Special role for “inner” operators

Definition: V is *inner* when causal, unitary (respect. *conjugate inner*)

Property: operators with state representations $\begin{bmatrix} A & B \\ C & D \end{bmatrix}$ such that

(1) spectral radius $\sigma(AZ) < 1$ - hence bounded inverse $(I-AZ)^{-1}$

(2) $\begin{bmatrix} A & B \\ C & D \end{bmatrix}$ unitary

are inner (spectral radius of AZ is $\lim_{n\to\infty} \left\| (ZA)^n \right\|^{1/n}$)

For “uniformly exponential stable systems” (i.e. systems with $\sigma(AZ) < 1$) we have a special role for inner operators:

(1) Put T in output normal form, then there exists an inner V such that $T'V$ is causal actually, a realization for V is found by unitary completion of the observability data: $V \approx \begin{bmatrix} A & B_V \\ C & D_V \end{bmatrix}$

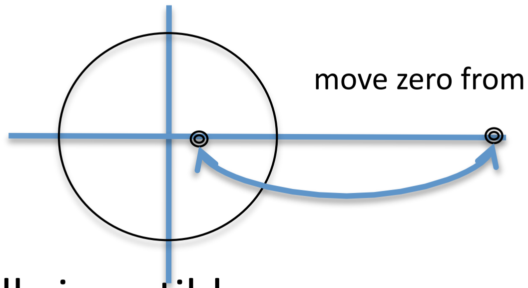
(2) [Square root algorithm] Given T with ker(T)=0, then there exists an inner V, such that $T = V \begin{bmatrix} T_o \\ 0 \end{bmatrix}$, V inner and T_o outer (i.e. has causal inverse)

Inner-outer factorization

The LTI (Linear time invariant) case for stable transfer functions

Example:

$$T(z)=\frac{3-z}{1-\frac{1}{2}z}=\underbrace{\frac{3-z}{1-\frac{1}{3}z}}_{\text{inner}}\;\underbrace{\frac{1-\frac{1}{3}z}{1-\frac{1}{2}z}}_{\text{outer}}$$



General property (Beurling): $T(z)=U(z)T_o(z)$

with *U(z)* causal unitary and $T_o(z)$ causally invertible

This generalizes to TV (Time-varying) systems: $T = UT_{o,r}$

with *U* causal isometric and $T_{o,r}$ causally right-invertible

to get invertibility, one needs also a right factorization: $T_{o,r} = T_o V$

now, *V* is co-isometric and T_o causally invertible

(compare with ULV factorization: same thing!)

Factorizations are found with the square root algorithm (see further)!

Geometric interpretation

Scalar LTI systems (Hardy theory)!

General input space: ℓ_2 = time series that are square summable,

"causal series": $\ell_2(0,+\infty)$

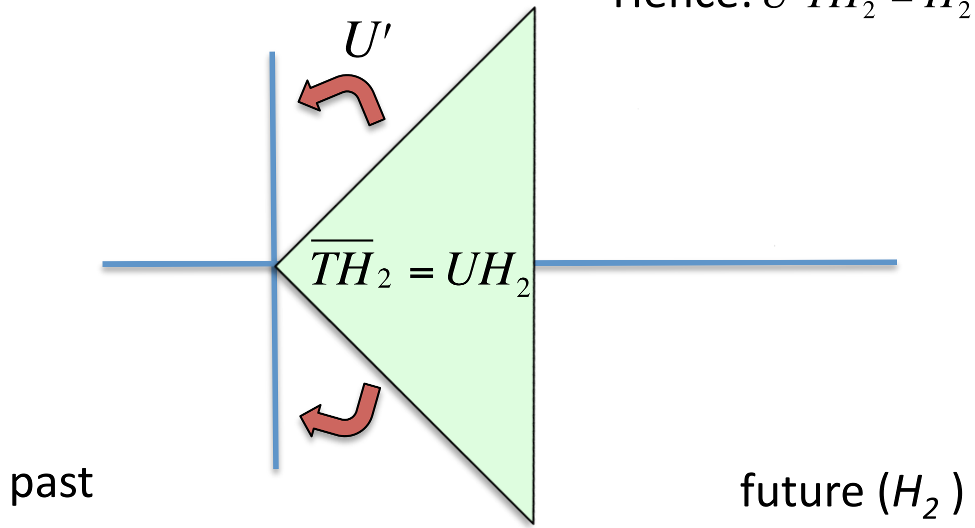
Fourier transform: H_2 = functions analytic in the unit disc, uniformly square integrable on circles of the type $re^{j\vartheta}$ *with* $r \leq 1$ are Fourier Transforms of causal series

Causal system: TH_2 is contained in H_2 , Beurling shows: $\overline{TH_2} = UH_2$

for some inner U (pure phase function)

(the upper bar means "closure" = "take limits")

Hence: $\overline{U'TH_2} = H_2 \Rightarrow U'T = T_o$ is causally invertible



The TV case

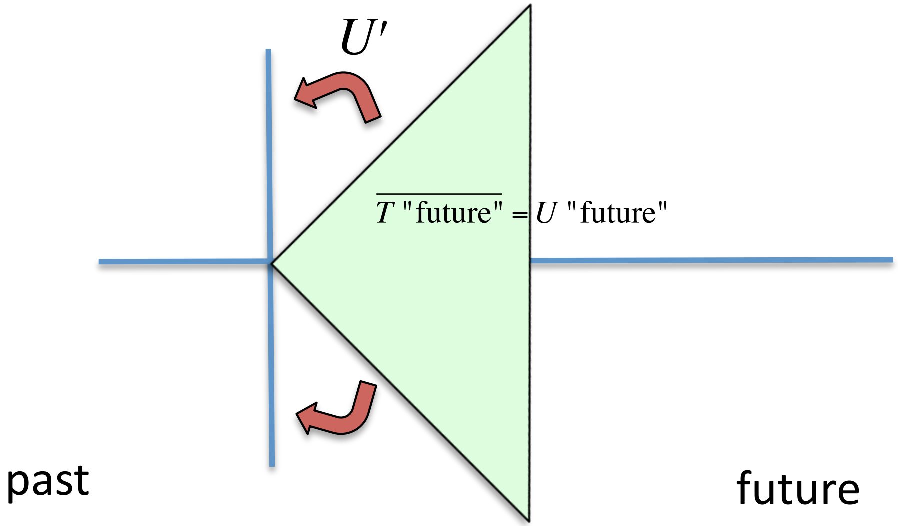
No more Fourier transforms, zeros or poles! What then?

When $\begin{bmatrix} A & B \\ C & D \end{bmatrix}$ is isometric, then $T \approx \begin{bmatrix} A & B \\ C & D \end{bmatrix}$ is isometric as well!

The realization can be completed to unitary: $\begin{bmatrix} A & B & B_W \\ C & D & D_W \end{bmatrix}$

then $\begin{bmatrix} U & W \end{bmatrix} \approx \begin{bmatrix} A & B & B_W \\ C & D & D_W \end{bmatrix}$ will be inner iff spectral radius $\sigma(AZ) < 1$
(otherwise just isometric still!)

Generalized Beurling theorem: $T = UT_{o,r}$
with U isometric and $T_{o,r}$ right causally invertible
(because $\overline{T_{o,r}\,\text{"future"}} = \text{"future"}!$)



inner-outer = square root algo.

$T_{o,r} = U'T$ tries to take out the isometric part

$$T_{o,r} = \left[D'_U + B'_U(I - Z'A'_U)^{-1}Z'C'_U\right]\left[D + CZ(I - AZ)^{-1}B\right] =$$

$$= D'_U D + B'_U(I - Z'A'_U)^{-1}Z'C'_U D + D'_U CZ(I - AZ)^{-1}B +$$

$$+ B'_U\left\{(I - Z'A'_U)^{-1}Z'C'_U CZ(I - AZ)^{-1}\right\}B$$

$$= (I - Z'A'_U)^{-1}Z'A'_U Y + Y + YAZ(I - AZ)^{-1}$$

partial fraction! Dichotomy!
where $ZYZ' = C'_U C + A'_U YA$

Require upper part to be zero: $C'_U D + A'_U YB = 0!$

Produces: $T_{o,r} = (D'_U D + B'_U YB) + (D'_U C + B'_U YA)Z(I - AZ)^{-1}B := D_o + C_o Z(I - AZ)^{-1}B$

Component-wise backward square root QR-algorithm computes the matrices:

$$\begin{bmatrix} Y_k B_k & Y_k A_k \\ D_k & C_k \end{bmatrix} = \begin{bmatrix} B_{U,k} & A_{U,k} & B_{W,k} \\ D_{U,k} & C_{U,k} & D_{W,k} \end{bmatrix} \begin{bmatrix} D_{o,k} & C_{o,k} \\ 0 & Y_{k-1} \\ 0 & 0 \end{bmatrix}$$

QR-factorization

(note: $\ker(\bullet D_{o,k}) = \ker(\bullet Y_{k+1}) = 0$*)*

Result: $T = \begin{bmatrix} U & W \end{bmatrix} \begin{bmatrix} T_o \\ 0 \end{bmatrix}, T_o \approx \begin{bmatrix} A & B \\ C_o & D_o \end{bmatrix}, \begin{bmatrix} U & W \end{bmatrix} = \begin{bmatrix} D_U & D_W \end{bmatrix} + C_U Z(I - A_U Z)^{-1}\begin{bmatrix} B_U & B_W \end{bmatrix}$