Fundamental limitations in control over networks with communication constraints

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TU Munchen, March 17th, 2015

Networked control

- Two fields of control and communication meet
- Recently, some level of interest from the comm people
- For real-time control: Machine to machine
- High reliability and robustness for wireless networks
- Applications: Factory automation, Automobiles,

  Medical devices, …

Cyber-physical systems, Industry 4.0, …

Control under communication constraint

Constraints due to shared channels

- Even if the total rate is large, each component may use only a (small) portion.

Challenges

- Modeling of communication constraints in networked control
- Allocating bandwidth to each transmission

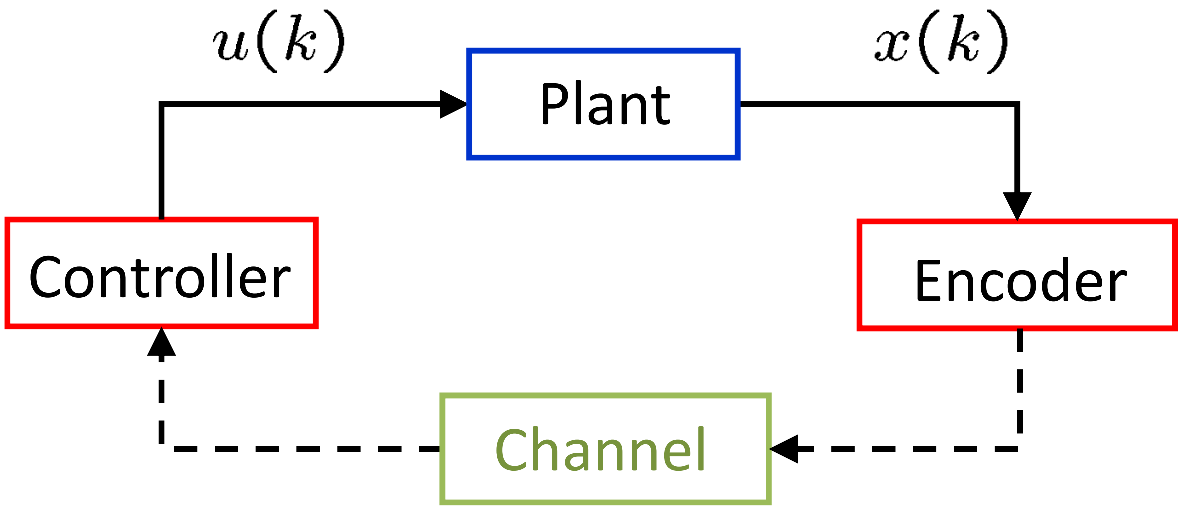
What is the necessary level of communication for control?

In this talk

- Data rate limited control
  - Uncertain systems case
  - Proposed approach: Scalar systems
  - Extensions to general order systems
- Bode integral for networked control systems

Data rate limited control for uncertain systems

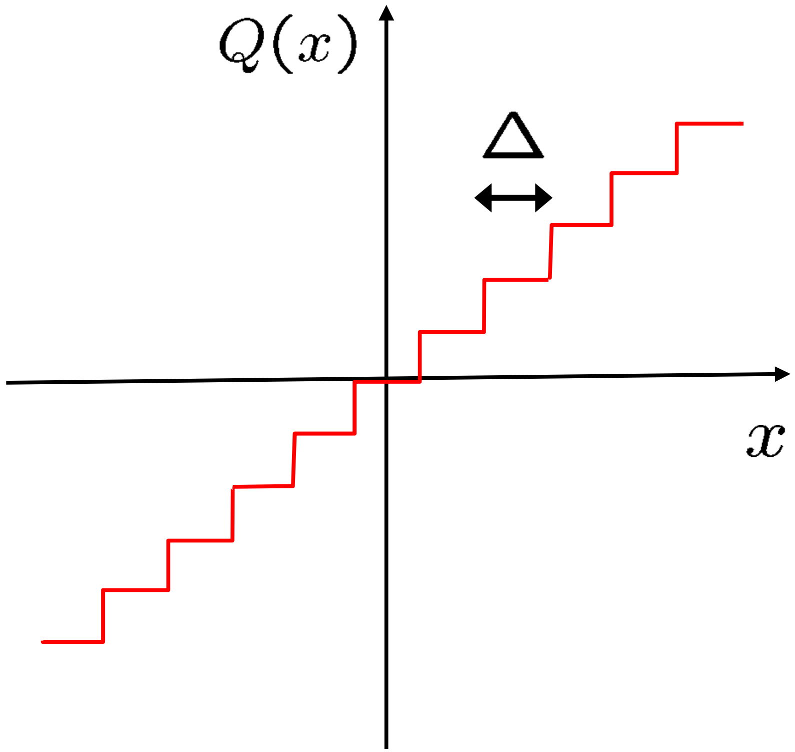
Problem setup



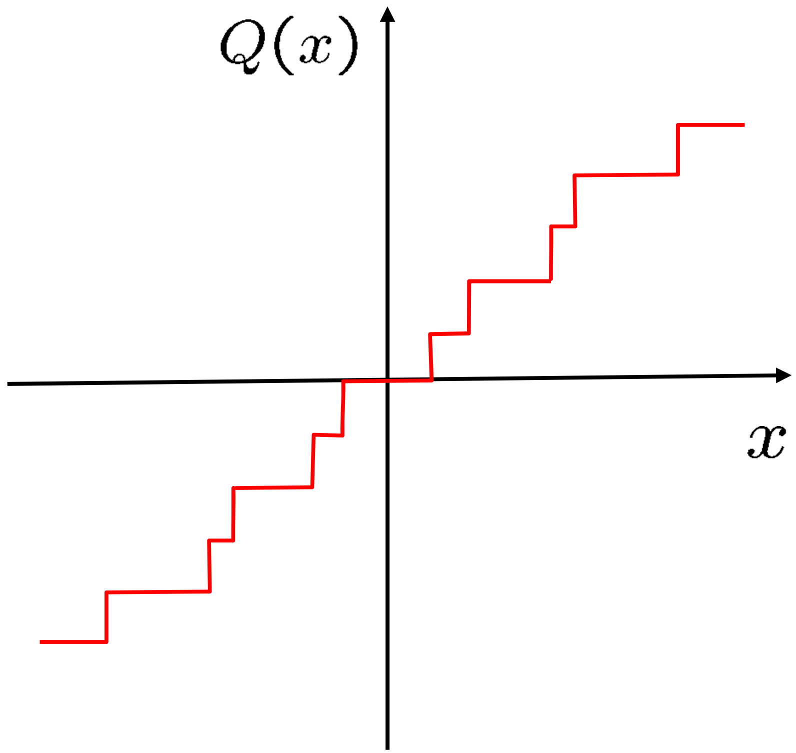
- Plant $x(k+1) = Ax(k) + Bu(k) \qquad x(k) \in \mathbb{R}^n$
  - Discrete-time LTI system: No uncertainty
  - Unstable, but stabilizable
- Channel: Finite bit rate, No error/delay

Quantizer

- Uniform

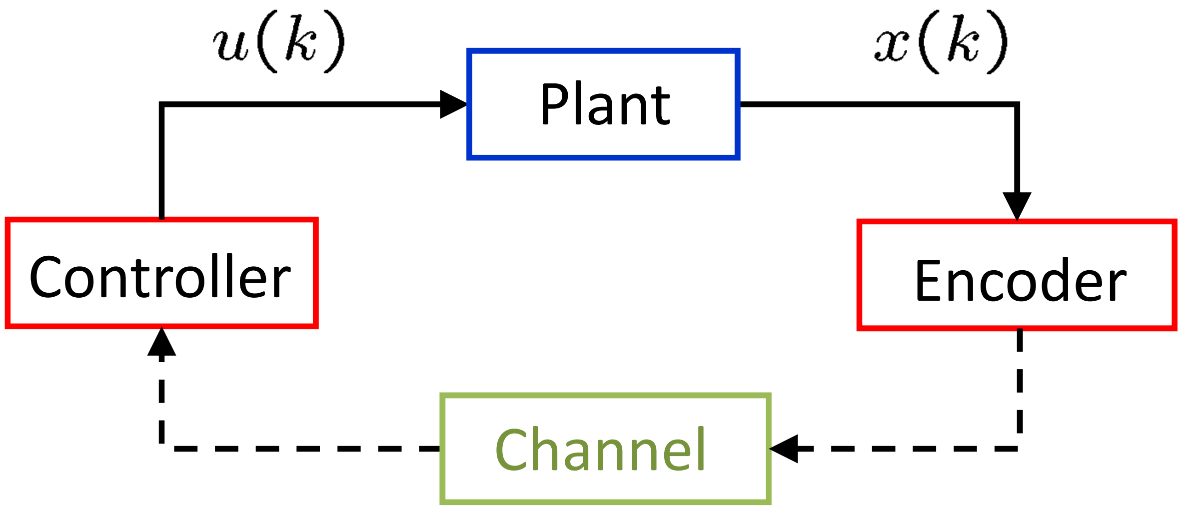


- General



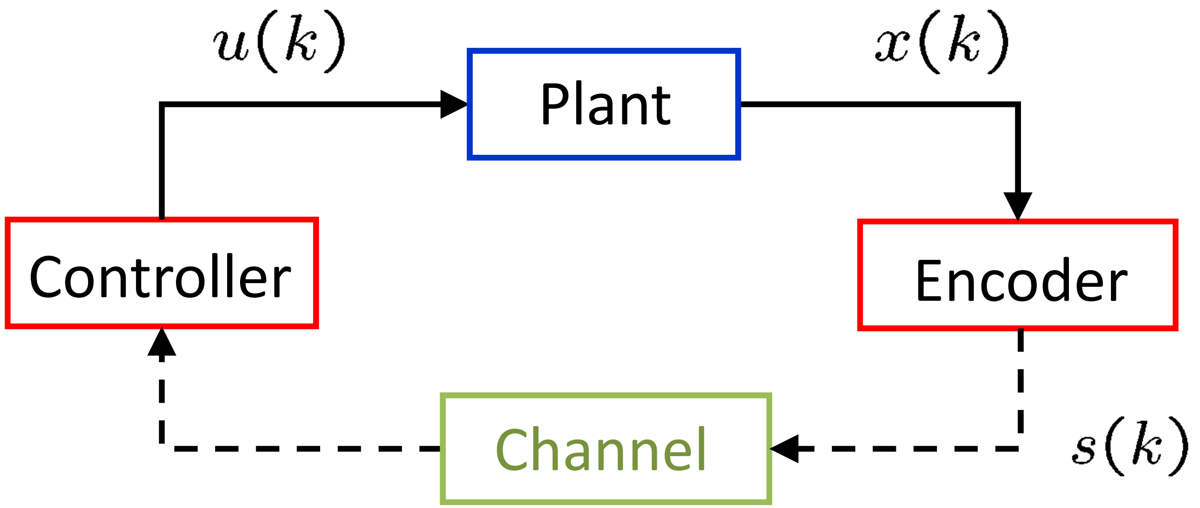
Are there quantizer structures suitable for control?

Data rate problem



- Total # of discrete values = N
- Data rate = $\lceil \log_2 N \rceil$ [bits/sample]
- To achieve stabilization, how much data rate is needed?
- First formulated by Wong & Brockett (1999)

Data rate problem: General setup



- Encoder $s(k) = E_k(x(k), \ldots, x(0), s(k-1), \ldots, s(0))$

  $N(k) =$ # of code words at time k

- Controller $u(k) = K_k(s(k-1), \ldots, s(0))$

- Average data rate

$$R = \limsup_{k \to \infty} \frac{1}{k} \sum_{t=0}^{k-1} \log_2 N(t) \quad \text{[bits/sample]}$$

The minimum data rate

Theorem

$$x(k) \to 0 \text{ as } k \to \infty \quad \forall x(0)$$

$$\Leftrightarrow \text{ Ave data rate } \quad R > \textstyle\sum_i \log_2 |\lambda_i^u(A)|$$

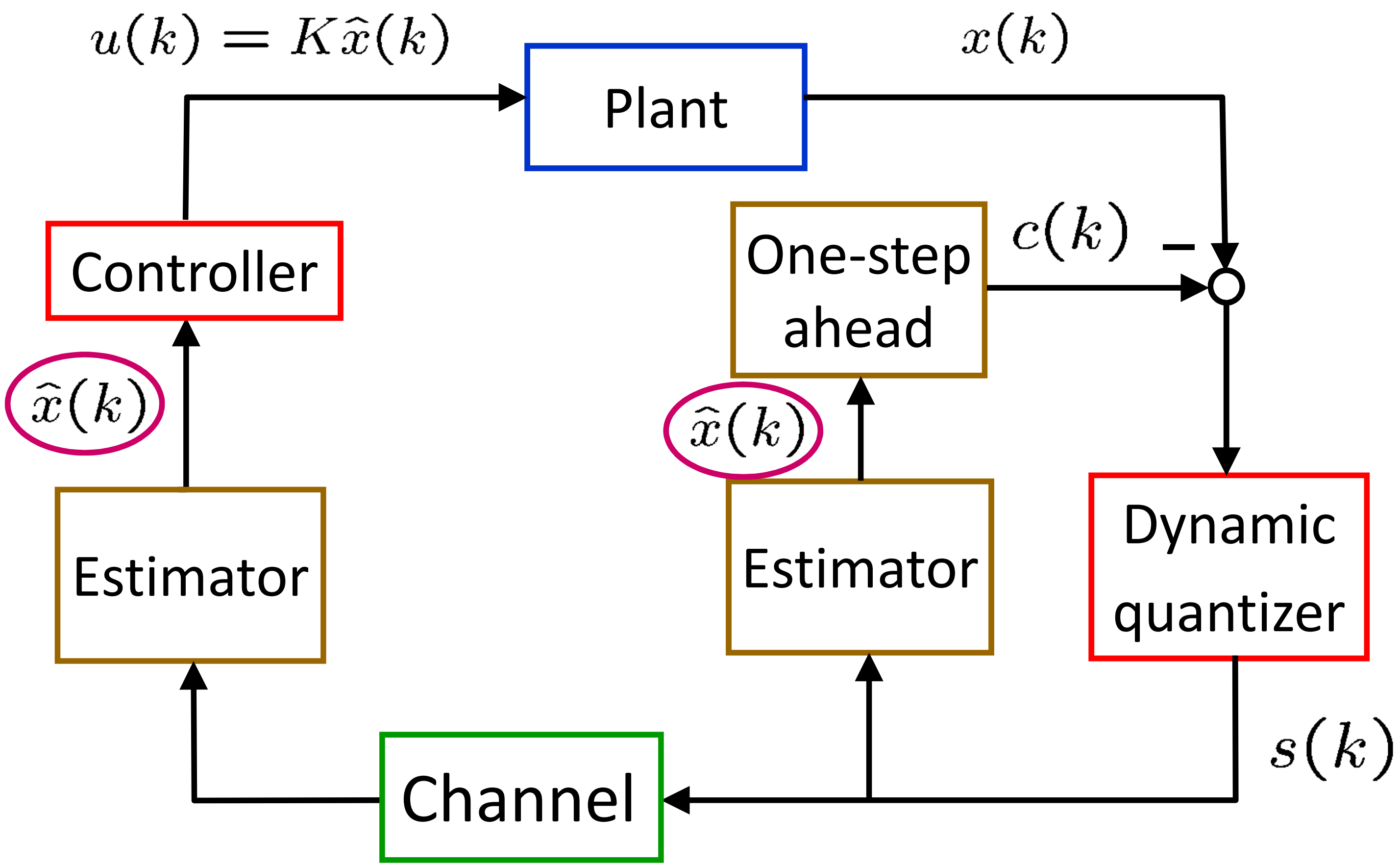
- Control is impossible if the bound is not met.
- Bound is determined by the product of unstable poles.
- Extensions to stochastic/nonlinear/multi-channel settings
- Proof by construction: Quantizer, transmission scheme, controller

Nair & Evans (2004), Tatikonda and Mitter (2004), Matveev & Savkin (2004),

De Persis (2005), Yuksel & Basar (2006), Minero, Franceschetti, Dey, & Nair (2009),

You & Xie (2010),...

Structure of the controller



$\hat{x}(k)$: Coarse estimate of state from quantized signal

$c(k)$: Estimate of one-step ahead

$$c(k+1) = A\hat{x}(k) + Bu(k), \quad c(0) = 0$$

Uncertain systems case

- Existing results are conservative
- Certain quantizer structures are assumed
- They provide only upper bounds on data rates

Issues & Difficulties

- How to decompose stable/unstable dynamics in the plant
- How to keep the unstable eigenvalues representation
- Class of uncertain systems

Phat, Jiang, Savkin, & Petersen (2004), Martins, Dahleh, & Elia (2006)

Our approach

- Deal with parametric uncertainties
- Uncertain plant: SISO ARX system

$$y_{k+1} = \sum_{i=1}^{n} a_{i,k} y_{k-i+1} + \sum_{i=1}^{l} b_i u_{k-i+1}$$

where the uncertain parameters are

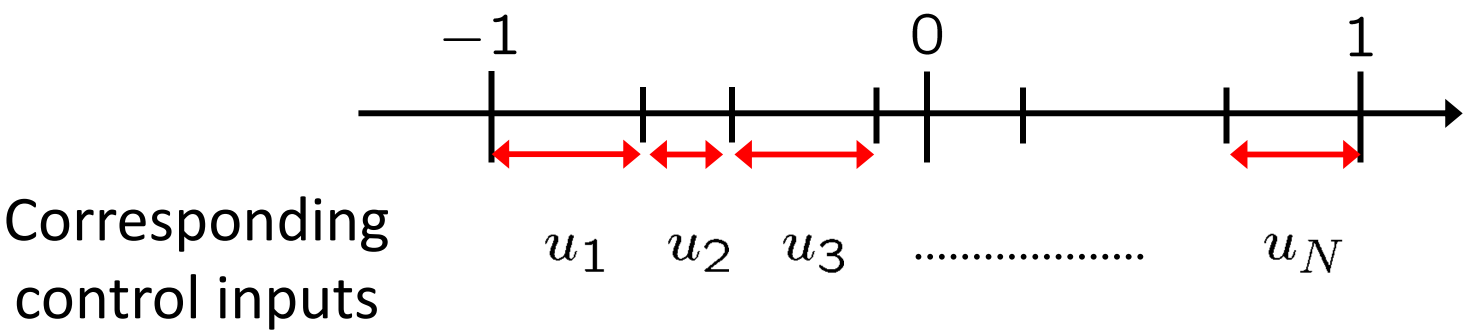
$$a_{i,k} \in [a_i^* - \epsilon_i, a_i^* + \epsilon_i], \ \epsilon_i \geq 0$$

- Obtain rate bounds in terms of the “nominal” dynamics

Minimum data rate: Scalar case

- Plant $$y(k+1) = a_k y(k) + u(k), \quad a_k \in [a^* - \epsilon, a^* + \epsilon]$$ $$u(k) = Q(y(k)), \quad y(0) \in [-1, 1]$$

- Quantizer: Partitions $[-1, 1]$ into N cells



To keep $y(k)$ within $[-1, 1]$ how large should N be?

Minimum data rate: Scalar case

Theorem

$$y(k) \in [-1, 1], \quad \forall k \geq 0$$

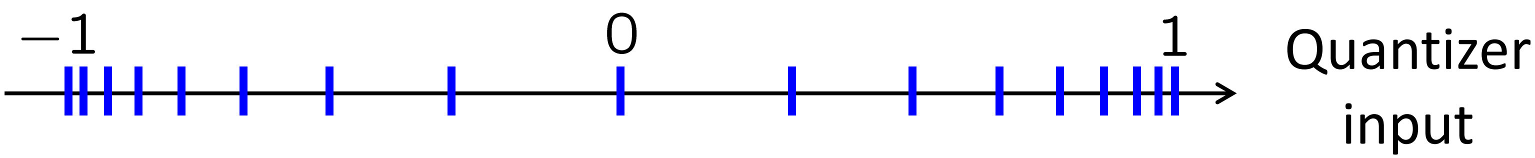
$\Leftrightarrow$ Data rate $R \geq \log \dfrac{\log \frac{1}{(1-\epsilon)^2}}{\log \frac{|a^*|+\epsilon}{|a^*|-\epsilon}} \longrightarrow \log |a^*|$, $\epsilon \to 0$

Uncertainty bound $0 \leq \epsilon < 1$

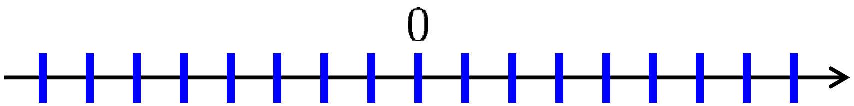
- More unstable nominal plant implies higher rate.
- For any nominal plant, high uncertainty level may require arbitrarily large rate: $\epsilon \to 1 \Rightarrow R \to \infty$
- A class of nonuniform quantizers arises in the derivation.

Okano & Ishii (IEEE Trans. Contr. Network Syst. 2014)

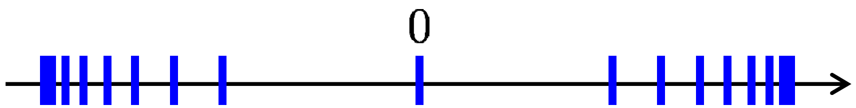
Optimal quantizer



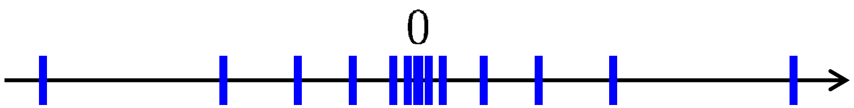
- For inputs with larger magnitude, quantization is finer.
- Nonuniformity depends on level of uncertainty:
  - $\epsilon \to 0$: Uniform



  - $\epsilon \to 1$



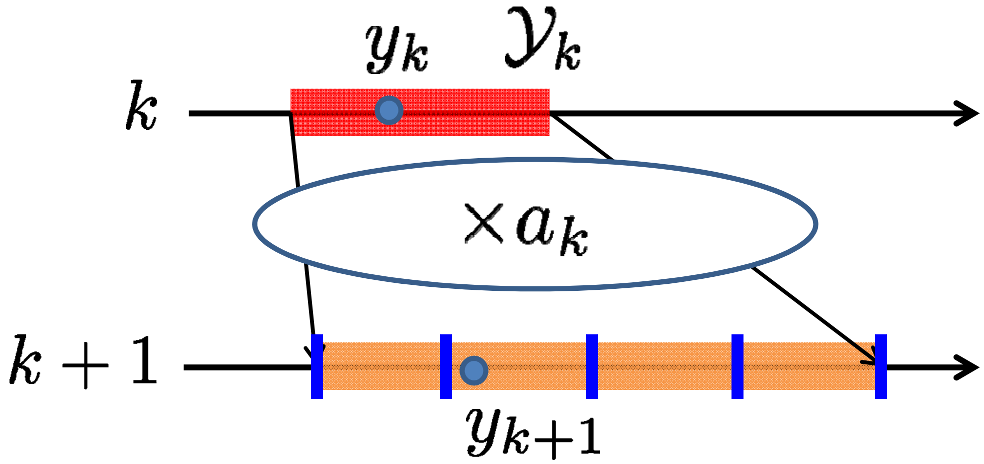
- Cf: Logarithmic quantizer



Elia & Mitter (2004)

Scalar plants case

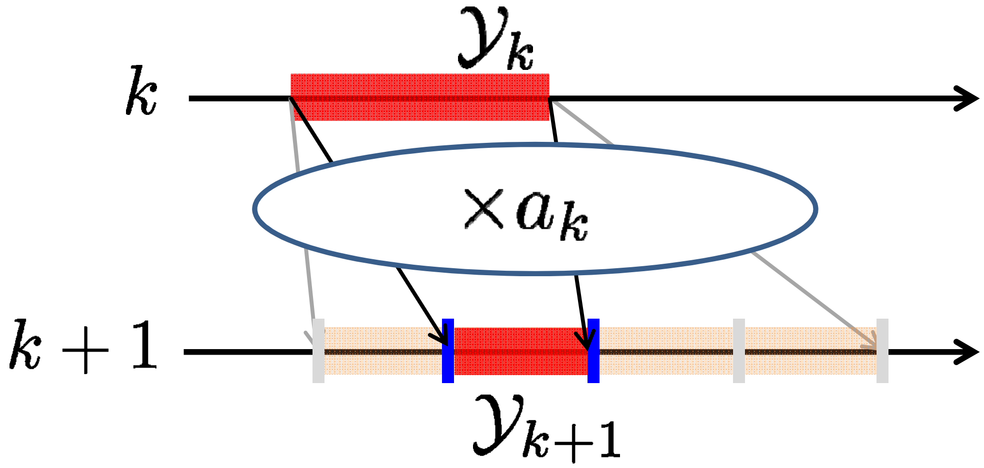
$$y_{k+1} = a_k y_k + u_k,$$
$$a_k \in [a^* - \epsilon, a^* + \epsilon]$$



- State is estimated to be contained in the interval $\mathcal{Y}_k$.
- It expands by the unstable dynamics.
- Quantizer finds the subinterval the state lies in.

Scalar plants case

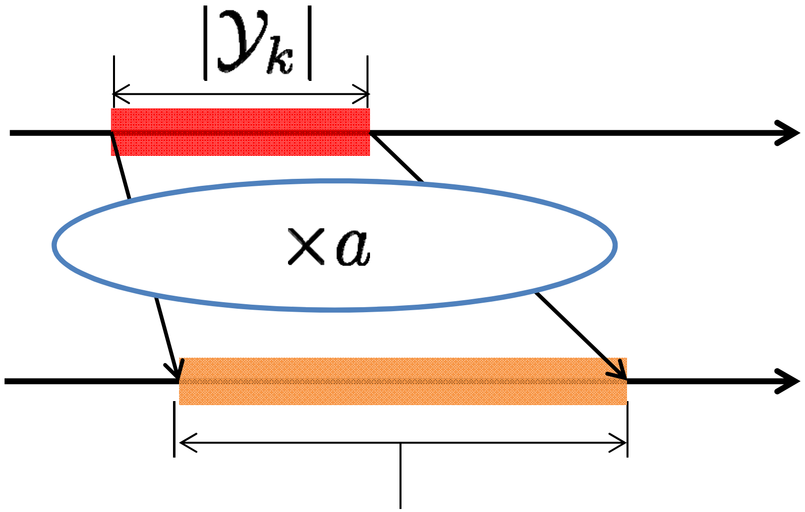
$$y_{k+1} = a_k y_k + u_k,$$
$$a_k \in [a^* - \epsilon, a^* + \epsilon]$$



- The interval $\mathcal{Y}_k$ shrinks based on quantized information.
- Stabilization is achieved if the interval shrinks over time.

Expansion of estimation intervals

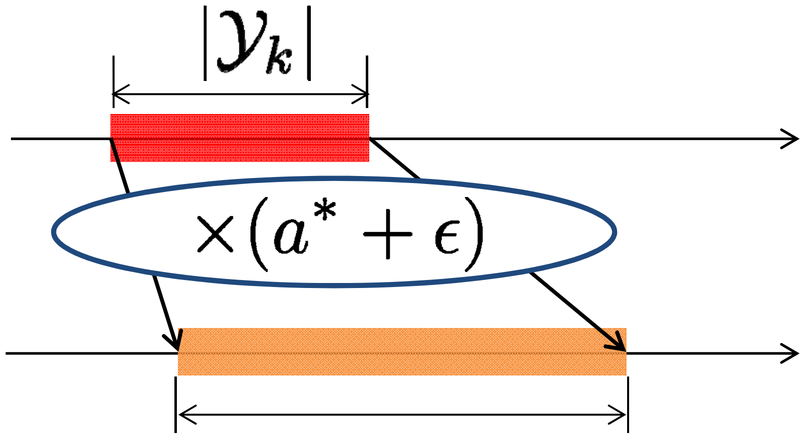
- When the exact parameter is known:

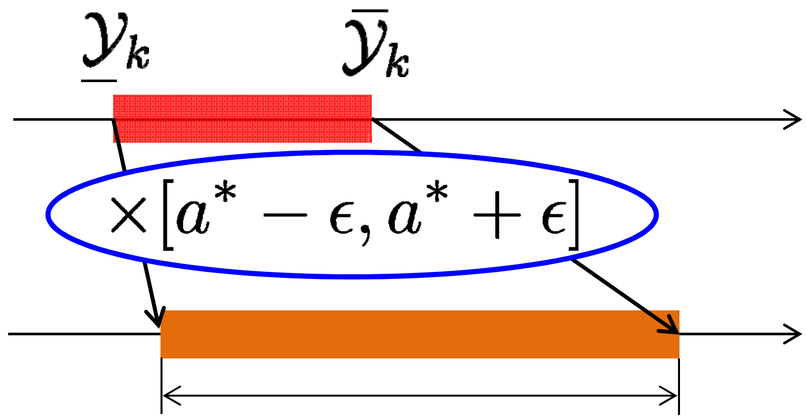


Width of the interval: $a|\mathcal{Y}_k|$

Expansion of estimation intervals

- When the parameter is uncertain: $a_k \in [a^* - \epsilon, a^* + \epsilon]$
  - The most unstable a_k
  - All possible a_k

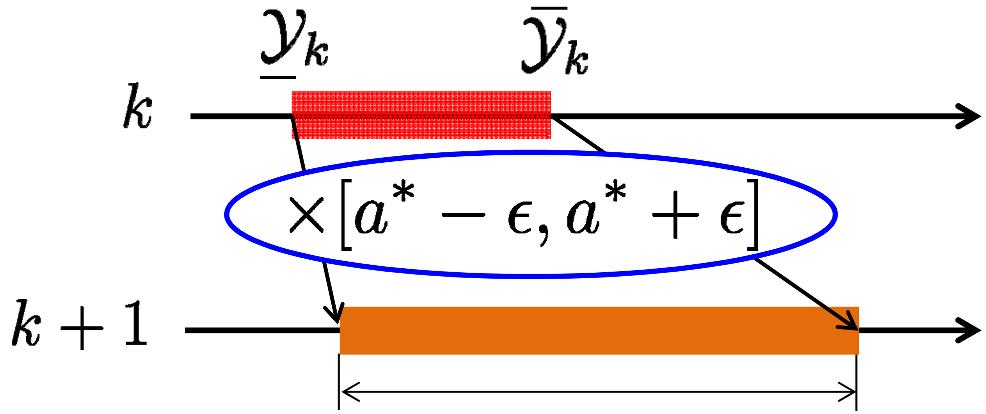




Width: $(a^* + \epsilon)|\mathcal{Y}_k| \quad < \quad \left|[a^* - \epsilon, a^* + \epsilon] \times [\underline{\mathcal{Y}}_k, \overline{\mathcal{Y}}_k]\right|$

$\Longrightarrow$ Strictly larger data rate is required

Effects of uncertainty



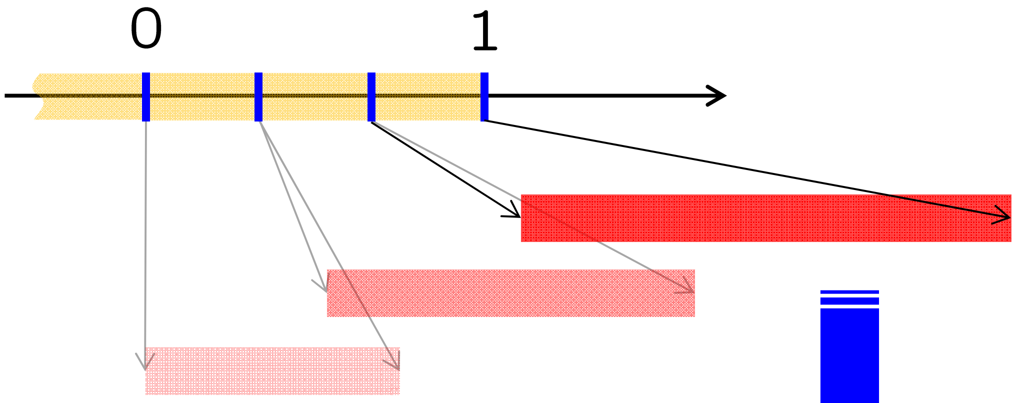
- The width can be obtained via an interval product:

$$\left|[a^* - \epsilon, a^* + \epsilon] \times [\underline{\mathcal{Y}}_k, \bar{\mathcal{Y}}_k]\right| = a^* |\mathcal{Y}_k| + \epsilon(\underline{\mathcal{Y}}_k + \bar{\mathcal{Y}}_k)$$

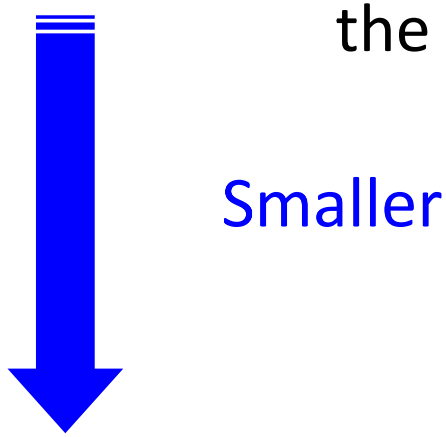
$\Longrightarrow$ Larger $\mathcal{Y}_k$ implies larger quantization error

Effects of uncertainty

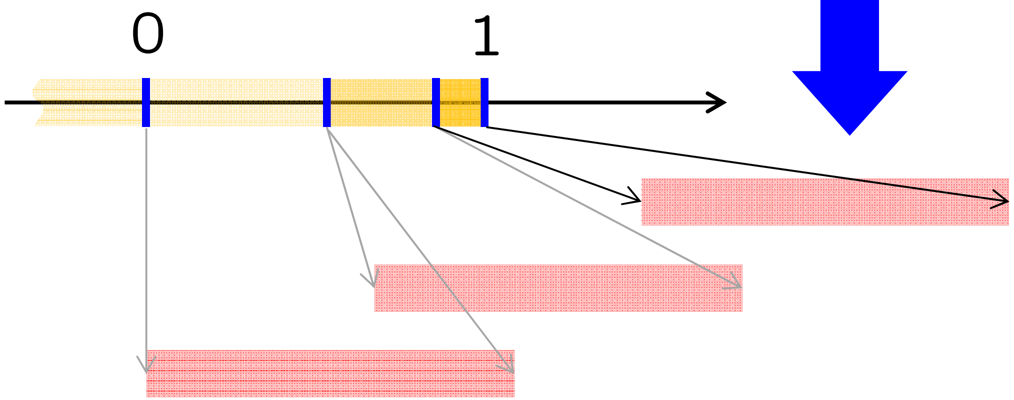
- Uniform quantizer



Cells further away from the origin expands more



- Optimal quantizer



Same width for all cells

- To keep $y(k) \in [-1, 1]$, this width must be ≤ 2.

Comparison

Previous results: Deal with different uncertain systems

- Norm bounded uncertainties

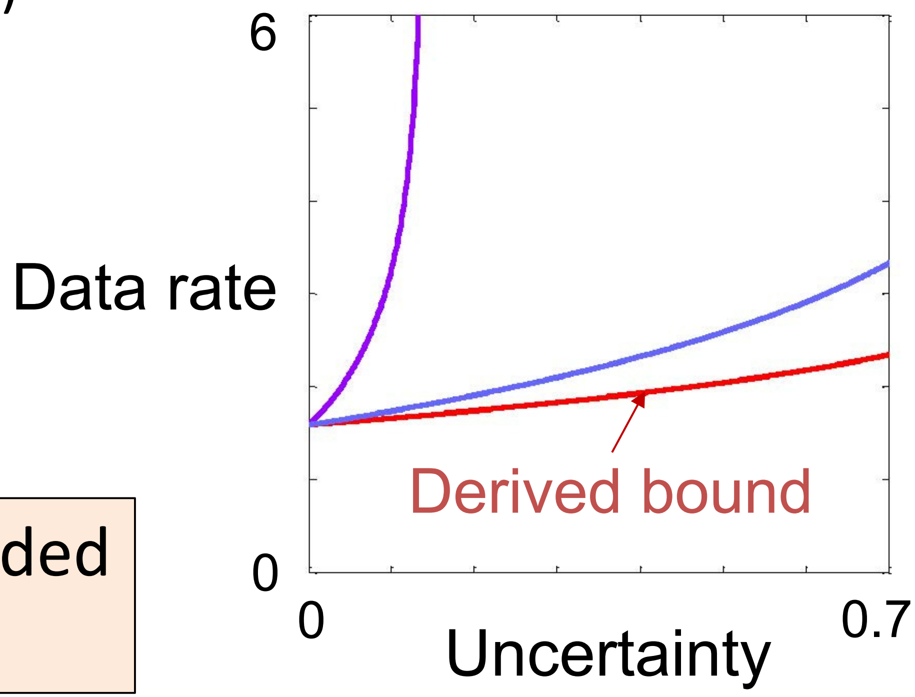
  Phat, Jiang, Savkin, & Petersen (2004)

- Stochastic, nonlinear uncertainties

  Martins, Dahleh, & Elia (2006)

- For scalar systems:

  Comparison is possible



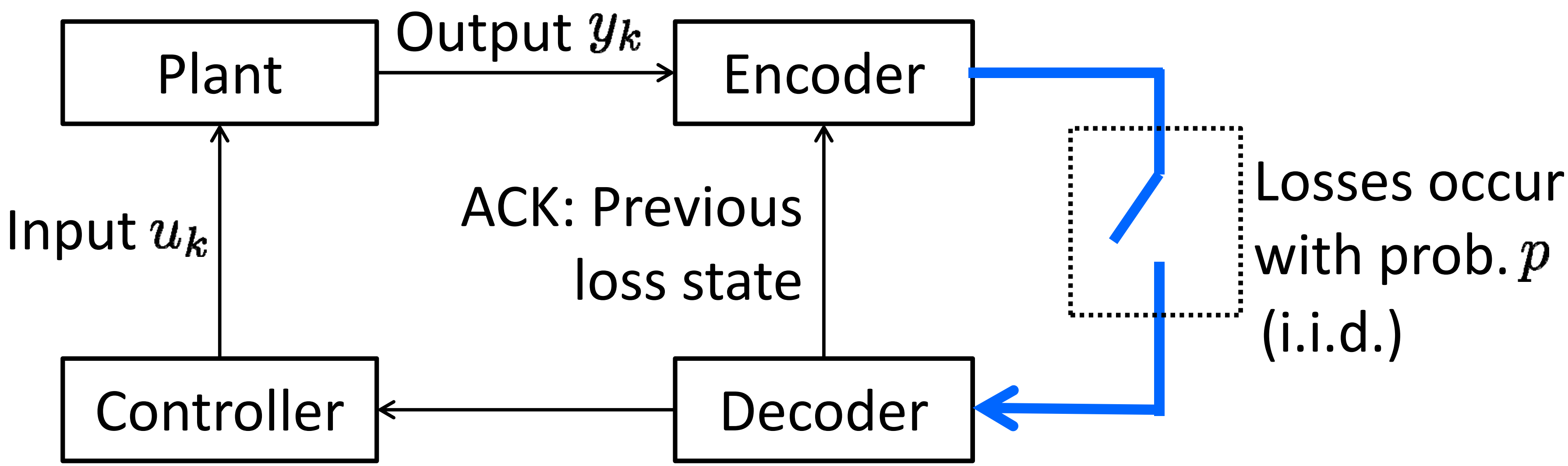
How can the limitation be extended to higher order systems?

General order plants case

ARX form $$y_{k+1} = \sum_{i=1}^{n} a_{i,k} y_{k-i+1} + \sum_{i=1}^{l} b_i u_{k-i+1},$$

- Necessary condition
  - Key parameter: Product of eigenvalues $= a_{n,k}$
  - An argument similar to the scalar case holds
- Sufficient condition
  - Take account of the effect of each parameter $a_{i,k}, \forall i$ on the expansion of state estimation sets
- We also consider random data losses in the channel.

Problem setup



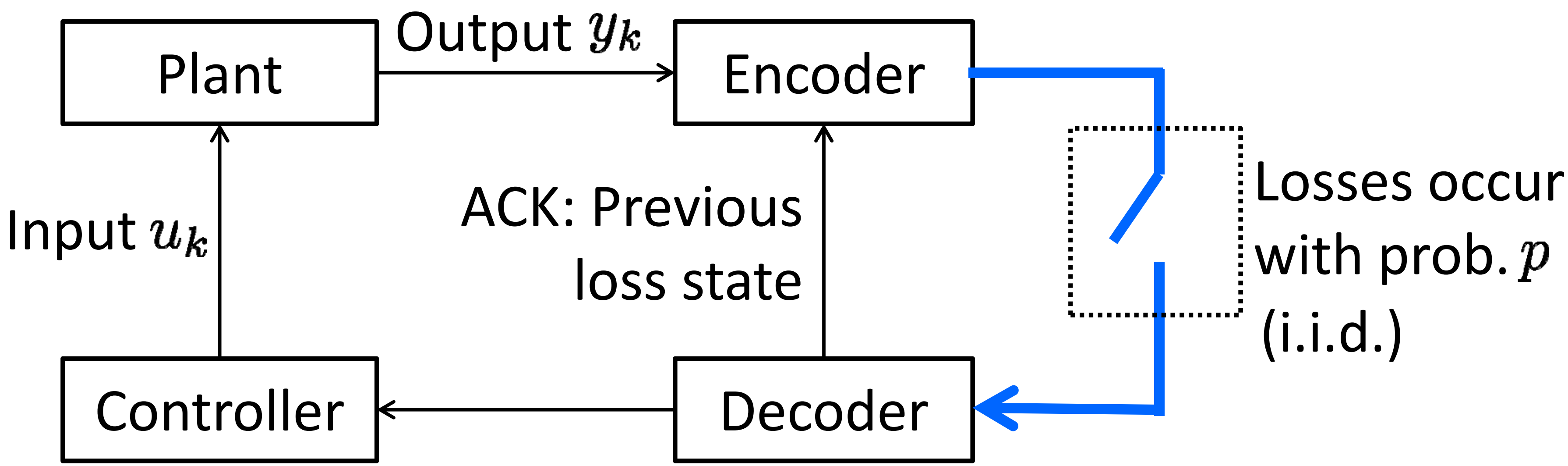
Plant: SISO ARX system, unstable

$$y_{k+1} = \sum_{i=1}^{n} a_{i,k} y_{k-i+1} + \sum_{i=1}^{l} b_i u_{k-i+1},$$

$a_{i,k}$: Uncertain, $a_{i,k} \in [a_i^* - \epsilon_i, a_i^* + \epsilon_i]$, $\epsilon_i \geq 0$,

b_i : Known

Problem setup



- Control objective: Mean square (MS) stability

$$\mathsf{E}[y_k^2] \to 0, \ \ k \to \infty \ \ \text{for all possible } a_{i,k}$$

- Encoder with a general quantizer

$$s_k = Q\left(\frac{y_k}{\sigma_k}\right) \text{ and a scaling parameter } \ \sigma_k > 0$$

Necessary condition

Theorem

Feedback system is MS stable

$\Longrightarrow$

Data rate $R > \log \dfrac{\log(1-\epsilon_n \nu)^2}{\log(|\lambda_\Pi^*| - \epsilon_n)/(|\lambda_\Pi^*| + \epsilon_n)}$,

Loss prob. $p < \dfrac{1-\epsilon_n^2}{(|\lambda_\Pi^*| + \epsilon_n)^2 - \epsilon_n^2}$,

Uncertainty bound $0 \le \epsilon_n < 1.$

$$\nu \triangleq \sqrt{\frac{1-p}{1-p(|\lambda_\Pi^*| + \epsilon_n)^2}},$$

$\lambda_\Pi^* \triangleq$ Product of the eigenvalues of the nominal plant

Necessary condition

$$\text{MS Stable} \Longrightarrow \begin{cases} R > \log \dfrac{\log(1-\epsilon_n \nu)^2}{\log(|\lambda_\Pi^*| - \epsilon_n)/(|\lambda_\Pi^*| + \epsilon_n)}, \\ p < \dfrac{1-\epsilon_n^2}{(|\lambda_\Pi^*| + \epsilon_n)^2 - \epsilon_n^2}, \quad 0 \le \epsilon_n < 1. \end{cases}$$

- *Without* uncertainty $\xrightarrow{\epsilon_n \to 0}$ $\begin{cases} R > \log(|\lambda_\Pi^*|\nu), \\ p < \dfrac{1}{|\lambda_\Pi^*|^2}. \end{cases}$
  - Coincides with existing results

[You & Xie (2010)]

- *With larger* uncertainty $\xrightarrow{\epsilon_n \to 1}$ $\begin{cases} R \to \infty, \\ p \to 0. \end{cases}$
  - Higher requirements in comm.

Sufficient condition

Theorem

Feedback system is MS stable

$\Longleftarrow$ Spectral radius of $F < 1$

F : A matrix depends on the data rate and the loss probability

$$F \triangleq F_1 F_2, \quad F_1 \triangleq P^T \otimes I_{n^2},$$
$$F_2 \triangleq \mathrm{diag}(H^{(1)} \otimes H^{(1)}, \ldots, H^{(2^n)} \otimes H^{(2^n)}),$$

P : A matrix defined by p

H_k : A random variable matrix depending on the loss states during past n steps

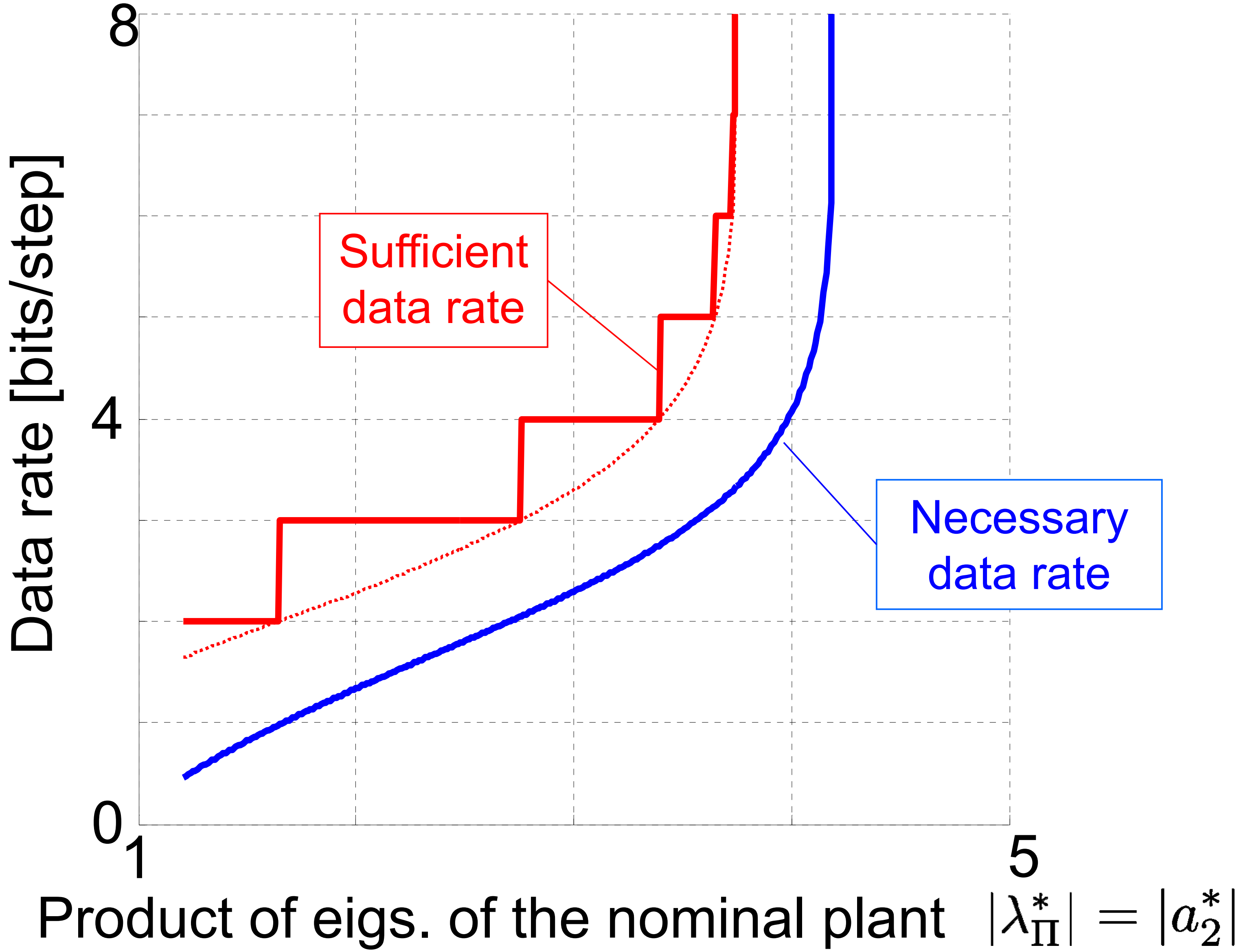
$H^{(i)}$: Realizations of H_k

Based on a stability test for Markov jump systems

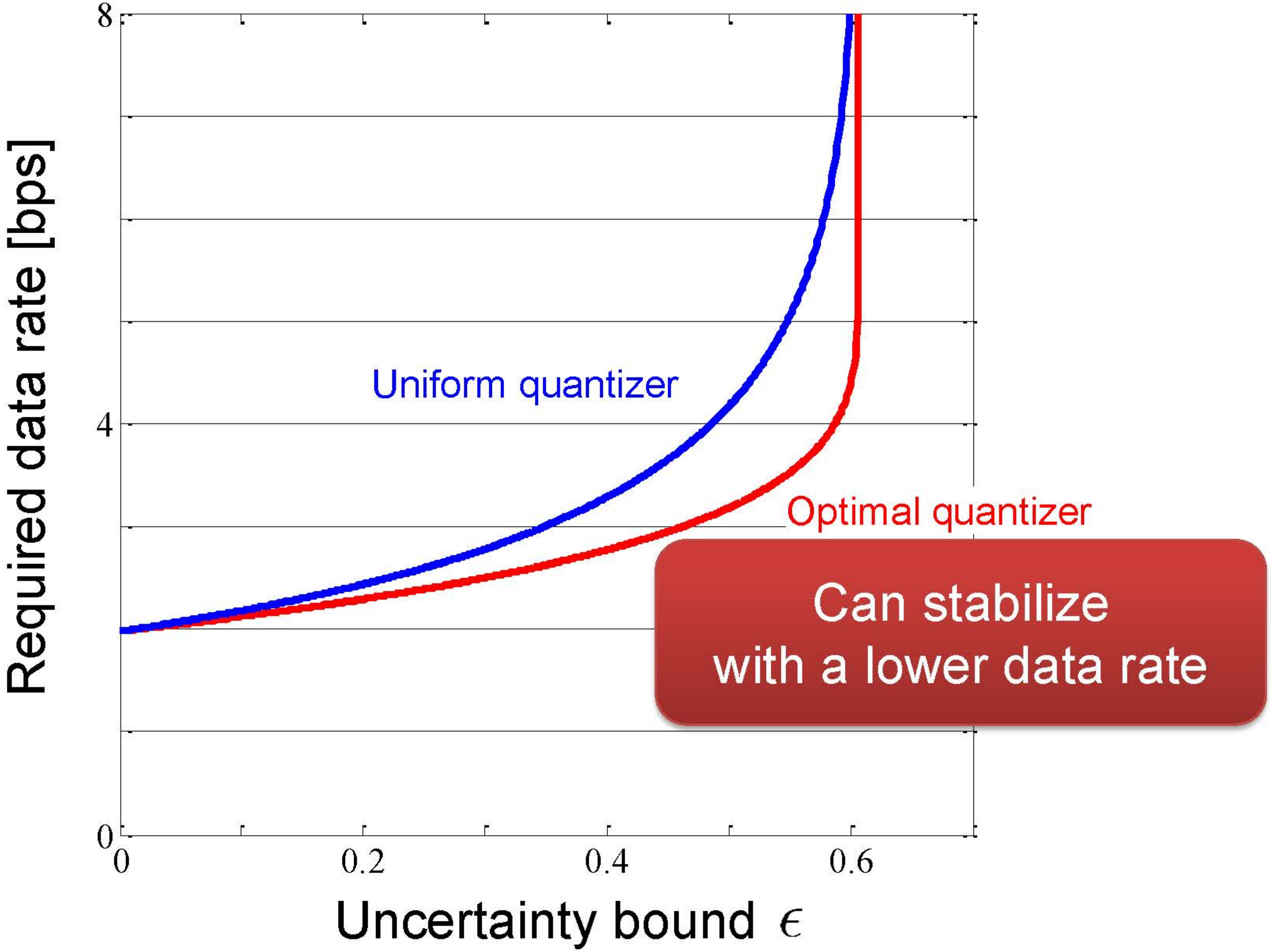
Costa, Fragoso, & Marques (2005)

Gap between upper/lower bounds

$$y_{k+1} = a_{1,k} y_k + a_{2,k} y_{k-1} + u_k, \quad p, a_1^*, \epsilon_1, \epsilon_2 \text{ — fixed}$$



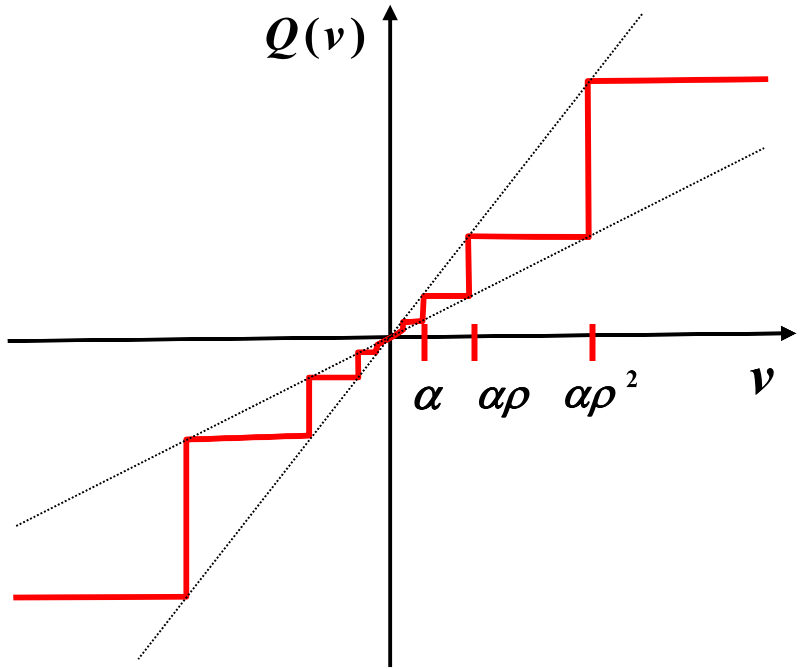
Optimal vs Uniform



Quantized control: The "coarsest" quantizer

- To achieve quadratic stabilization (via static quantization), what is the quantization with the least dense structure?
- The "coarsest" is logarithmic
- Parameter $\rho > 1$ is bounded by unstable eigenvalues:

$$\rho^* = \frac{\prod_i |\lambda_i^u|^2 + 1}{\prod_i |\lambda_i^u|^2 - 1}$$



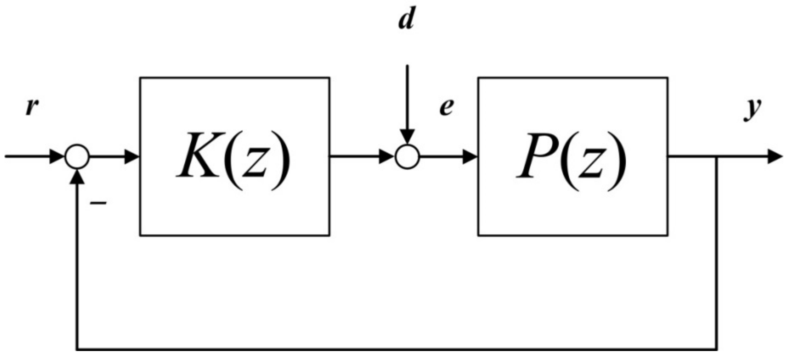
- Tight bounds for ARX systems

Kang & Ishii (Automatica 2015)

Elia & Mitter (2001), Tsumura, Ishii, & Hoshina (2009), Qiu, Gu, & Chen (2013), …

Bode integral for networked control systems

Bode integral: Discrete-time SISO case

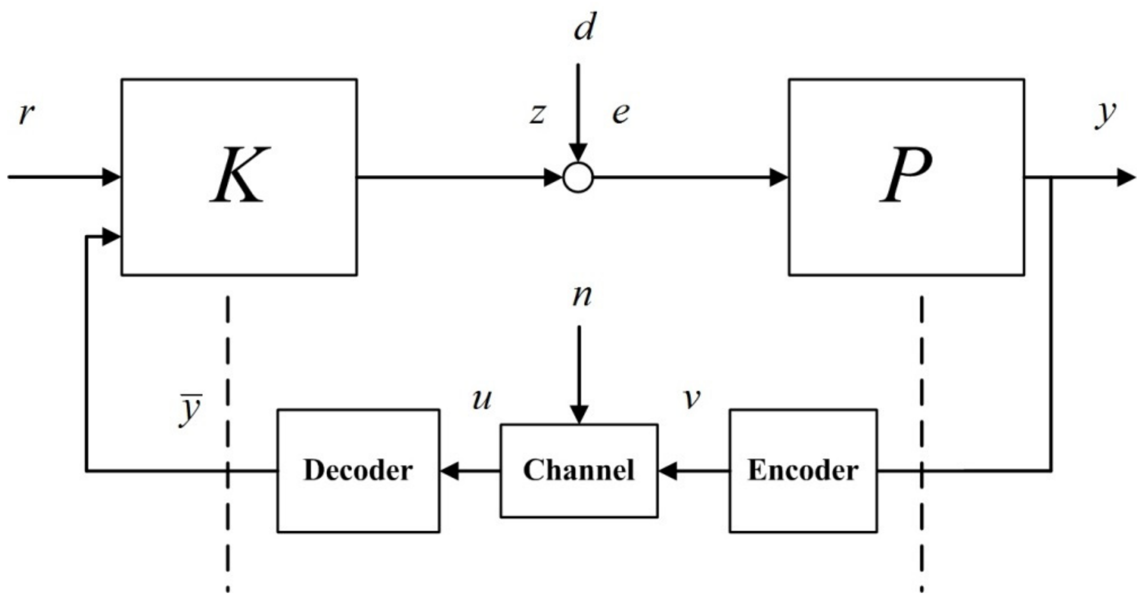


$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\log\left|\frac{1}{1+K(e^{j\omega})P(e^{j\omega})}\right|d\omega=\sum_{\lambda\in UP_A}\log|\lambda|$$

- *P, K*: LTI systems
- Closed-loop is stable
- UP_{KP}: Unstable poles of *P(z)K(z)*
- Fundamental limitation and tradeoffs in design of feedback control systems

Bode (1945), Freudenberg & Looze (1988), Sung & Hara (1988)

Challenges in networked control



- Presence of communication channels in the feedback loop
- Does something like the classical Bode integral exist?
- Information theoretic approach: Systems → Signals

Martins & Dahleh (2008)

Problem setup

- Plant: $x_{k+1} = Ax_k + be_k$

  $y_k = cx_k$

- x_0: Random vector with finite entropy $h(x_0)$
- Controller, encoder, decoder: <u>Causal</u>
- Disturbance $\{d_k\}$: Asymptotically stationary with zero mean, and independent of x_0
- $\{e_k\}$: Asymptotically stationary with zero mean
- Closed-loop system is stable: $\sup E\left(x_k^T x_k\right) < \infty$

Bode-type integral

Theorem

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \log \sqrt{\frac{S_e(e^{j\omega})}{S_d(e^{j\omega})}}\, d\omega \geq \sum_{\lambda \in UE_A} \log|\lambda| - J_\infty(d) + I_\infty(n; e)$$

- $S_d(e^{j\omega}), S_e(e^{j\omega})$: Asymptotic power spectrum
- UE_A: Unstable eigenvalues of system matrix A

Fang, Ishii, & Chen (IFAC WC, 2014)

Bode-type integral

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\log\sqrt{\frac{S_e(e^{j\omega})}{S_d(e^{j\omega})}}\,d\omega \geq \sum_{\lambda\in UE_A}\log|\lambda| - \underline{J_\infty(d)} + \underline{I_\infty(n;e)}$$

Three elements:

1. Unstable poles of plant (more unstable → worse)

2. Negentropy rate of disturbance

   Degree of Gaussianity (more Gaussian → worse)

3. Blurredness of channel

   Degree of noisiness (noisier → worse)

Effect of disturbance

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\log\sqrt{\frac{S_e(e^{j\omega})}{S_d(e^{j\omega})}}d\omega \geq \sum_{\lambda\in UE_A}\log|\lambda| - \underline{J_\infty(d)} + I_\infty(n;e)$$

Negentropy rate

$$J_\infty(x) = \int_{-\pi}^{\pi}\log\sqrt{2\pi e S_x(e^{j\omega})}\,d\omega - h_\infty(x)$$

- $S_x(e^{j\omega})$: Asymptotic power spectrum, $h_\infty(x)$: Entropy rate
- $J_\infty(x) \geq 0$, and $J_\infty(x) = 0$ iff $\{x_k\}$ is Gaussian
- Extension of negentropy of a random variable x with variance σ^2: $J(x) = \log\sqrt{2\pi e S_x(e^{j\omega})} - h(x)$

Effect of the channel

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\log\sqrt{\frac{S_e(e^{j\omega})}{S_d(e^{j\omega})}}d\omega \geq \sum_{\lambda\in UE_A}\log|\lambda| - J_\infty(d) + \underline{I_\infty(n;e)}$$

Theorem:

$$\max_d I_\infty(n;e) \geq B$$

Channel blurredness:

$$B = \min_{p(v)} I(n;u)$$

- Comparison with channel capacity:

$$C = \max_{p(v)} I(v;u)$$

Conclusion

- Networked control with communication constraints
- Minimum data rate for stabilization for uncertain systems
- Bode integral based on information theoretic approach

Fundamental limitations in networked control

- Resilient control against cyber attacks

- Acknowledgements:
  - Kunihisa Okano and Xile Kang (Tokyo Tech)
  - Song Fang and Jie Chen (City U Hong Kong)

References

- K. Okano and H. Ishii, Stabilization of uncertain systems with finite data rates and Markovian packet losses, IEEE Trans. Control of Network Systems, 2014.
- X. Kang and H. Ishii, Coarsest quantization for networked control of uncertain systems, Automatica, 2015.
- S. Fang, H. Ishii, and J. Chen, Control over additive white Gaussian noise channels, Proc. 19th IFAC World Congress, 2014.

- http://www.sc.dis.titech.ac.jp/ishii/