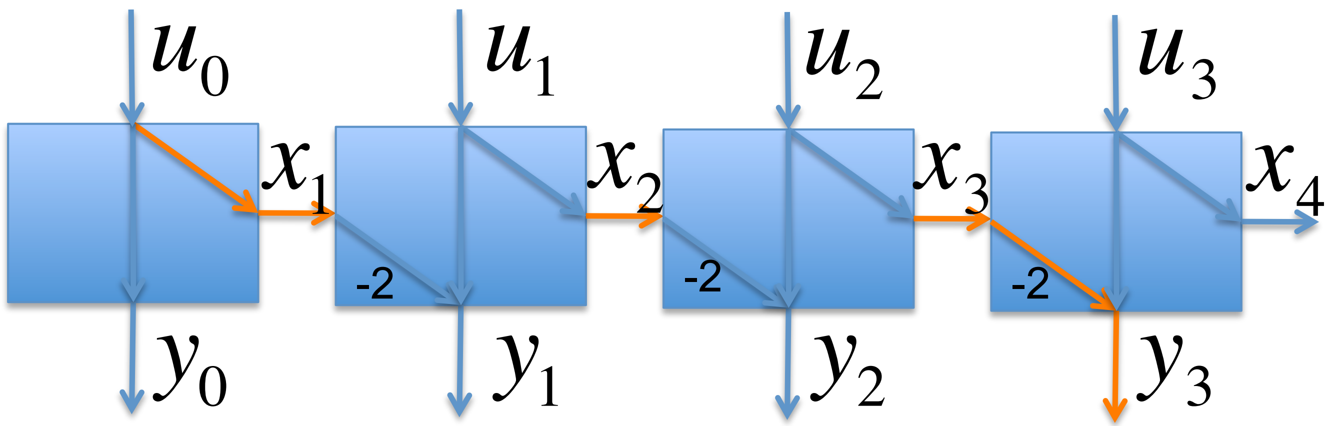
What is then the Moore-Penrose inverse?
(definition: $T^{\dagger}y = \min\{\arg\min_v \|Tv - y\|_2\}$
typically used in principal component analysis)
As T has a left inverse, it is already left outer. What about the right side? To convert T to fully outer, we need to find an inner-outer factorization: $T=UT_o$. The corresponding square root algorithm is:

$$\begin{bmatrix} Y_k A_k & Y_k B_k \\ C_k & D_k \end{bmatrix} = \begin{bmatrix} B_{ak} & A_{Uk} & B_{Uk} \\ D_{ak} & C_{Uk} & D_{Uk} \end{bmatrix} \begin{bmatrix} \text{"0"} & \text{"0"} \\ Y_{k-1} & 0 \\ C_{ok} & D_{ok} \end{bmatrix}$$

(it is a backward recursion now). As the system is LTI in the far future, this provides for a starting point of the recursion

Moore-Penrose inverse (3)

A realization for *T* is easily found (e.g. INF):



$$T \sim_c \operatorname{diag}\left(\begin{bmatrix} \cdot & \cdot \\ \cdot & \cdot \end{bmatrix}, \boxed{\begin{bmatrix} | & 1 \\ | & 1 \end{bmatrix}}, \begin{bmatrix} 0 & 1 \\ -2 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ -2 & 1 \end{bmatrix}, \cdots \right)$$

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Moore-Penrose (4)

Running the square root algorithm produces:

$$U \sim_c \operatorname{diag}\left(\begin{bmatrix} | & | \\ \cdot & \cdot \end{bmatrix}, \begin{bmatrix} 1/2 & \sqrt{3}/2 \\ -\sqrt{3}/2 & 1/2 \end{bmatrix}, \begin{bmatrix} 1/2 & \sqrt{3}/2 \\ -\sqrt{3}/2 & 1/2 \end{bmatrix}, \cdots \right)$$

$$T_o \sim_c \operatorname{diag}\left(\begin{bmatrix} \cdot & \cdot \\ \cdot & \cdot \end{bmatrix}, \begin{bmatrix} | & 1 \\ | & 2 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ -1 & 2 \end{bmatrix}, \cdots \right)$$

and finally

$$U = \begin{bmatrix} \boxed{1/2} & & & \\ -3/4 & 1/2 & & \\ -3/8 & -3/4 & 1/2 & \\ \vdots & \ddots & \ddots & \ddots \end{bmatrix}$$

$$T_o = \begin{bmatrix} \boxed{2} & & & \\ -1 & 2 & & \\ 0 & -1 & 2 & \\ \vdots & \ddots & \ddots & \ddots \end{bmatrix}$$

$$T^{\dagger} = \frac{1}{4} \begin{bmatrix} 1 & -3/2 & -3/4 & -3/8 & \cdots \\ 1/2 & 1/4 & -15/8 & -15/16 & \cdots \\ 1/4 & 1/8 & 1/16 & -63/32 & \cdots \\ 1/8 & 1/16 & 1/32 & 1/64 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

Envoy

We are now going to use our new knowledge on stochastic state estimation (Kalman filtering)!